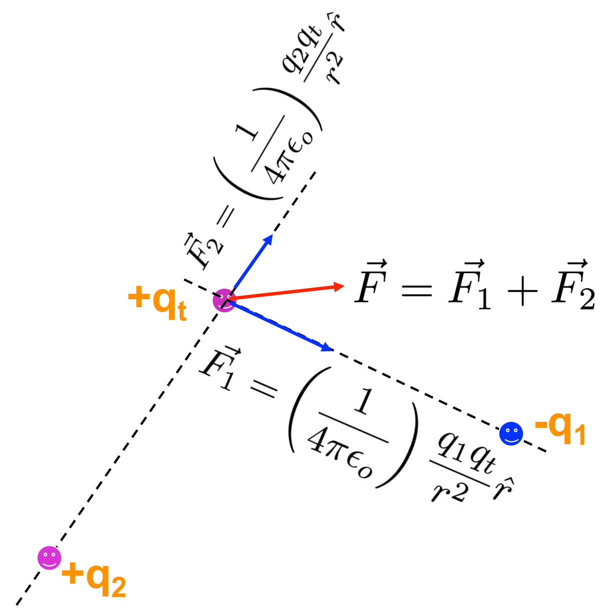
Fields

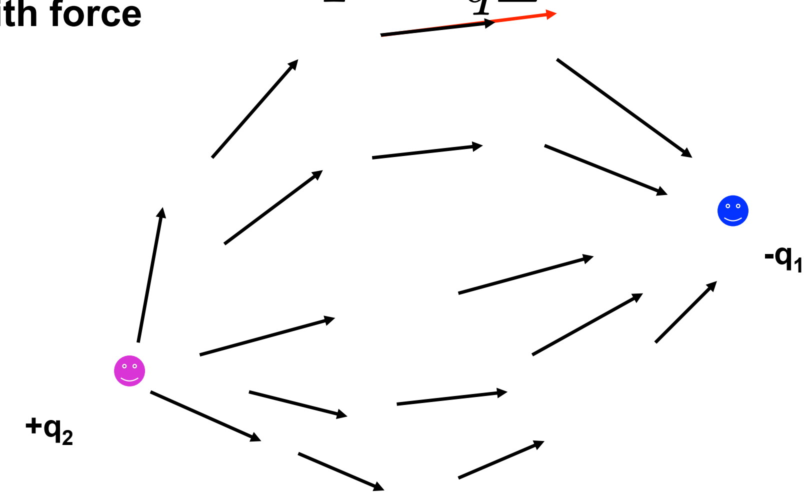
Coulomb's Law



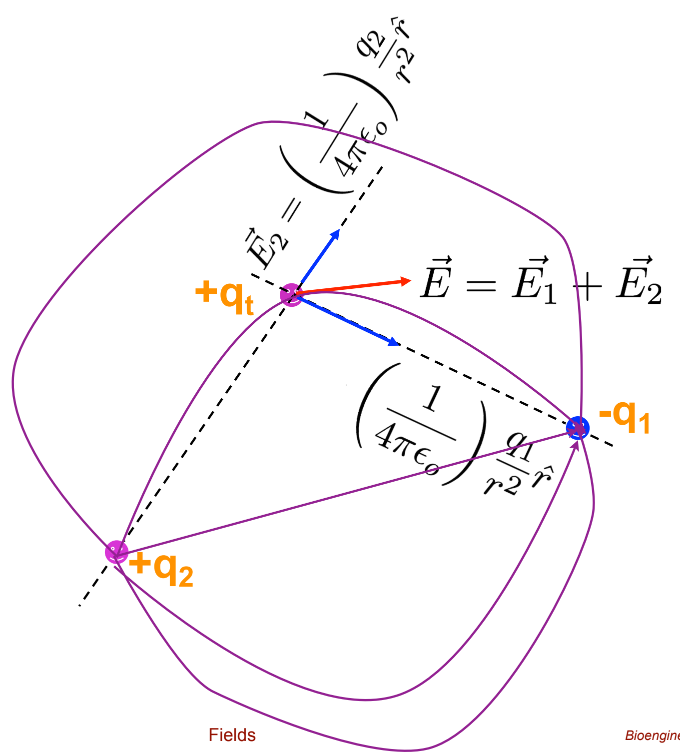
Electric Field

- **Vector valued function**
- **Aligned with force**

$$\vec{F} = q\vec{E}$$



Superposition of Electric Field

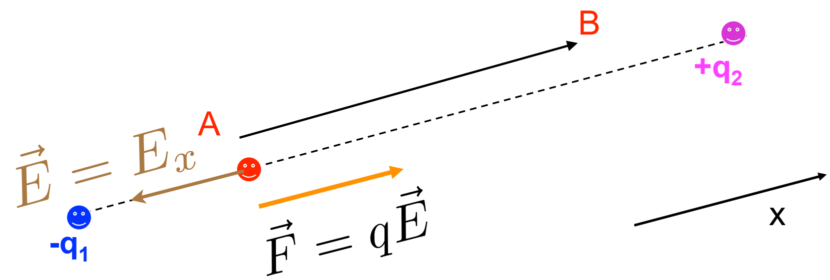


Potential Energy

$$\Delta U = U(B) - U(A) = -q \int_A^B E_x dx$$

$$dU = F_x\, dx = qE_x\, dx$$

$$\Delta U = F_x\, \Delta x = qE_x\, \Delta x$$



Electric Potential

- Scalar field
- Generalization of potential energy as E is a generalization of F
- Always relative measure
- Note sign convention

Potential Energy

$$\Delta U = U(B) - U(A) = -q \int_A^B E_x dx$$

Electric Potential

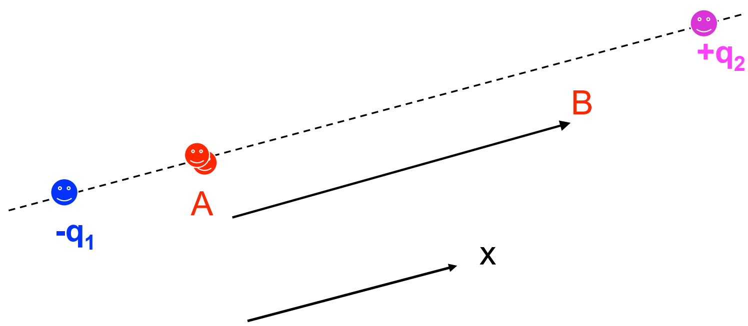
$$\Delta u = -\int_A^B E_x\, dx$$

Another definition of Electric Field

$$E_x = -\frac{\partial u}{\partial x}$$

Electric Potential Field

- Scalar valued field
- Related to energy and work
- Independent of path



Ohm's Law in a Volume Conductor

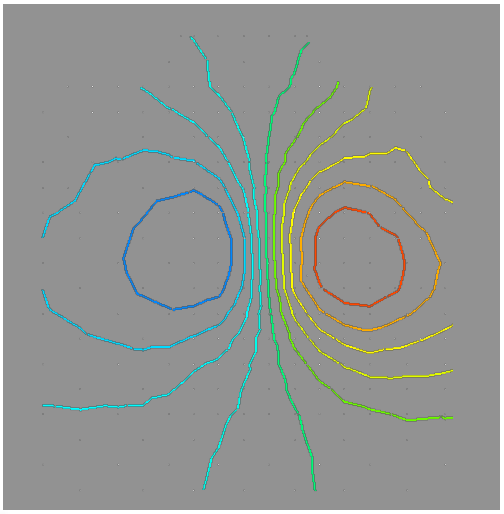
- Continuous form of discrete version
- Expressed in terms of field parameters

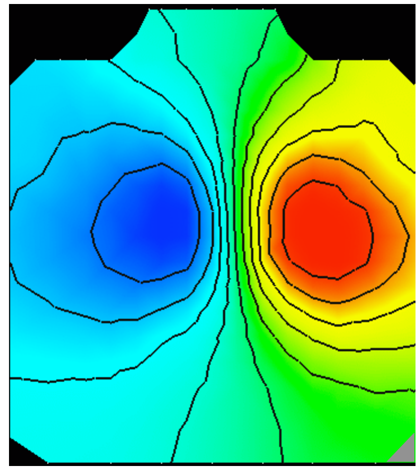
$$j = \frac{i}{A} = \frac{v(B) - v(A)}{RA}$$

$$j_x = -\sigma \frac{\partial v}{\partial x}$$

$$\vec{j} = -\sigma \vec{E}$$

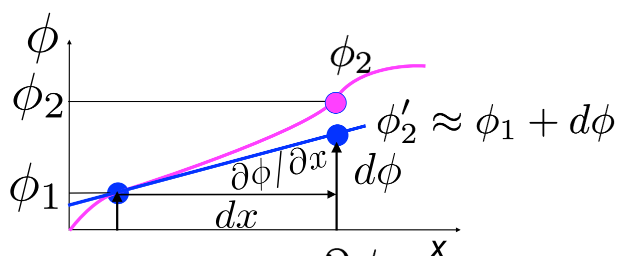
Scalar Fields and Isocontours



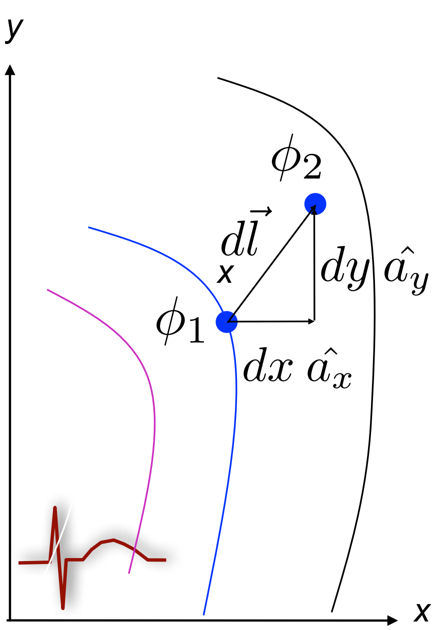


Each line represents a region of constant scalar value

Scalar Difference



$$\phi_2' = \phi_1 + \frac{\partial \phi}{\partial x} dx \; (+ \ldots)$$



$$\phi_2 = \phi_1 + \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \; (+ \ldots .)$$

The Gradient

We wrote for the scalar difference:

$$\begin{aligned}\phi_2 - \phi_1 &= d\phi \\ &= \partial\phi/\partial x\, dx + \partial\phi/\partial y\, dy\end{aligned}$$

Now with

$$d\vec{l} = dx\,\vec{a_x} + dy\,\vec{a_y}$$

we can generalize this as a scalar product by defining the vector operator:

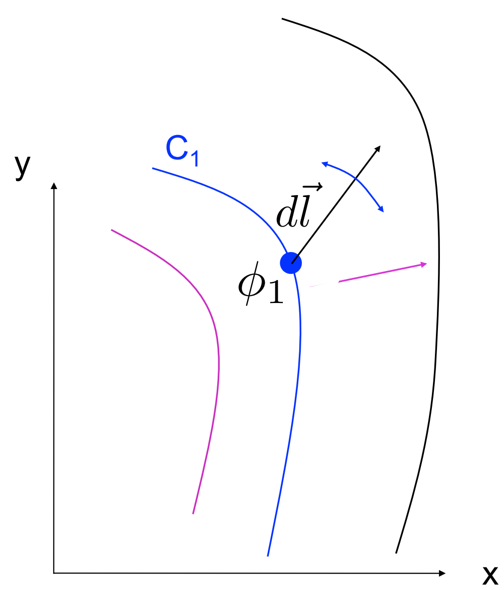
$$\nabla\phi = \partial\phi/\partial x\,\vec{a_x} + \partial\phi/\partial y\,\vec{a_y}$$

So that the product becomes

$$d\phi = \nabla\phi \cdot d\vec{l}$$

We refer to $\nabla\phi$ as the gradient of the scalar field ϕ. (It is another vector field, i.e., associated with all points in space.

Features of the gradient



$$d\phi = \nabla\phi \cdot d\vec{l}$$

Along which *dl* is *d*ϕ a maximum?
 - a minimum?

What is the direction of $\overline{\nabla\phi}$ relative to the iso-value contour?

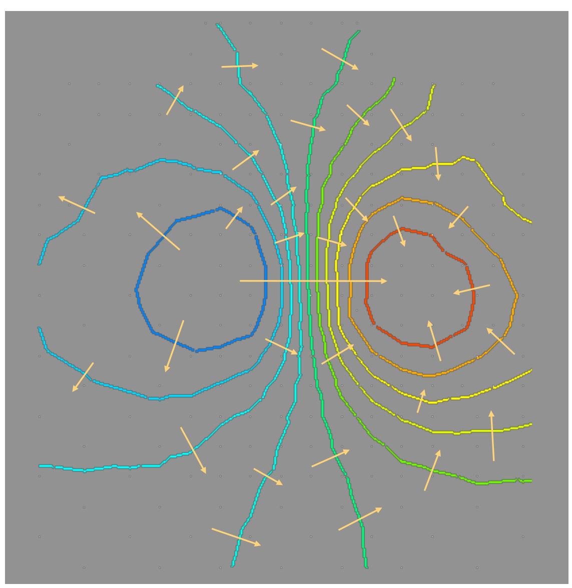
If ϕ is electric potential, what is - $\overline{\nabla\phi}$?

Computing the Gradient

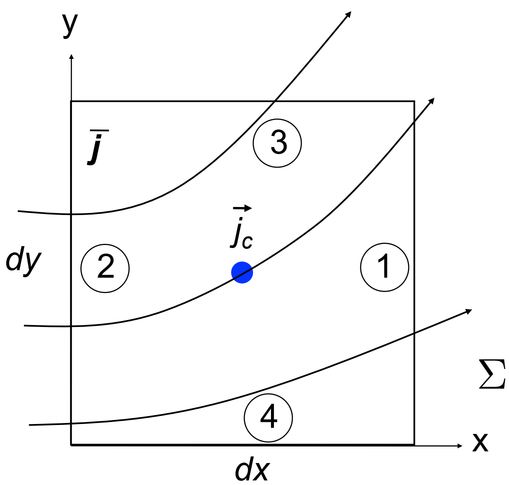
$$\nabla \phi = \frac{\partial \phi}{\partial x} \vec{a_x} + \frac{\partial \phi}{\partial y} \vec{a_y}$$

Only in Cartesian coordinates!

Gradient Example



Divergence



$$\mathrm{Flux}_1 = \left(j_{cx} + \frac{1}{2}\frac{\partial j}{\partial x}\, dy + ...\right)$$

$$\mathrm{Flux}_2 = -\left(j_{cx} - \frac{1}{2}\frac{\partial j}{\partial x}\, dy + ...\right)$$

$$\mathrm{Flux}_3 = \left(j_{cy} + \frac{1}{2}\frac{\partial j_y}{\partial y}\, dx + ...\right)$$

$$\mathrm{Flux}_4 = -\left(j_{cy} - \frac{1}{2}\frac{\partial j_y}{\partial y}\, dx + ...\right)$$

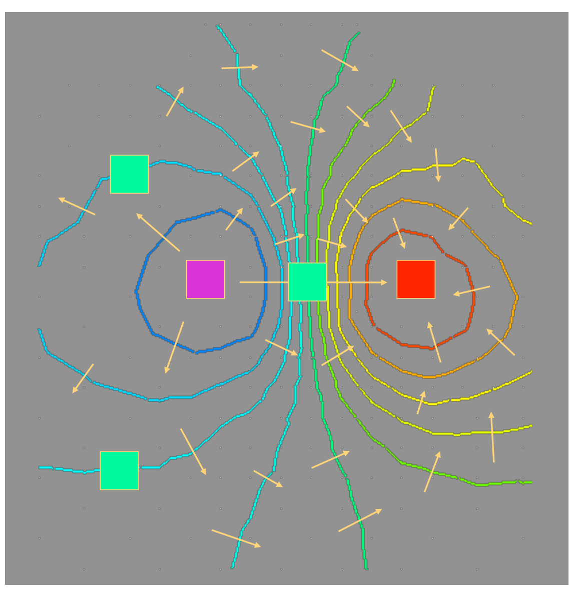
$$\Sigma\mathrm{Flux} = (\partial j_x/\partial x + \partial j_y/\partial y)\, dxdy$$

But the value of flux depends on dxdy, so we define a normalized quantity:

$$Div\, \vec{j} = \lim_{xy \to 0} \frac{\Sigma\mathrm{Flux}}{dxdy} = \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} = \nabla \cdot \vec{j}$$

Divergence Example

- Div = 0
- Div > 0
- Div < 0



Gradient to Laplacian

Gradient
$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\vec{a_x} + \frac{\partial \phi}{\partial y}\vec{a_y}$$

Divergence
$$\nabla \cdot \vec{j} = \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y}$$

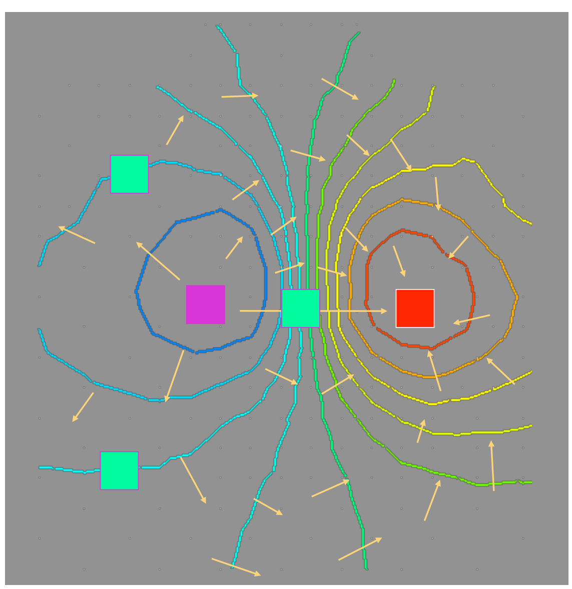
Divergence of the gradient = Laplacian
$$\nabla \cdot \vec{\nabla}\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi$$

Example of Laplacian

Lap = 0

Lap > 0

Lap < 0



Back to (Bioelectricity) Basics

- Apply the vector calculus to currents, electric potentials
  - extend to three dimensions
  - note: formulas apply only to Cartesian coordinates
- Bioelectric volume conductors
  - for there to be current, there must be potential difference
  - quasi static assumption (impedance = resistance)
  - may be uniform, piecewise uniform, or anisotropic
  - “regular” circuits are discrete/lumped version of volume conductors

Electric Potential

- Scalar field
- Generalization of potential energy as E is a generalization of F
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Potential Energy

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Electric Potential

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Another definition of Electric Field

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Potentials, Fields, Currents in Volume Conductors

$$E_x = -\frac{\partial \phi}{\partial x} \qquad \vec{E} = -\vec{\nabla}\phi$$ Definition of *E*

$$\vec{J} = \sigma \vec{E} = -\sigma \vec{\nabla}\phi$$ Ohm's Law

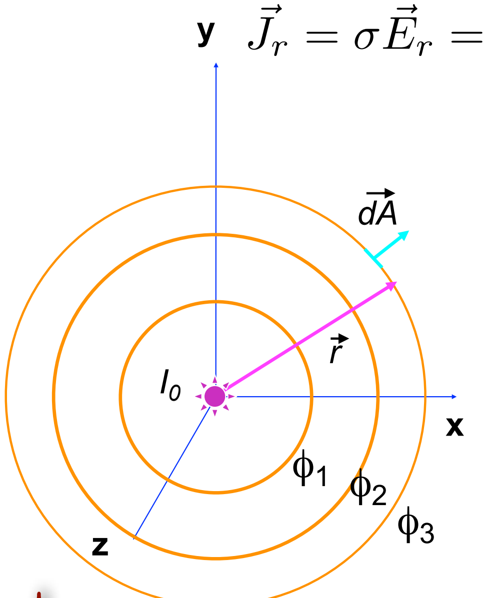
$$\nabla \cdot \vec{J} = -\nabla \cdot \sigma \vec{\nabla}\phi = -\sigma \nabla^2 \phi = I_v$$ I_v is source density

$$\nabla^2 \phi = -\frac{I_v}{\sigma}$$ Poisson's Equation

$$\nabla^2 \phi = 0$$ Laplace's Equation

Current Monopole Field

$$\vec{J}_r = \sigma \vec{E}_r = \sigma \frac{\partial \phi}{\partial r} \qquad I_v = \int \vec{J}_r \cdot d\vec{A} = J_r 4\pi r^2$$



$$\sigma \frac{\partial \phi}{\partial r} = -\frac{I_v}{4\pi r^2} \vec{a_r}$$

$$\frac{\partial \phi}{\partial r} = -\frac{I_v}{4\pi \sigma r^2} \vec{a_r}$$

$$\phi_m = -\int \frac{I_v}{4\pi \sigma r^2} \vec{a_r} \cdot \vec{dr}$$

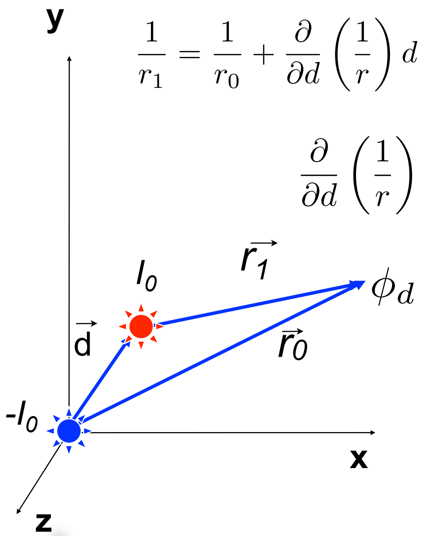
$$\phi_m = \frac{I_v}{4\pi \sigma r} + C$$

Current Dipole Field

$$\phi_m = \frac{I_v}{4\pi\sigma r} + C \qquad \phi_d = -\frac{I_v}{4\pi\sigma r_0} + \frac{I_v}{4\pi\sigma r_1}$$

$$\frac{1}{r_1} = \frac{1}{r_0} + \frac{\partial}{\partial d}\left(\frac{1}{r}\right) d + \dots \qquad \phi_d = -\frac{I_v}{4\pi\sigma r_0} + \frac{I_v}{4\pi\sigma}\left[\frac{1}{r_0} + \frac{\partial}{\partial d}\left(\frac{1}{r}\right) d\right]$$

$$\frac{\partial}{\partial d}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right)\cdot \vec{a_d} \qquad \phi_d = \frac{I_0}{4\pi\sigma}\frac{\partial}{\partial d}\left(\frac{1}{r}\right) d + \dots$$



$$\phi_d = \frac{I_0}{4\pi\sigma}\left[\nabla\left(\frac{1}{r}\right)\cdot \vec{a_d}\right] d$$

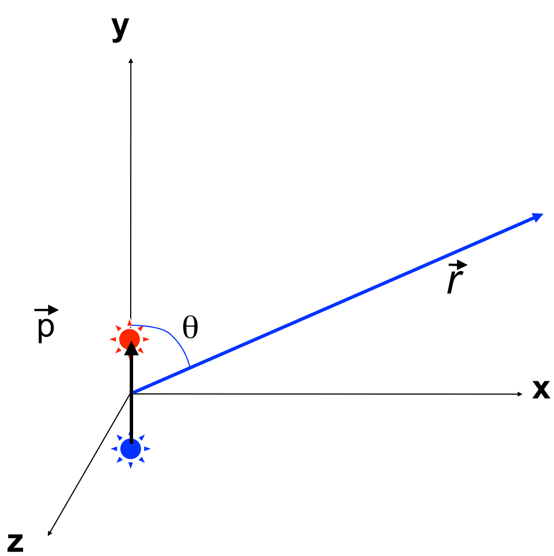
$$I_0\, \vec{a_d}\, d = \vec{p}$$

$$\phi_d = \frac{1}{4\pi\sigma}\nabla\left(\frac{1}{r}\right)\cdot \vec{p}$$

Assumes that d --> 0, “mathematical dipole”

Example

$$\phi_d = \frac{I_0}{4\pi\sigma}\nabla\left(\frac{1}{r}\right)\cdot \vec{p}$$



$$\phi_d = \frac{p\cos\theta}{4\pi\sigma r^2}$$

Potential from a dipole in an infinite homogeneous medium

Why Dipoles?

- Represent bioelectric sources
  - Membrane currents
  - Coupled cells
  - Activation wavefront
  - Whole heart