Polyhedron separation

John Edwards

Department of Computer Science
Computational Visualization Center
Institute of Computational Engineering and Sciences
University of Texas at Austin

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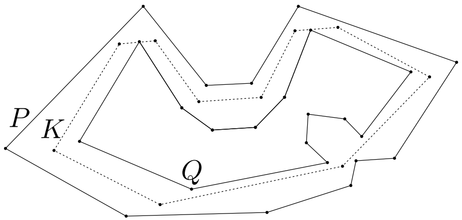
Outline

1. Problem statement and motivating example
2. Separation of convex polyhedra
3. Separation of arbitrary polyhedra
4. Summary and moving forward

Outline

Problem statement

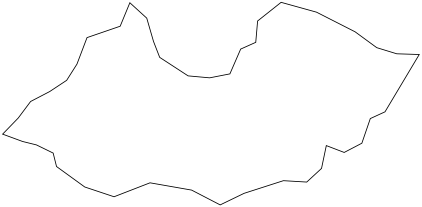


Definition

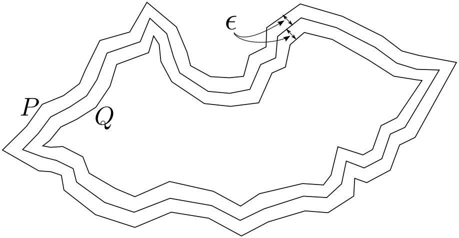
A polyhedron K separates polyhedra P and Q if any path from a point on ∂Q to a point on ∂P passes through ∂K.

Problem statement

Given arbitrary polyhedra P and Q where $\partial Q \cap \partial P = \emptyset$, find a separating polyhedron K with the minimum number of facets possible. K is called an optimal separating polyhedron.



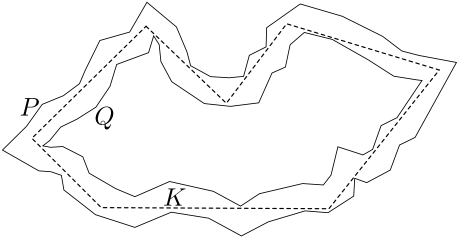
We wish to decimate a polyhedron to as few facets as possible while guaranteeing that any point on the decimated polyhedron is no more than ϵ from the original polyhedron.



First offset the polyhedron by ϵ and $-\epsilon$.

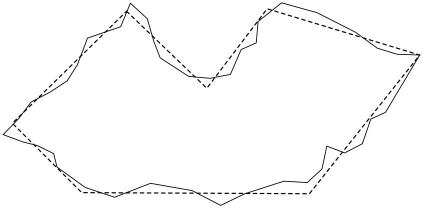
Note: This is not strictly correct. We should really be using the Minkowski sum with a ball of radius ϵ.

Motivation - decimation



Find an optimal separating polyhedron K.

Motivation - decimation



K is within ϵ of the original polyhedron and (hopefully) has fewer facets.

2D vs 3D separation

- In 2D the problem has been solved for convex polygons [Aggarwal et al., 1985] and general simple polygons [Wang and Chan, 1986, Wang, 1991], both in $O(n \log n)$ time.
- In other words, an optimal separating polygon can be found in $O(n \log n)$ time.

2D vs 3D separation

- The 3D case of finding a separating polyhedron between outer P and nested Q is NP-hard [Das, 1990, Das and Goodrich, 1997], even when both polyhedra are convex. But there are approximation algorithms for convex polygons:
  - Clarkson [Clarkson, 1993] gives a randomized algorithm for P and Q convex.
  - Mitchell and Suri [Mitchell and Suri, 1995] give a deterministic algorithm (heretofore referred to as SEPARATE(P,Q)) that works when either of P or Q are non-convex.
  - Both algorithms find a solution $O(\log n)$ times the optimal, where n is the number of faces in P and Q. In other words, if G is the optimal separating polyhedron, then

$$|K| \leq |G| \log(|P| + |Q|)$$

    where $|P|$ is the number of facets, or facet complexity, of polyhedron P.
- This has been stated as an open problem in computational geometry [Mitchell and O'Rourke, 2001].

Outline

Notes

- We will restrict our discussion to polyhedra P and Q such that Q is nested inside P ($Q \subseteq P$). This results in no loss of generality.
- This section discusses Mitchell and Suri's algorithm and proofs [Mitchell and Suri, 1995].
- Diagrams will mostly be in 2D for simplicity.

Canonical separator

- Either P or Q must be convex.
  - If P is convex, then we separate P and CH(Q), the convex hull of Q.
  - If at least Q is convex, then we separate P and Q.
  - From now on we'll assume that Q and P are both convex.
  - We'll see later that this restriction can be relaxed slightly.
- Reminder: a convex polyhedron P can be defined as the intersection of a set of halfspaces. Let $\mathscr{H}(P)$ denote the set of hyperplanes bounding the facets of P.

Definition

A *canonical separator* is a convex separator polyhedron such that the hyperplanes bounding its facets are a subset of the hyperplanes bounding ∂Q.

Canonical separator

Lemma

There exists a canonical separator of P and Q whose facet-complexity is at most three times the facet-complexity of an optimal separator

Note: the authors implicitly assume that the optimal separators are convex.

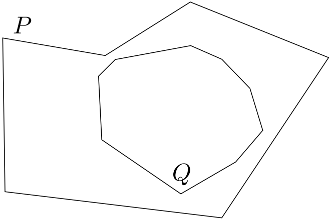
Proof

- Let G be a minimum separator
- Translate each facet of G towards Q until it is incident with a vertex of Q
- Let k' be a translated facet incident with vertex $v \in Q$ and let $q_1, q_2, \ldots, q_s$ be hyperplanes of Q passing through v.
- $m^+ \supseteq \bigcap_{i=1}^{s} q_i^+$ which implies that $m^- \subseteq \bigcup_{i=1}^{s} q_i^-$.
- By Caratheodory's Theorem, $m^- \subseteq q_j^- \cup q_k^- \cup q_l^-$.
- Thus three halfspaces induced by facets of Q can be substituted for each facet of G.

The algorithm

SEPARATE(P,Q)

$\mathscr{C} \leftarrow \mathscr{C}(\mathscr{H}(Q))$
$\mathscr{H}(K) \leftarrow \emptyset$
while $\mathscr{C} \neq \emptyset$ **do**
    Select a plane $q_i \in \mathscr{H}(Q)$ that maximizes $|\mathscr{C} \cap S(q_i)|$
    $\mathscr{H}(K) \leftarrow \mathscr{H}(K) \cup \{q_i\}$
    $\mathscr{C} \leftarrow \mathscr{C} - S(q_i)$
end while
return $\mathscr{H}(K)$



SEPARATE(P,Q)

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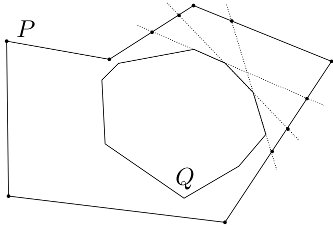
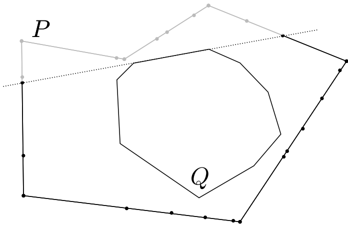
7: **end while**
8: return $\mathscr{H}(K)$



SEPARATE(P,Q)

```
𝒞 ← 𝒞(ℋ(Q))
ℋ(K) ← ∅
while 𝒞 ≠ ∅ do
    Select a plane q_i ∈ ℋ(Q) that maximizes |𝒞 ∩ S(q_i)|
    ℋ(K) ← ℋ(K) ∪ {q_i}
    𝒞 ← 𝒞 − S(q_i)
end while
return ℋ(K)
```



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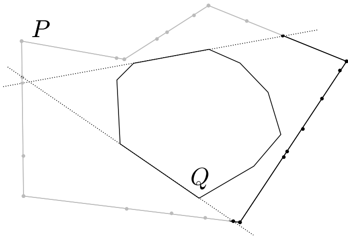

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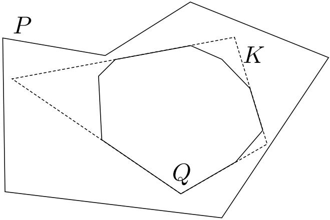


Lemma

K is a separating polyhedron, or equivalently, $Q \subseteq K \subseteq P$.

Proof

Let $(h_1, \ldots, h_t) = \mathscr{H}(K)$. Since $\bigcup_{i=1}^{t} h_i^- \supset \partial P$ we know that $\bigcap_{i=1}^{t} h_i^+ = K \subseteq P$. Since also $\mathscr{H}(K) \subseteq \mathscr{H}(Q)$ we see that $Q \subseteq K$.

Canonical separator

Theorem

K is an $O(\log n)$ approximation of an optimal separator G, or in other words, $|K| = O(|G| \log n)$.

Proof (idea)

It all boils down to the set cover problem. The greedy approach has been shown to be an $O(\log n)$ approximation of the optimal.

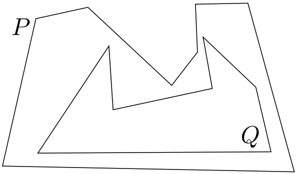
Outline

- Let $\mathrm{CH}(Q)$ be the convex hull of Q.
- Let C be the maximal connected subsets of $\mathrm{CH}(Q) - Q$.
- We define the *mortise regions* of P and Q as $\mathscr{R}(P,Q) = \{M \in C | M \cap \overline{P} \neq \emptyset\}$.
- Given a mortise region R, we define the associated *mortise* as $M(R) = \partial R - \partial \mathrm{CH}(Q)$. We let $\mathscr{M}(P,Q)$ be the set of all mortises.
- Define $\mathscr{F}(P)$ to be the set of all facets of polyhedron P.
- Given a mortise M, we define its associated tenon $T(M) = \{f \in \mathscr{F}(P) | f \cap \mathrm{CH}(M) \neq \emptyset\}$. We will use T_i as shortcut notation for $T(M_i)$.
  - Note that a given tenon may not be connected, i.e., P may protrude into the mortise multiple times.
- Note: now we can loosen the restrictions for the convex algorithm: both polyhedra may be non-convex if there are no mortises.

The Algorithm

ArbitrarySeparate(P,Q)

$Q' \leftarrow \partial \mathrm{CH}(Q)$
$P' \leftarrow \partial P - \bigcup_{T \in \mathscr{T}(P,Q)} \mathscr{F}(T)$
$K^H \leftarrow$ Separate(P', Q')
$K^M = \emptyset$
for all $M \in \mathscr{M}(P,Q)$ **do**
    $K_i^M \leftarrow$ ArbitrarySeparate$(M, T(M))$
    $K^M \leftarrow K^M \cup K_i^M$
end for
$K \leftarrow K^H - K^M$
return K



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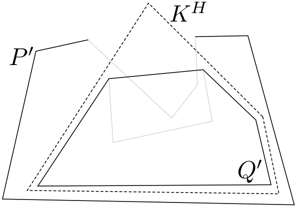

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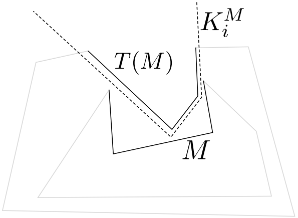

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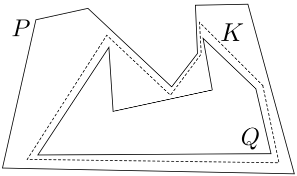


Assume no tenons have "subtenons." Let G^H be the optimal separator of P' and Q'. Similarly, let G_i^M be the optimal separator of M_i and $T(M_i)$. Clearly,

$$|G^H|, |G_i^M| \leq |G|$$

Claim

If all edges in the intersection region of two polyhedra A and B are convex then $|A - B| \leq |A| + |B|$.

If the claim is true, then

$$|G^H - \bigcup_i G_i^M| = |G^H - G^M| \leq |G^H| + |G^M| \leq m|G| \tag{1}$$

where m is the number of mortise regions.

Facet complexity of K

Let $f = \mathscr{F}(\mathrm{CH}(Q)) - \mathscr{F}(Q)$, i.e. the number of facets in the convex hull of Q that are not on Q. Also let $\tau = \sum_i |T_i|$ and $\mu = \sum_j |M_j|$. By [Mitchell and Suri, 1995], we have a bound on facet complexity of a separator given convex P and Q. Using this bound we derive the following:

$$|K^H| = O(G^H \log(|P| - \tau + |Q| - \mu + f))$$

$$|K_i^M| = O(G_i^M \log(|T_i| + |M_i|))$$

Rearranging we get

$$\frac{|K^H|}{c^H \log(|P| - \tau + |Q| - \mu + f)} \leq |G^H|$$

$$\frac{|K_i^M|}{c^M \log(|T_i| + |M_i|)} \leq |G_i^M|$$

for constants c_H and c_M.

Facet complexity of K

Now we find a relation of G^M with K^M:

$$
\begin{aligned}
|G^M| &= \sum_i \frac{|K_i^M|}{c^M \log(|T_i| + |M_i|)} \\
&\geq \frac{\sum_i |K_i^M|}{mc^M \sum_i \log(|T_i| + |M_i|)} \\
&\geq \frac{\sum_i |K_i^M|}{mc^M \log(\sum_i |T_i| + |M_i|)} \\
&\geq \frac{\sum_i |K_i^M|}{mc^M \log(\tau + \mu)}
\end{aligned}
$$

Facet complexity of K

$$
\begin{aligned}
|G^H| + |G^M| \geq & \frac{|K^H|}{c^H \log\left(|P| - \tau + |Q| - \mu + f\right)} \\
& + \frac{\sum_i |K_i^M|}{c^M \log\left(\tau + \mu\right)} \\
= & \frac{|K^H| c^M \log^2\left(\tau + \mu\right) + \sum_i |K_i^M| c^H \log\left(|P| + |Q| + f\right)}{c^M c^H \log\left(|P| + |Q| + f + \tau + \mu\right)} \\
\geq & \frac{|K^H| c^M \log\left(\tau + \mu\right) + \sum_i |K_i^M| c^H \log\left(\tau + \mu\right)}{c^M c^H \log\left(|P| + |Q| + f + \tau + \mu\right)} \\
= & \frac{|K^H| c^M + \sum_i |K_i^M| c^H}{c^M c^H \log\left(|P| + |Q| + f\right)}
\end{aligned}
$$

$$
|K^H| + \sum_i |K_i^M| \leq |K| = O((|G^H| + |G^M|) \log\left(|P| + |Q| + f\right))
$$

$$
|K| = O(m|G| \log\left(|P| + |Q| + f\right)) \tag{2}
$$

Facet complexity of K

Bound for convex polyhedra:

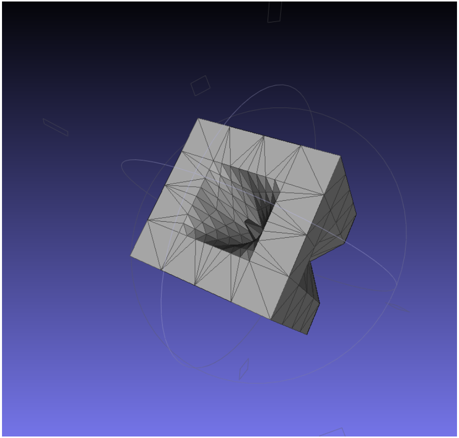
$$|K| = O(|G| \log (|P| + |Q|))$$

Bound for arbitrary polyhedra:

$$|K| = O(m|G| \log (|P| + |Q| + f))$$

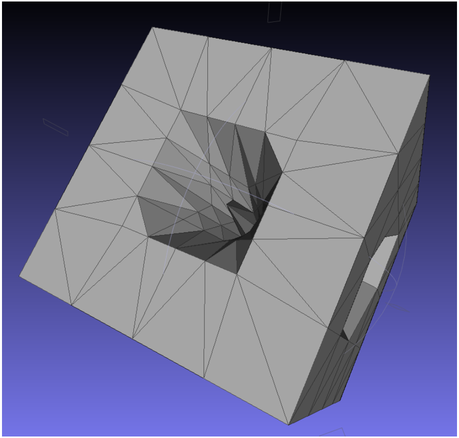
Reminder:

- m = the number of mortises
- $|G|$ = the number of facets of the optimal separator
- f = the number of facets in CH(Q) that are not in Q



- $|\mathbf{Q}| = \mathbf{176}$
- $|P| = 80$
- $|K| = 28$

Implementation



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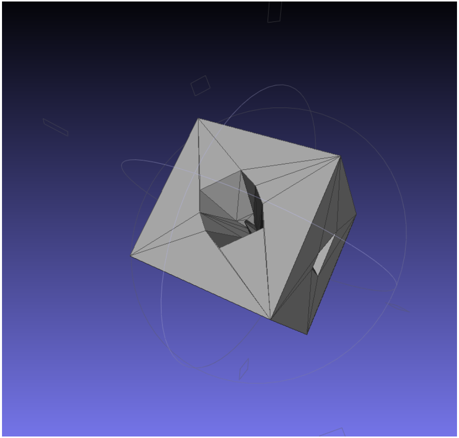

- $|Q| = 176$
- $|P| = 80$
- $|\mathbf{K}| = \mathbf{28}$

Outline

- Finding separating polyhedra is important in error-bounded decimation
- Current published work finds bounded separating polyhedra in 3D only if one or both of the polyhedra are convex
- Our algorithm finds a separator a factor of $m \log(|P| + |Q| + f)$ from the optimal where:
  - m = the number of mortises
  - f = the number of facets in CH(Q) that are not in Q
- We believe the algorithm is roughly $O(n^3)$ but this needs to be derived

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Caratheodory's Theorem [Schrijver, 1986]

Theorem

Let $a_1, \ldots, a_m, b$ be vectors in n-dimensional space. Then either:

I b is a nonnegative linear combination of linearly independent vectors from $a_1, \ldots, a_m$, or

II there exists a hyperplane x|cx=0, containing $t-1$ linearly independent vectors from $a_1, \ldots, a_m$, such that $cb < 0$ and $ca_1, \ldots, ca_m \geq 0$, where t is rank$(a_1, \ldots, a_m, b)$.

Application: translate m and $q_1, \ldots, q_s$ to the origin. The normals span $\mathbb{R}^3$. By Caratheodory's Theorem we can say that

$$m = \lambda_1 q_j + \lambda_2 q_k + \lambda_3 q_l$$

where $\lambda_1, \lambda_2, \lambda_3 \geq 0$. Thus, any point $x \in m^-$, or in other words, $m^T x < 0$, also satisfies $\lambda_1 q_j^T x + \lambda_2 q_k^T x + \lambda_3 q_l^T x < 0$. Since $\lambda_1, \lambda_2, \lambda_3 \geq 0$, there must be at least one q_i such that $q_i^T x < 0$. Thus, $m^- \subseteq q_j^- \cup q_k^- \cup q_l^-$.