Polyhedron separation

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1. Problem statement and motivating example

2. Separation of convex polyhedra

3. Separation of arbitrary polyhedra

4. Summary and moving forward
Outline

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3. Separation of arbitrary polyhedra
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Problem statement

Given arbitrary polyhedra $P$ and $Q$ where $\partial Q \cap \partial P = \emptyset$, find a separating polyhedron $K$ with the minimum number of facets possible. $K$ is called an optimal separating polyhedron.

Definition

A polyhedron $K$ separates polyhedra $P$ and $Q$ if any path from a point on $\partial Q$ to a point on $\partial P$ passes through $\partial K$. 
We wish to decimate a polyhedron to as few facets as possible while guaranteeing that any point on the decimated polyhedron is no more than $\epsilon$ from the original polyhedron.
First offset the polyhedron by $\epsilon$ and $-\epsilon$.

Note: This is not strictly correct. We should really be using the Minkowski sum with a ball of radius $\epsilon$. 
Find an optimal separating polyhedron $K$. 
$K$ is within $\epsilon$ of the original polyhedron and (hopefully) has fewer facets.
In 2D the problem has been solved for convex polygons [Aggarwal et al., 1985] and general simple polygons [Wang and Chan, 1986, Wang, 1991], both in $O(n \log n)$ time.

In other words, an optimal separating polygon can be found in $O(n \log n)$ time.
The 3D case of finding a separating polyhedron between outer $P$ and nested $Q$ is NP-hard [Das, 1990, Das and Goodrich, 1997], even when both polyhedra are convex. But there are approximation algorithms for convex polygons:

- Clarkson [Clarkson, 1993] gives a randomized algorithm for $P$ and $Q$ convex.
- Mitchell and Suri [Mitchell and Suri, 1995] give a deterministic algorithm (heretofore referred to as $\text{SEPARATE}(P,Q)$) that works when either of $P$ or $Q$ are non-convex.
- Both algorithms find a solution $O(\log n)$ times the optimal, where $n$ is the number of faces in $P$ and $Q$. In other words, if $G$ is the optimal separating polyhedron, then

\[ |K| \leq |G| \log (|P| + |Q|) \]

where $|P|$ is the number of facets, or facet complexity, of polyhedron $P$.

- This has been stated as an open problem in computational geometry [Mitchell and O’Rourke, 2001].
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We will restrict our discussion to polyhedra $P$ and $Q$ such that $Q \subseteq P$. This results in no loss of generality.

This section discusses Mitchell and Suri’s algorithm and proofs [Mitchell and Suri, 1995].

Diagrams will mostly be in 2D for simplicity.
Either $P$ or $Q$ must be convex.

- If $P$ is convex, then we separate $P$ and $\text{CH}(Q)$, the convex hull of $Q$.
- If at least $Q$ is convex, then we separate $P$ and $Q$.
- From now on we’ll assume that $Q$ and $P$ are both convex.
- We’ll see later that this restriction can be relaxed slightly.

Reminder: a convex polyhedron $P$ can be defined as the intersection of a set of halfspaces. Let $\mathcal{H}(P)$ denote the set of hyperplanes bounding the facets of $P$.

**Definition**

A *canonical separator* is a convex separator polyedron such that the hyperplanes bounding its facets are a subset of the hyperplanes bounding $\partial Q$.
Lemma
There exists a canonical separator of $P$ and $Q$ whose facet-complexity is at most three times the facet-complexity of an optimal separator.

Note: the authors implicitly assume that the optimal separators are convex.

Proof

- Let $G$ be a minimum separator.
- Translate each facet of $G$ towards $Q$ until it is incident with a vertex of $Q$.
- Let $k'$ be a translated facet incident with vertex $v \in Q$ and let $q_1, q_2, \ldots, q_s$ be hyperplanes of $Q$ passing through $v$.
- $m^+ \supseteq \bigcap_{i=1}^s q_i^+$ which implies that $m^- \subseteq \bigcup_{i=1}^s q_i^-$. 
- By Caratheodory’s Theorem, $m^- \subseteq q_j^- \cup q_k^- \cup q_l^-$. 
- Thus three halfspaces induced by facets of $Q$ can be substituted for each facet of $G$. 
The algorithm

\textbf{SEPARATE}(P,Q)

1: \( C \leftarrow C(H(Q)) \)
2: \( H(K) \leftarrow \emptyset \)
3: \textbf{while} \( C \neq \emptyset \) \textbf{do}
4: \hspace{1em} Select a plane \( q_i \in H(Q) \) that maximizes \( |C \cap S(q_i)| \)
5: \hspace{1em} \( H(K) \leftarrow H(K) \cup \{q_i\} \)
6: \hspace{1em} \( C \leftarrow C - S(q_i) \)
7: \textbf{end while}
8: return \( H(K) \)
The algorithm

\textsc{Separate}(P,Q)

1: \( C \leftarrow C(\mathcal{H}(Q)) \)
2: \( \mathcal{H}(K) \leftarrow \emptyset \)
3: \textbf{while} \( C \neq \emptyset \) \textbf{do}
4: \hspace{1em} Select a plane \( q_i \in \mathcal{H}(Q) \) that maximizes \( |C \cap S(q_i)| \)
5: \hspace{1em} \( \mathcal{H}(K) \leftarrow \mathcal{H}(K) \cup \{q_i\} \)
6: \hspace{1em} \( C \leftarrow C - S(q_i) \)
7: \hspace{1em} \textbf{end while}
8: \textbf{return} \( \mathcal{H}(K) \)
The algorithm

SEPARATE(P,Q)
1: \( C \leftarrow C(\mathcal{H}(Q)) \)
2: \( \mathcal{H}(K) \leftarrow \emptyset \)
3: while \( C \neq \emptyset \) do
4: Select a plane \( q_i \in \mathcal{H}(Q) \) that maximizes \( |C \cap S(q_i)| \)
5: \( \mathcal{H}(K) \leftarrow \mathcal{H}(K) \cup \{q_i\} \)
6: \( C \leftarrow C - S(q_i) \)
7: end while
8: return \( \mathcal{H}(K) \)
**The algorithm**

**Separate** $(P, Q)$

1. $C \leftarrow C(H(Q))$
2. $H(K) \leftarrow \emptyset$
3. **while** $C \neq \emptyset$ **do**
   4. Select a plane $q_i \in H(Q)$ that maximizes $|C \cap S(q_i)|$
   5. $H(K) \leftarrow H(K) \cup \{q_i\}$
   6. $C \leftarrow C - S(q_i)$
4. **end while**
5. return $H(K)$
The algorithm

\textbf{SEPARATE}(P, Q)

1: \( C \leftarrow C(\mathcal{H}(Q)) \)
2: \( \mathcal{H}(K) \leftarrow \emptyset \)
3: \textbf{while} \( C \neq \emptyset \) \textbf{do}
4: \text{Select a plane } q_i \in \mathcal{H}(Q) \text{ that maximizes } |C \cap S(q_i)|
5: \( \mathcal{H}(K) \leftarrow \mathcal{H}(K) \cup \{q_i\} \)
6: \( C \leftarrow C - S(q_i) \)
7: \textbf{end while}
8: \text{return } \mathcal{H}(K)
Lemma

$K$ is a separating polyhedron, or equivalently, $Q \subseteq K \subseteq P$.

Proof

Let $(h_1, \ldots, h_t) = \mathcal{H}(K)$. Since $\bigcup_{i=1}^{t} h_i^- \supset \partial P$ we know that $\bigcap_{i=1}^{t} h_i^+ = K \subseteq P$. Since also $\mathcal{H}(K) \subseteq \mathcal{H}(Q)$ we see that $Q \subseteq K$. 
Theorem

\( K \) is an \( O(\log n) \) approximation of an optimal separator \( G \), or in other words, \( |K| = O(|G| \log n) \).

Proof (idea)

It all boils down to the set cover problem. The greedy approach has been shown to be an \( O(\log n) \) approximation of the optimal.
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Let \( \text{CH}(Q) \) be the convex hull of \( Q \).

Let \( C \) be the maximal connected subsets of \( \text{CH}(Q) - Q \).

We define the **mortise regions** of \( P \) and \( Q \) as
\[
\mathcal{R}(P, Q) = \{ M \in C | M \cap \overline{P} \neq \emptyset \}.
\]

Given a mortise region \( R \), we define the associated **mortise** as
\[
M(R) = \partial R - \partial \text{CH}(Q).
\]
We let \( \mathcal{M}(P, Q) \) be the set of all mortises.

Define \( \mathcal{F}(P) \) to be the set of all facets of polyhedron \( P \).

Given a mortise \( M \), we define its associated tenon
\[
T(M) = \{ f \in \mathcal{F}(P) | f \cap \text{CH}(M) \neq \emptyset \}.
\]
We will use \( T_i \) as shortcut notation for \( T(M_i) \).

- Note that a given tenon may not be connected, i.e., \( P \) may protrude into the mortise multiple times.

- Note: now we can loosen the restrictions for the convex algorithm: both polyhedra may be non-convex if there are no mortises.
**Algorithm**

ArbitrarySeparate\((P,Q)\)

1: \( Q' \leftarrow \partial\text{CH}(Q) \)
2: \( P' \leftarrow \partial P - \bigcup_{T \in \mathcal{T}(P,Q)} \mathcal{F}(T) \)
3: \( K^H \leftarrow \text{Separate}(P', Q') \)
4: \( K^M = \emptyset \)
5: **for all** \( M \in \mathcal{M}(P, Q) \) **do**
6: \( K^M_i \leftarrow \text{ArbitrarySeparate}(M, T(M)) \)
7: \( K^M \leftarrow K^M \cup K^M_i \)
8: **end for**
9: \( K \leftarrow K^H - K^M \)
10: return \( K \)
**ArbitrarySeparate**(P,Q)

1: \( Q' \leftarrow \partial \text{CH}(Q) \)
2: \( P' \leftarrow \partial P - \bigcup_{T \in \mathcal{T}(P, Q)} \mathcal{F}(T) \)
3: \( K^H \leftarrow \text{Separate}(P', Q') \)
4: \( K^M = \emptyset \)
5: **for all** \( M \in \mathcal{M}(P, Q) \) **do**
6: \( K^M_i \leftarrow \text{ArbitrarySeparate}(M, T(M)) \)
7: \( K^M \leftarrow K^M \cup K^M_i \)
8: **end for**
9: \( K \leftarrow K^H - K^M \)
10: **return** \( K \)
**The Algorithm**

**ArbitrarySeparate**(P,Q)

1: Q′ ← ∂CH(Q)
2: P′ ← ∂P − ∪_{T \in \mathcal{T}(P,Q)} \mathcal{F}(T)
3: K^H ← Separate(P′, Q′)
4: K^M = ∅
5: for all M ∈ \mathcal{M}(P, Q) do
6:   K^M_i ← ArbitrarySeparate(M, T(M))
7:   K^M ← K^M ∪ K^M_i
8: end for
9: K ← K^H − K^M
10: return K
ArbitrarySeparate(P,Q)

1: \( Q' \leftarrow \partial \text{CH}(Q) \)
2: \( P' \leftarrow \partial P - \bigcup_{T \in \mathcal{T}(P,Q)} \mathcal{F}(T) \)
3: \( K^H \leftarrow \text{Separate}(P', Q') \)
4: \( K^M = \emptyset \)
5: for all \( M \in \mathcal{M}(P, Q) \) do
6: \( K^M_i \leftarrow \text{ArbitrarySeparate}(M, T(M)) \)
7: \( K^M \leftarrow K^M \cup K^M_i \)
8: end for
9: \( K \leftarrow K^H - K^M \)
10: return \( K \)
Assume no tenons have “subtenons.” Let $G^H$ be the optimal separator of $P'$ and $Q'$. Similarly, let $G_i^M$ be the optimal separator of $M_i$ and $T(M_i)$. Clearly,

$$|G^H|, |G_i^M| \leq |G|$$

**Claim**

If all edges in the intersection region of two polyhedra $A$ and $B$ are convex then $|A - B| \leq |A| + |B|$.

If the claim is true, then

$$|G^H - \bigcup_i G_i^M| = |G^H - G^M| \leq |G^H| + |G^M| \leq m|G|$$  \(1\)

where $m$ is the number of mortise regions.
Let $f = \mathcal{F}(\text{CH}(Q)) - \mathcal{F}(Q)$, i.e. the number of facets in the convex hull of $Q$ that are not on $Q$. Also let $\tau = \sum_i |T_i|$ and $\mu = \sum_j |M_j|$. By [Mitchell and Suri, 1995], we have a bound on facet complexity of a separator given convex $P$ and $Q$. Using this bound we derive the following:

\[ |K^H| = O(G^H \log (|P| - \tau + |Q| - \mu + f)) \]
\[ |K_i^M| = O(G_i^M \log (|T_i| + |M_i|)) \]

Rearranging we get

\[ \frac{|K^H|}{c^H \log (|P| - \tau + |Q| - \mu + f)} \leq |G^H| \]
\[ \frac{|K_i^M|}{c_i^M \log (|T_i| + |M_i|)} \leq |G_i^M| \]

for constants $c_H$ and $c_M$. 
Facet complexity of $K$

Now we find a relation of $G^M$ with $K^M$:

$$|G^M| = \sum_i \frac{|K^M_i|}{c^M \log (|T_i| + |M_i|)}$$

$$\geq \frac{\sum_i |K^M_i|}{mc^M \sum_i \log (|T_i| + |M_i|)}$$

$$\geq \frac{\sum_i |K^M_i|}{mc^M \log (\sum_i |T_i| + |M_i|)}$$

$$\geq \frac{\sum_i |K^M_i|}{mc^M \log (\tau + \mu)}$$
Facet complexity of $K$

$$|G^H| + |G^M| \geq \frac{|K^H|}{c^H \log (|P| - \tau + |Q| - \mu + f)}$$

$$+ \frac{\sum_i |K^M_i|}{c^M \log (\tau + \mu)}$$

$$= \frac{|K^H| c^M \log^2 (\tau + \mu) + \sum_i |K^M_i| c^H \log (|P| + |Q| + f)}{c^M c^H \log (|P| + |Q| + f + \tau + \mu)}$$

$$\geq \frac{|K^H| c^M \log (\tau + \mu) + \sum_i |K^M_i| c^H \log (\tau + \mu)}{c^M c^H \log (|P| + |Q| + f + \tau + \mu)}$$

$$= \frac{|K^H| c^M + \sum_i |K^M_i| c^H}{c^M c^H \log (|P| + |Q| + f)}$$

$$|K^H| + \sum_i |K^M_i| \leq |K| = O((|G^H| + |G^M|) \log (|P| + |Q| + f))$$

$$|K| = O(m|G| \log (|P| + |Q| + f))$$

(2)
Facet complexity of $K$

Bound for convex polyhedra:

$$|K| = O(|G| \log (|P| + |Q|))$$

Bound for arbitrary polyhedra:

$$|K| = O(m|G| \log (|P| + |Q| + f))$$

Reminder:

- $m$ = the number of mortises
- $|G|$ = the number of facets of the optimal separator
- $f$ = the number of facets in CH($Q$) that are not in $Q$
\[ |Q| = 176 \]
\[ |P| = 80 \]
\[ |K| = 28 \]
- $|Q| = 176$
- $|P| = 80$
- $|K| = 28$
• $|Q| = 176$
• $|P| = 80$
• $|K| = 28$
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Finding separating polyhedra is important in error-bounded decimation.

Current published work finds bounded separating polyhedra in 3D only if one or both of the polyhedra are convex.

Our algorithm finds a separator a factor of $m \log (|P| + |Q| + f)$ from the optimal where:

- $m =$ the number of mortises
- $f =$ the number of facets in $\text{CH}(Q)$ that are not in $Q$

We believe the algorithm is roughly $O(n^3)$ but this needs to be derived.
Finding minimal convex nested polygons.
In Proceedings of the first annual symposium on Computational geometry pp. 296–304, ACM ACM.

Algorithms for polytope covering and approximation.
Algorithms and Data Structures 709, 246–252.

Approximation schemes in computational geometry.

On the complexity of optimization problems for 3-dimensional convex polyhedra and decision trees.
Computational Geometry 8, 123–137.

Computational geometry column 42.

Separation and approximation of polyhedral objects.
Computational Geometry 5, 95–114.

Theory of linear and integer programming.
Wiley.

Finding minimal nested polygons.
References

Finding the minimum visible vertex distance between two non-intersecting simple polygons.
In Proceedings of the second annual symposium on Computational geometry pp. 34–42, ACM ACM.
Theorem

Let $a_1, \ldots, a_m, b$ be vectors in $n$-dimensional space. Then either:

I. $b$ is a nonnegative linear combination of linearly independent vectors from $a_1, \ldots, a_m$, or

II. there exists a hyperplane $x|cx=0$, containing $t - 1$ linearly independent vectors from $a_1, \ldots, a_m$, such that $cb < 0$ and $ca_1, \ldots, ca_m \geq 0$, where $t$ is rank($a_1, \ldots, a_m, b$).

Application: translate $m$ and $q_1, \ldots, q_s$ to the origin. The normals span $\mathbb{R}^3$. By Caratheodory’s Theorem we can say that

$$m = \lambda_1 q_j + \lambda_2 q_k + \lambda_3 q_l$$

where $\lambda_1, \lambda_2, \lambda_3 \geq 0$. Thus, any point $x \in m^-$, or in other words, $m^T x < 0$, also satisfies $\lambda_1 q_j^T x + \lambda_2 q_k^T x + \lambda_3 q_l^T x < 0$. Since $\lambda_1, \lambda_2, \lambda_3 \geq 0$, there must be at least one $q_i$ such that $q_i^T x < 0$. Thus, $m^- \subseteq q_j^- \cup q_k^- \cup q_l^-$. 