Multiscaling in the presence of indeterminacy: wall-induced turbulence

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Abstract

This paper provides a multiscale analytical study of steady incompressible turbulent flow through a channel, of either Couette or pressure-driven type. Mathematically, the paper's two most novel features are that (1) the analysis begins with an underdetermined singular perturbation problem, and (2) it leads to the existence of an infinite number of length scales. (These two features are probably linked, but the linkage will not be pursued.) A good portion of the paper is devoted to developing, in a systematic manner, credible assumptions of a physical or mathematical nature which, when added to the initial underdetermined problem, result in a knowledge of the complete layer (scaling) structure of the mean velocity and Reynolds stress profiles. This structure in turn provides much important information about those profiles themselves. The possibility of almost-logarithmic sections of the mean velocity profile is given special attention. Most traditional analyses of the mean profile problem are based on only the inner and outer scales, together with a hypothesis that (1) there exists a region wherein both the inner and outer approximations are valid, and (2) the profile is monotone increasing in the overlap region. (The second of these is crucial but not usually stated.) No such overlap hypothesis is used in the present paper.

1 Introduction

Flow problems at large Reynolds number are a fertile ground for multiscale methods. Qualitative techniques of this sort are particularly useful, since they may provide insights that are unavailable from other sources. For example, direct numerical simulations of the exact governing equations are presently impossible for even moderately high Reynolds numbers. In any case although simulations, together with empirical observations, may reveal important flow behaviors, they often need to be supplemented with qualitative methods in order to cast further light on the all-important question of how those behaviors are related to the underlying differential equations.

Scaling approaches to problems in wall-induced turbulence, however, are typically applied to some time-averaged form of the Navier-Stokes equations, which is inadequate to provide a full exact solution. The process of averaging erases many details. Therefore when this is done, what results is an equation or system of equations for certain averaged quantities. Of necessity, such equations are underdetermined. In the wall-bounded turbulence problems to be considered here, for instance, there will be a single equation for two unknowns (averaged flow quantities). Being underdetermined, the problem can yield no unique solution. This conflicts with the usual image of asymptotic methods providing a sequence of more and more accurate approximations to a unique exact solution. Given this, one must therefore ask what the role of multiscale methods is when confronting underdetermined problems. That is the primary question to be explored in this paper, within the context of wall-induced turbulence.

The approach will be by means of the concept of a "scaling patch". One seeks to determine sites (patches) within the flow domain, and within which the unknown variables are most naturally considered to be regular functions of a certain rescaled variable. The rescaling, of course, varies from patch to patch. When all possible scaling patches are found, together they provide a composite overview of the flow structure throughout nearly the entire flow domain. As it turns out, they will also provide an approximate picture of the unknown functions themselves.

How, then, can one obtain so much information about the flow when the basic mathematical problem is underdetermined? The answer is, that this is done by adding, to the basic ill-posed problem, some assumptions about the qualitative nature of the solutions. The additional assumptions should, of course, be as reasonable and minimal as possible. Of primary importance among these hypotheses are criteria under which one may surmise the existence and locations of scaling patches. This, and the additional assumptions, should be based ultimately on the physical nature of turbulent flow—facts which may have been obscured by the process of averaging, but which seem clearly to be true.

The scaling approach to these underdetermined problems, then, will be developed in this paper within the context of turbulent flow next to walls, the flow being driven by the action of some forcing mechanism. Following [4], we begin with the simplest such scenario: turbulent Couette flow in a 2D channel. After that, a convenient transformation will enable us to treat pressure-driven turbulent flow, again through a channel.

The development given here follows concepts and results which are either found in basic form in [4, 14] or are extensions of ideas in those papers.

2 Introduction to steady turbulent Couette flow

The physical picture is that of a viscous incompressible fluid sandwiched between two infinite parallel horizontal plates (walls) a distance $2h$ apart. The lower wall is stationary and the upper one moves with constant speed V. After sufficient time has elapsed, the fluid's motion attains a statistically stationary state. In this state, the temporal averages, over a long enough time interval, of the velocity components and products of them, are independent of the time interval chosen.

Moreover, the fact that the two walls are parallel suggests some symmetries. The flow scenario is invariant when the horizontal axis is translated, so that the average flow quantities may be taken to be independent of the horizontal coordinate; similarly, they are independent of time. As a result of this and the conservation of mass, the vertical component of the average velocity vanishes. We shall denote dimensional variables such as U^* and y^* with asterisks. If a new reference frame is envisaged in horizontal motion with velocity $V/2$ with respect to the original one, so that the upper wall has velocity $V/2$ in the new system and the lower wall has velocity $-V/2$, then another symmetry is suggested: the mean horizontal velocity U^* in the new system is odd with respect to the centerline $y^* = h$ and to the value of U^* there: $U^*(h+\xi) - U^*(h)$ is an odd function of ξ; hence $\frac{d^2U^*}{dy^{*2}} = 0$ at the centerline. Due to the oddness, the problem, in the original frame, may be effectively reduced to one in the half-channel $0 < y^* < h$ with this derivative condition holding at the upper boundary $y^* = h$. The velocity $U^* = 0$ at the fixed wall $y^* = 0$.

2.1 The averaged differential equations

Taking the time average of the Navier-Stokes equation and applying these symmetry conditions, we obtain the following, where u' and v' are the horizontal and vertical velocity fluctuations and U^* is the average of the streamwise velocity u^*. Thus $u^* = U^* + u'$. The average of the product $u'v'$ is denoted by $\langle u'v' \rangle$. The parameters involved in the problem are the viscosity μ, the width of the channel $2h$, the velocity V of the upper wall, and the fluid density ρ_m. Under the condition (which is known to hold in this context) that the average vertical velocity component is zero, there results the single equation

$$\mu \frac{d^2 U^*}{dy^{*2}} - \rho_m \frac{d}{dy^*} \langle u'v' \rangle = 0. \qquad \boxed{\text{p1}} \qquad (1)$$

Besides V, there is another characteristic velocity which plays a critical role in the study of wall-induced turbulence. In fact the combination $\frac{\mu}{\rho_m} U^*_{y^*}$ can be seen to have the dimensions of velocity squared, so that we may define the "friction velocity"

$$u_\tau = \sqrt{\frac{\mu}{\rho_m} \, U^*_{y^*} \Big|_{y^*=0}} \qquad \boxed{\text{p50}} \qquad (2)$$

Thus, it depends on the gradient of U^* at the fixed wall (either wall, actually). In fact, when all parameters except u_τ and V are fixed, it is intuitively clear that u_τ and V should vary together in a monotone fashion: one is an increasing function of the other. Generally, u_τ is taken to be the more basic parameter, so that part of the solution of the Couette problem is to find V as a function of u_τ (and the other parameters).

2.2 Nondimensionalizations

The equation (1) can be nondimensionalized in many ways. In examining the relationships of the resulting dimensionless variables with one another, the Reynolds number,

$$h^+ = \frac{u_\tau h \rho_m}{\mu}, \qquad \boxed{\text{p51}} \qquad (3)$$

based on the friction velocity is seen to play a central role. (There are also Reynolds numbers based on other velocities in use; we shall not consider them here.)

The two most common ways to nondimensionalize the problem are by using "inner" and "outer" variables. In both cases, u_τ is taken as the characteristic velocity, producing a dimensionless mean streamwise velocity $U = U^*/u_\tau$. Similarly, $\langle u'v' \rangle$ (called the Reynolds shear stress) is nondimensionalized by setting $T = -\frac{\langle u'v' \rangle}{u_\tau^2}$. The first way is with use of h as characteristic vertical distance. This provides nondimensional versions of y, U^* and $\langle u'v' \rangle$ of the form

$$\eta = \frac{y^*}{h}, \quad U = \frac{U^*}{u_\tau}, \quad T = -\frac{\langle u'v' \rangle}{u_\tau^2}. \qquad \boxed{\text{p2}} \qquad (4)$$

These are called the **outer** variables, more appropriate (as we shall see) near the centerline.

The second common nondimensionalization is with $\ell = \frac{\mu}{\rho_m u_\tau}$ as characteristic length, the unknowns U and T remaining as in (4):

$$y = \frac{y^*}{\ell}, \quad U = \frac{U^*}{u_\tau}, \quad T = -\frac{\langle u'v' \rangle}{u_\tau^2}. \qquad \boxed{\text{p3}} \qquad (5)$$

These are the **inner** variables, valid close to the wall. The traditional designation for inner-scaled variables (y^* scaled with ℓ and velocities with u_τ) is with a superscript "+"; thus we are using the notation y in place of y^+ and U in place of U^+.

Note that the only difference between the outer and inner variables is the characteristic lengths in the two cases. Their ratio is

$$\frac{h}{\mu/\rho_m u_\tau} = h^+.$$

This previously defined Reynolds number is our basic large parameter, and

$$\epsilon = (h^+)^{-1/2} \qquad \boxed{\text{p6}} \tag{6}$$

is our basic small parameter. Thus

$$y = \epsilon^{-2}\eta. \qquad \boxed{\text{p8}} \tag{7}$$

When the two nondimensionalizations are performed, the basic differential equation (1) assumes two different forms.

In terms of the traditional **inner** or **wall** variables (5), which are appropriate scaled variables to use near the wall, the averaged conservation equation for streamwise momentum (1) is

$$\frac{d^2U}{dy^2} + \frac{dT}{dy} = 0. \qquad \boxed{\text{a1}} \tag{8}$$

Associated with (8) are some known boundary values,

$$U = T = \frac{dT}{dy} = 0 \text{ and } \frac{dU}{dy} = 1 \text{ at } y = 0. \qquad \boxed{\text{a2}} \tag{9}$$

Note that the definition of u_τ was fashioned so as to ensure that the latter boundary condition is satisfied. The integrated form of (8), (9) is

$$\frac{dU}{dy} = 1 - T. \qquad \boxed{\text{a1''}} \tag{10}$$

As was brought out before, the problem (8) and (9) is underdetermined, consisting of a single DE for two unknown functions $U(y)$ and $T(y)$. Because of the underdetermined nature of the problem, there exist many solutions; for example, if $\zeta(y)$ is any smooth function vanishing for values of y near the walls and the centerline, one could replace U by $U + \zeta$ and T by $T - \frac{d\zeta}{dy}$, which is still a solution of the DE and boundary conditions. However, only one of the many solutions represents the correct physical U and T profiles. Our task will be to inject additional considerations into the reasoning which, together with (8) and (9), will provide useful information about these functions. In particular, we aim to determine their multiscaling and order of magnitude properties.

The traditional **outer** variables are (4). (Alternatively, the variable U in (4) is often replaced by the defect velocity $U_c - U$, where U_c is the (inner normalized) value of U at $\eta = 1$.) It is well accepted that they are appropriate near the centerline. (The centerline is at $y = h$, i.e. $\eta = 1$.) However, it is by no means obvious from the outer equation (11), for example, that η is the natural length scale for U near the centerline; this issue will be resolved in Sec. 4.3. Equation (8) becomes

$$\frac{dT}{d\eta} + \epsilon^2 \frac{d^2U}{d\eta^2} = 0, \qquad \boxed{\text{a3}} \tag{11}$$

with boundary conditions

$$\frac{dT}{d\eta} = \frac{d^2U}{d\eta^2} = 0 \text{ at } \eta = 1. \qquad \boxed{\text{a4}} \tag{12}$$

In fact, the second of (12) was already derived on the basis of symmetries, and the first then follows from (11).

It was mentioned before that one can scale the variables in the averaged momentum balance equation any number of ways, creating an infinite number of versions of it; (8) and (11) are only two of the many choices. Although all versions are mathematically equivalent, only some of them reflect the behavior of the actual functions U and T (which are uniquely defined but unknown), and then only in restricted regions of the flow. Our main objective will be to determine which scalings are realistic in this sense, and where they are expected to be representative of the flow quantities. With this knowledge, the appropriate differential equations satisfied by the scaled flow quantities will be obtained immediately by rescaling (8) (or (11)).

In a somewhat broader context, a brief discussion relating to the empirical determination of scaling behaviors is warranted. For this we consider the typical case in which profiles of a velocity field statistic are acquired over a range of Reynolds numbers. In their dimensional form, these statistical profiles (empirically determined functions) can generally vary widely in their magnitude and shape. For each point in the profile, the statistic and the y value are made non-dimensional according to the normalization being tested (e.g., inner or outer). If the different Reynolds number profiles (or more likely a portion of the profiles) merge to a single curve under this normalization, then the scaling is said to be appropriate (successful) over the indicated sub-domain. Operationally, the successful scaling must therefore stretch or compress the statistical function and its independent variable, y^*, such that the differences due to a variation in Reynolds number are effectively removed. At a minimum this requires that the normalized amplitude of the function and its variation in terms of the normalized y variable remain the same order of magnitude as the Reynolds number is varied. Note that having the normalized functions and their variations in the normalized dependent variable remain the same order of magnitude does not necessarily guarantee that the profiles will merge to a single curve. If they do not remain of the same order, however, it is a surety that they will not merge. The criteria set forth herein reflect this minimal requirement, and thus constitute a mild assumption relative to the empirical test.

2.3 Rationale for the inner scaling

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As an introduction to our methods, let us recall why (5), which leads to (8), is expected to provide us with the valid scaled variables in regions next to the wall. Our argument will be based on mathematical considerations, although in the past most discussions have also relied on physical and intuitive reasoning; a typical source is e.g. [12, Sec. 5.2]. Our approach will introduce the formal definitions in section 3.

All order of magnitude relations below are meant to hold as $\epsilon \to 0$. In particular, "$a(\epsilon) = O(1)$" means that there exist positive constants c and C, independent of ϵ, such that $c < a(\epsilon) < C$ for all small enough ϵ. The "nominal" order of magnitude of a term in an equation is defined to be the order of magnitude based only on the appearance in that term of the parameter ϵ. Thus, for example, the two terms in (11) have nominal orders 1 and ϵ^2, respectively, irrespective of what their actual numerical values are. In contrast, the "numerical" order of magnitude of a term takes into consideration that the derivative appearing in the term may take on values which are not $O(1)$.

Consider, then, the scaling (5). To check whether it is the appropriate scaling near the wall, we have to define what "appropriate" is in this context, and then examine whether this particular scaling satisfies those criteria.

Appropriate will mean, for one thing, that all relevant derivatives of U and T with respect to y are numerically $\leq O(1)$, i.e. bounded as $\epsilon \to 0$, in the region being considered. This is necessary if (5) are to be considered the natural scaled variables in that region. Secondly, the basic momentum balance law (1), when written in the scaled variables, retains its relevance as a meaningful balance of two forces or similar kinds of quantities. Such a balance is the most fundamental ingredient of the physical law (1), and this aspect should not be lost by any physically relevant rescaling. In this case, the rescaled version is (8), and it clearly does express a balance between two $O(1)$ scaled forces: one derived from viscous effects, $\frac{d^2U}{dy^2}$, and the other one coming from the action of the turbulence, $\frac{dT}{dy}$. If this rescaling had, e.g., resulted in an equation like (8), but with a small parameter ϵ multiplying one term but not the other, then the equation would not express such a balance.

In addition to the rescaling preserving a balance, there are also compatibility conditions to be satisfied. The compatibility criteria are as follows. Let us collect all the derivatives of various orders (including the undifferentiated forms) of U and T with respect to y that we have any information about. In this collection, there will be the two derivatives in (8), which we know balance each other, and the quantities in (9). In all, we have five such derivatives. It is first to be shown that they are all numerically $\leq O(1)$ at some point y_0 in the proposed region of validity, namely near the wall. The clear choice is $y_0 = 0$, because that is where the values (9) are assumed. All these values are indeed $\leq O(1)$. Moreover, the boundary value $\frac{dT}{dy}(0) = 0$ further implies that both terms in (8) vanish there. Therefore all derivatives in our set are $\leq O(1)$, and one is nontrivially $O(1)$. In fact, all are 0 except $\frac{dU}{dy}$, which is 1; the inner scaling length ℓ in (5) was chosen precisely to guarantee that normalization. It is important that at least one derivative be $O(1)$, not smaller, in order for the variation of U and T with respect to the scaled variable y to be nontrivial.

These order of magnitude conditions on the set of derivatives at $y_0 = 0$ will be called *compatibility conditions* between the scaling and the known values of the set of derivatives. In this case this compatibility of the scaling was verified only at $y = 0$ and only for five derivatives. However, these derivatives at other points in the scaling domain, as well as all other derivatives there, are implicitly connected with the five derivatives at y_0 through the fluid dynamics which was obscured by the averaging, i.e. the process that produced (1) from the original Navier-Stokes equations. It can be assumed that this connection allows the compatibility of orders of magnitude to be extended from the five derivatives at $y_0 = 0$ to the other derivatives and locations mentioned above.

Therefore (8) is the proper scaled DE governing the flow near the wall. How near to the wall should this scaling be expected to hold? In a scaling patch adjacent to $y = 0$, U and T are regular functions of y, implying that higher derivatives with respect to y are $\leq O(1)$. Therefore it would take an interval $\Delta y \geq O(1)$ for the two derivatives in (8) to grow to be $> O(1)$. Therefore we further surmise that the width of the region of validity is at least $O(1)$ in the variable y. In short, the terms in (8) remain $\leq O(1)$ at least for $0 \leq y \leq O(1)$. This provides a lower bound on the width of the inner layer.

All this concerned the inner scaling. The validity of the outer scaling near the centerline, on the other hand, is actually not completely obvious, and will be established later in Sec. 4.3.

Now let us generalize this procedure by formalizing the criteria alluded to above.

3 Formalization of the scaling procedure

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In order to obtain the most complete structural picture of turbulent Couette flow, we shall introduce an extra small parameter ρ, in addition to ϵ, by the definition

$$T^{\rho}(y) = T(y) - \rho y. \qquad \boxed{\text{d10}} \tag{13}$$

In particular cases, ρ will be taken to be ϵ^{ν} for some specific positive exponent ν. These new functions will be called adjusted Reynolds stresses. They satisfy

$$\frac{dT^{\rho}}{dy} = \frac{dT}{dy} - \rho, \qquad \boxed{\text{q1}} \tag{14}$$

and from (8),

$$\frac{d^2U}{dy^2} + \frac{dT^{\rho}}{dy} + \rho = 0. \qquad \boxed{\text{d11}} \tag{15}$$

We now have three force-like terms in the basic equation, rather than the two in the previously considered one, (8). Nevertheless the reasoning used there can be easily adapted to this case.

The development to follow outlines the search for scaling patches for (15), and therefore for (8).

Def. A differential scaling is a transformation of differentials

$$dy = \alpha d\hat{y}, \; dT^{\rho} = \beta d\hat{T}^{\rho}, \; dU = d\hat{U}, \qquad \boxed{\text{rr1}} \tag{16}$$

where α and β are positive functions of ϵ or ρ. (It would be natural to include a flexible coefficient with $d\hat{U}$ as well, but in fact we will not need it.) The ρ-dependence of $\hat{y}$ and $\hat{U}$ is being suppressed.

A differential scaling induces a transformation of (15) to another equation of the form

$$K\left[\frac{1}{\alpha^2}\frac{d^2\hat{U}}{d\hat{y}^2} + \frac{\beta}{\alpha}\frac{d\hat{T}^{\rho}}{d\hat{y}} + \rho\right] = 0, \qquad \boxed{\text{a5}} \tag{17}$$

where with no loss of generality, the constant K is chosen so that the maximum nominal order of magnitude of the three terms in (17) is 1.

Def. A scaling patch is an interval I on the $y-$axis, together with a differential scaling (α, β) such that for some number $y_0 \in I$, the two functions

$$\hat{U}(\hat{y}) = U(y_0 + \alpha\hat{y}) - U(y_0), \quad \hat{T}^{\rho}(\hat{y}) = \frac{1}{\beta}[T^{\rho}(y_0 + \alpha\hat{y}) - T^{\rho}(y_0)] \qquad \boxed{\text{a6'}} \tag{18}$$

are, to dominant order in ρ and ϵ, regular functions of $\hat{y}$ for $y_0 + \alpha\hat{y} \in I$, i.e. functions whose derivatives with respect to $\hat{y}$ up to order 3 (say) are bounded in I independently of ϵ and ρ. Moreover, it is required that at least one of these derivatives is $O(1)$, so that it does not happen that all the derivatives $\to 0$ as ϵ and $\rho \to 0$. This is to ensure that $\hat{U}$ and $\hat{T}^{\rho}$ vary nontrivially with respect to $\hat{y}$. This definition is a way of expressing that (α, β) provides the natural scaling of U and T^{ρ} in I. Note that $\hat{U}(0) = \hat{T}^{\rho}(0) = 0$, and also that for points $y = y_0 + \alpha\hat{y}$ near y_0, $U(y) = U(y_0) + \hat{U}(\hat{y}), \; T^{\rho}(y) = T^{\rho}(y_0) + \beta\hat{T}(\hat{y})$.

Our object is to find a set of scaling patches that cover as much as possible of the entire range in y, i.e. $0 \leq y \leq 1/\epsilon^2$, or $0 \leq \eta \leq 1$. (Recall from (7) that the centerline is at $y = \epsilon^{-2}$.) If we are successful, this information will tell us the proper scaling of U and T at almost every location in the channel; in particular we could read off the layer structure of the flow.

Def. An admissible scaling is a differential scaling for which (17) has at least two terms of nominal order of magnitude 1. Since by choice of K this is the maximal order of magnitude, if there is another term, it will have a smaller nominal order of magnitude, and (17) will express an approximate balance of the two larger terms.

Main assumption. Given an admissible scaling and a point y_0, consider the set of all derivatives appearing in the terms of (17) of nominal order 1, evaluated at the point $\hat{y} = 0$. If each derivative in the set is known to be numerically $\leq O(1)$ and there exists a derivative, not necessarily in that set, which is $O(1)$, then that scaling, together with some interval I containing y_0, is a scaling patch. The maximal extent of I is undetermined at this point, except that it can be taken to include at least the interval $\{|\hat{y}| \leq O(1)\}$.

Comments explaining the main assumption. This assumption is a generalization of the assumptions made in Sec. 2.3, which explore it in the case of the classical inner scaling. As was mentioned before, our goal is to find scaling patches. Suppose we have a candidate scaling (α, β). One good indication that it might lead to a patch would be the presence of a "reasonable" version of the balance equation (15) in the scaled variables. Reasonable here will mean a DE (17) that, after terms that are nominally small ($o(1)$) have been neglected, expresses a nontrivial relation between (or among) at least two of the three terms in the equation. The reason is that otherwise only one term would be $O(1)$, and then the DE would give little information other than that that term is small (it being equal to the nominally small terms). This is why we operate only with admissible scalings. Note that admissible scalings use the same characteristic distance variable $\hat{y}$ for the two functions U and T. We are implicitly assuming that the turbulence mechanisms couple those two quantities in that way.

But a reasonable DE is not enough; if there were a scaling patch associated with it, we still would not immediately know where it is located. That is why it is necessary also to determine some location y_0 where it is known that the appropriate (the ones designated in the main assumption) derivatives of the scaled variables $\hat{U}$ and $\hat{T}^\rho$ with respect to the scaled space variable $\hat{y}$ are compatible with the scaling under consideration. Compatible is taken to mean, for one thing, that these derivatives are $O(1)$ or smaller. But to preclude weak or trivial dependence of $\hat{U}$ and $\hat{T}^\rho$ on $\hat{y}$, at least one derivative should be $O(1)$. The values of the derivatives are monotone functions of α and β, so it is clear that at least one scaling can achieve that requirement. We leave aside the question of whether it is unique. If this sort of compatibility occurs at one point, then as before the obscured coupling should, we argue, maintain the compatibility for at least some interval containing the point. This is the rationale for the various parts of the main assumption.

This approach to finding the natural scaling in specific regions is not really new, since many justifications for the inner scaling and the law of the wall given in the past at least touched on these ideas. However, it more precisely identifies the requirements for an appropriate (successful) scaling. In the special case of the inner scaling, we have $\alpha = \beta = 1$ and (17) with $\rho = 0$ is the same as (8). The point y_0 in the main assumption is the wall, $y_0 = 0$. At that point, (9) shows that all relevant derivatives of $\hat{U} = U$ and $\hat{T} = T$ vanish, to lowest order, except for $\frac{dU}{dy} = \frac{d\hat{U}}{d\hat{y}}$, which is unity. Applying the main assumption, we surmise that there is a scaling patch near the origin, extending at least a distance $O(1)$ (measured by the inner scale $\hat{y} = y$) into the interior of the flow.

In the case of the outer scaling (4) $\alpha = 1/\epsilon^2$, $\beta = 1$, and $y_0 = 1/\epsilon^2$, which is the centerline $\eta = 1$. The justification of this scaling will proceed in Sec. 4.3 indirectly through the hierarchy constructed below in Section 4.

To reiterate, given any particular scaling, the basic averaged differential equation for the quantities $\hat{U}$ and $\hat{T}^\rho$ will consist of three terms (in exceptional cases, just two). This equation will be a meaningful relation between those quantities only if it consists of at least two terms which are nominally $O(1)$ (e.g., do not involve the small parameter ϵ explicitly), plus possibly a third term which is nominally of smaller order of magnitude. One surmises that this smaller term, if it exists, can be neglected in the part of the flow region pertinent to that scaling. There will thus be an approximate balance of at least two terms. If this is the case, then we call the scaling "admissible".

Again for any particular admissible scaling and some specific point, if one can demonstrate independently that some derivative of $\hat{U}$ or $\hat{T}^\rho$ with respect to the rescaled space variable $\hat{y}$, or the undifferentiated variable itself, is $O(1)$ and the other derivatives, of the orders appearing in the equation, are $\leq O(1)$, then the DE arising from the admissible scaling is, we surmise, satisfied approximately in some range including the point in question. That range will be called the "scaling domain" or "patch" for that scaling. If this can be done for two different points, then we surmise that generally the scaling domain includes at least the interval between them.

In the following, we shall use this criterion to obtain a fairly complete qualitative picture of the profiles of the functions U and T almost entirely across the channel.

4 A continuum of patches

cont

For each number ρ in an interval to be specified, it will be shown that there exists a corresponding scaling patch (layer) L_ρ with specific admissible scaling, depending on ρ. The location of L_ρ will be at the point $y_m(\rho)$ where $T^\rho(y)$ (13) attains a maximum. It will be important actually to know, at least approximately, the function $y_m(\rho)$; that will be addressed in Sec. 4.2. In L_ρ, the rescaled basic equation assumes a form in which all three terms have equal nominal orders of magnitude; in fact the rescaled equation has no explicit dependence on ρ or ϵ, suggesting certain invariances as we pass from one layer to another.

There is a striking connection between the continuum of patches and the profiles of U and T; that relation is explored in Sec. 5.

4.1 The existence of patches

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The details of the origin and properties of the patch L_ρ are now explained. We are interested in values of ρ for which T^ρ has a maximum, and will show that there exists a scaling patch containing the location $y_m(\rho)$ of that maximum. We construct a differential scaling which gives all three terms in (15) the same nominal order of magnitude. For coefficients α and β, to be determined depending on ρ, one sets

$$dy = \alpha d\hat{y}, \quad dT^\rho = \beta d\hat{T}^\rho, \quad \text{and} \quad dU = d\hat{U}. \qquad \boxed{\text{d12}} \qquad (19)$$

Under this transformation, the first two terms in (15) become $\alpha^{-2}\frac{d^2\hat{U}}{d\hat{y}^2}$ and $\frac{\beta}{\alpha}\frac{d\hat{T}^\rho}{d\hat{y}}$ respectively. They must match, in formal order of magnitude, the third term, ρ. This requires $\alpha = \rho^{-1/2}$ and $\beta = \rho^{1/2}$. Therefore

$$dy = \rho^{-1/2} d\hat{y}, \quad dT^\rho = \rho^{1/2} d\hat{T}^\rho, \quad dU = d\hat{U} \qquad \boxed{\text{d12'}} \tag{20}$$

The equations (20) can be integrated with integration constants chosen such that $\hat{y} = 0$ when $y = y_m(\rho)$, $\hat{T}^\rho = 0$ when $T^\rho = T^\rho_m(\rho)$, and $\hat{U} = 0$ when $U = U_m(\rho)$, where T^ρ_m and U_m are the values of T^ρ and U at $y = y_m(\rho)$. In terms of (18), $y_0 = y_m(\rho)$, $U_0 = U_m(\rho)$, $T^\rho_0 = T^\rho_m(\rho)$. Then

$$y = y_m(\rho) + \rho^{-1/2}\hat{y}, \quad T^\rho = T^\rho_m(\rho) + \rho^{1/2}\hat{T}^\rho(\hat{y}) \quad U = U_m(\rho) + \hat{U}(\hat{y}). \qquad \boxed{\text{d13}} \tag{21}$$

The basic equation (15) then becomes

$$\frac{d^2\hat{U}}{d\hat{y}^2} + \frac{d\hat{T}^\rho}{d\hat{y}} + 1 = 0. \qquad \boxed{\text{d14}} \tag{22}$$

This makes it an admissible scaling.

To complete the verification of the main assumption, we examine the compatibility criteria. We first choose a point y_0. The natural choice is $y_0 = y_m(\rho)$. In fact at that point, the middle term in (22) vanishes, so that the first term $\frac{d^2\hat{U}}{d\hat{y}^2}$ equals -1, and we can evaluate all terms, to verify that we satisfy the criteria of our Main Assumption. Certain boundary values, analogous to (9) in the case of the inner scaling, are also known at that point, namely $\hat{U} = \hat{T}^\rho = 0$, $\frac{d\hat{T}^\rho}{d\hat{y}} = 0$. The last of these comes about because we have chosen T^ρ_m to be a local maximum of the function T^ρ. Again, all this is in accordance with the conditions of the Main Assumption, which leads to the existence of a scaling patch at that location. (At this point we do not know the actual location $y(\rho)$ of the patches; that issue will be addressed in the next section.)

We now characterize those values of ρ for which T^ρ has a maximum and therefore there exists a patch L_ρ. We use the following monotonicity properties of the function $T(y)$ and its derivative, well known from empirical data:

Empirical qualitative facts.

1. $T(y)$ is strictly increasing in the interval $(0, 1/\epsilon^2)$, with $T(0) = 0$.
2. $\frac{dT}{dy}$ is unimodal, vanishing at $y = 0$ and at $y = 1/\epsilon^2$, and positive in between.

We set $T_m = T(1/\epsilon^2)$, the maximal value of T. Let the maximum T'_m of $\frac{dT}{dy}$ be attained at $y = y'_m > 0$. Empirical data [4] on the function $\frac{dT}{dy}(y)$ indicate that $T'_m \sim .07$ and $y'_m \sim 7$.

From Emp. Fact 2 and (14), any ρ with $\rho < T'_m$ will be such that T^ρ has a maximum, and that maximum is attained at the value y_m where $\frac{dT}{dy}$ is decreasing and $\frac{dT}{dy}(y_m) = \rho$. However, if ρ is too large ($\rho > T'_m$), T^ρ will be a monotone decreasing function and no such maximum occurs. For reasons to be brought out later, we restrict $\rho \geq \epsilon^4$. The allowed range of ρ will therefore be $\epsilon^4 \leq \rho < T'_m$.

In view of the left part of (21), the characteristic length $\ell(\rho)$ in the layer can be taken as $\ell(\rho) = \rho^{-1/2}$. Since there exists a one-to-one correlation between values of ρ in the allowed interval $T'_m > \rho \geq \epsilon^4$ and values $y_m(\rho)$ in the interval $y'_m < y_m < \epsilon^{-2}$, the continuum of layers L_ρ can equally well be parameterized by their locations $y_m(\rho)$.

Basic result. To every ρ in the interval $\epsilon^4 \leq \rho < T'_m$, there is associated a scaling patch L_ρ with characteristic length $\ell(\rho) = \rho^{-1/2}$. Alternatively, to every y in the interval

$$y'_m < y \leq \epsilon^{-2}, \qquad \boxed{\text{yy20}} \tag{23}$$

there exists a scaling patch L_ρ located at $y_m(\rho) = y$.

At this point, it has been shown that for each value of ρ for which T^ρ has a max, there exists an interval L_ρ containing $y_m(\rho)$ within which $\hat{U}$ and $\hat{T}^\rho$ are regular functions of $\hat{y}$, so that with reference to the inner variable y, these functions vary with characteristic length $\rho^{-1/2}$.

If ρ_1 and ρ_2 are close to each other, L_{ρ_1} and L_{ρ_2} overlap. However, a discrete set of values of ρ may be chosen so that the associated layers do not overlap but nevertheless fill out the entire domain of the hierarchy. If this is done, the number of members in the ensemble increases indefinitely as $\epsilon \to 0$.

An important question remains as to how the unadjusted Reynolds stress T and velocity U scale in L_ρ. The answer comes from (13): $T = T^\rho + \rho y = T^\rho_m + \rho y_m + \rho^{1/2}(\hat{T}^\rho + \hat{y}) = T^\rho_m + \rho y_m + \rho^{1/2}\hat{T}_*(\hat{y})$, where this expression defines $\hat{T}_*(\hat{y}) = \hat{T}^\rho(\hat{y}) + \hat{y}$. It is a regular function of $\hat{y}$. Therefore the conclusion is that in L_ρ, T also scales with $\hat{y}$. In fact

$$T = T^\rho_m + \rho y_m + \rho^{1/2}\hat{T}_*, \qquad \boxed{\text{d19}} \tag{24}$$

where $\hat{T}_*$ is a regular function of $\hat{y}$ (i.e. its derivatives are bounded independently of ϵ or ρ). Of course, U is also a regular function of $\hat{y}$ in L_ρ. This result is self-consistently reinforced by the fact that (24) is analogous to the rescaling derived in (21).

In summary, layer L_ρ is characterized in part by the characteristic length (in inner units) of variation of U and T being $O(\rho^{-1/2})$ and

- $\frac{d\hat{U}}{d\hat{y}} = O(1)$; $\frac{dU}{dy} = O(\rho^{1/2})$;
- the higher derivatives of $\hat{U}$ and $\hat{T}^\rho$ with respect to $\hat{y}$ are $\leq O(1)$.

$\boxed{\text{summ}}$ (25)

The locations of the L_ρ will be considered below in Sec. 4.2.

The above constitutes the theoretical foundation for the scale hierarchy. Namely, it provides the existence of a scaling patch, L_ρ, for each allowed value of ρ.

4.2 The locations of the patches

$\boxed{\text{locs}}$

An important piece of information is still lacking. This relates to how the location $y_m(\rho)$ (which serves to pinpoint L_ρ) of the maximum of T^ρ depends on ρ. Once this is found, the behavior of the velocity $U(y)$ and the Reynolds stress $T(y)$ can in principle be obtained. It is argued, in fact, that for large $y_m(\rho)$, the characteristic extent of the layer has the order of magnitude of its distance $y_m(\rho)$ from the wall. This means that the layer occupies a fraction of the distance y from the wall to the center of the layer itself.

At the point $y_m(\rho)$, the left side of (14) vanishes, and the right side as well. It follows that

$$\frac{dT}{dy}(y_m(\rho)) = \rho. \qquad \boxed{\text{xx1}} \tag{26}$$

By differentiating (26) with respect to ρ, one obtains

$$\frac{d^2T}{dy^2}(y_m(\rho))\frac{dy_m}{d\rho} = 1. \qquad \boxed{\text{v2}} \tag{27}$$

This equation holds for all y_m for which $y_m(\rho)$ is defined. Also by (20)

$$\frac{d^2T}{dy^2} = \rho^{1/2}\frac{d^2T}{dyd\hat{y}} = \rho\frac{d^2T}{d\hat{y}^2} = \rho^{3/2}\frac{d^2\hat{T}^\rho}{d\hat{y}^2}. \qquad \boxed{\text{v20}} \tag{28}$$

In L_ρ, derivatives such as $\frac{d^2\hat{T}^\rho}{d\hat{y}^2}$ are $O(1)$ quantities or smaller (independent of ϵ to dominant order). We now define

$$A(\rho) = -\left(\frac{d^2\hat{T}^\rho}{d\hat{y}^2}\right)_{\hat{y}=0}. \qquad \boxed{\text{xx7}} \tag{29}$$

It will be reasoned in the next section that $A = O(1)$ except at the beginning of the hierarchy. Although it will generally depend somewhat on ρ, i.e. on y_m, its order of magnitude will not change, with the indicated exception. In (28), set $y = y_m(\rho)$. Then

$$\frac{d^2T}{dy^2}(y_m(\rho)) = -A(\rho)\rho^{3/2}. \qquad \boxed{\text{v3}} \tag{30}$$

Putting this into (27) gives

$$\frac{dy_m}{d\rho} = -\frac{1}{A}\rho^{-3/2}. \qquad \boxed{\text{v4}} \tag{31}$$

For most of the range of ρ (see Sec. 5.3), $A(\rho) = O(1)$, and it satisfies bounds of the form $0 < \alpha_1 < \frac{1}{A} < \alpha_2$. Therefore from (31), there is an integration constant C independent of ρ with $y_m(\rho) = C - \int \frac{1}{A}\rho^{-3/2}d\rho$, so that

$$2\alpha_1\rho^{-1/2} + C < y_m(\rho) < 2\alpha_2\rho^{-1/2} + C. \qquad \boxed{\text{xx10}} \tag{32}$$

In short, $y_m(\rho) = O(\rho^{-1/2}) \quad (\rho \to 0)$. And since $\rho^{-1/2}$ is the characteristic length in L_ρ, this establishes the claim that the characteristic length of L_ρ is asymptotically proportional to its distance $y_m(\rho)$ from the wall.

4.3 The case $\rho = \epsilon^4$ and the outer scaling

$\boxed{\text{1=2}}$ It is seen from (20) that in the case $\rho = \epsilon^4$

$$d\hat{y} = \epsilon^2 dy = d\eta, \qquad \boxed{\text{q80}} \tag{33}$$

where η is the traditional outer variable (7). Therefore $\hat{y}$ and η differ only in their origins:

$$\hat{y} = \eta - \eta_m, \qquad \boxed{\text{q81}} \tag{34}$$

where η_m is defined as the value of η where $T^{(\rho=\epsilon^4)}$ has its maximum. Therefore it is to be expected that the centerline $\eta = 1$ is in the layer $L_{\rho=\epsilon^4}$. In fact, the order of magnitude of $1 - \eta_m$ can be found. Both computational and empirical data show [4] that $1 - \eta_m \sim .1$.

In summary, when $\rho = \epsilon^4$, the location of the maximum adjusted Reynolds stress $T^{(\rho=\epsilon^4)}$ lies within a distance of about .1 (in η, i.e. in $\hat{y}$ for this value of ρ) of the maximum of T itself, which is at $\eta = 1$. Thus for $\rho = \epsilon^4$, except for a small shift of the order $\sim .1$, $\hat{y}$ and η are identical scaled distances. Thus the centerline $\eta = 1$ lies in the layer $L_{\rho=\epsilon^4}$, where $\hat{U}$ and $\hat{T}^\rho = \frac{T^\rho - T_m^\rho}{\epsilon^2}$ (21) are regular functions of η. This corroborates the assertion to that effect at the end of Sec. 2.3.

4.4 Balance exchanges

The following description of the balance exchange process, used also in [14, 4], provides binding evidence of the necessity for the patches L_ρ.

The scaling patch L_ρ was defined for all ρ values such that T^ρ has a maximum. At the maximum, the derivative $\frac{dT^\rho}{dy} = 0$ and the flow quantities vary with a length scale $O(\rho^{-1/2})$.

The condition for T^ρ to have a maximum was shown to be $\rho < T'_m$. However, let us now further require ρ to be so small that $T'_m \gg \rho$. Then there will be a point y_0 at which $\frac{dT}{dy} \gg \rho$, and hence from (14), $\frac{dT^\rho}{dy} \gg \rho$. But (15) tells us that the sum of the first two terms in that equation is $-\rho$, so that they balance, except for a small error term $-\rho$. This will continue to be true as y increases from y_0 up until $y_m(\rho)$ is approached. Eventually, we arrive at a location where $\frac{dT^\rho}{dy} = \rho$ (say) and all three terms of (15) will have the same numerical order of magnitude. We have now entered L_ρ. Proceeding to the point $y_m(\rho)$, where the first and last terms in (15) balance and the middle term is zero, we see that a balance exchange has occurred: the balance of the first two terms has been exchanged for the balance of the first and last. Thus the appearance of L_ρ is accompanied by a balance exchange of the terms in (15)

5 The profiles $U(y)$ and $T(y)$.

prof

5.1 The profiles are determined by the function $A(\rho)$

detd Knowledge of the characteristic function $A(\rho)$ (29) of the hierarchy leads rigorously and uniquely, up to integration constants, to the profiles of U and T. This is done by integrating (31), (26) and (10), which are written here in terms of the general coordinate $y = y_m$ in the hierarchy, representing the location of the maximal point of T^ρ:

$$\frac{dy}{d\rho} = -\frac{1}{A(\rho)}\rho^{-3/2}, \qquad \text{yy1} \tag{35}$$

$$\frac{dT}{dy} = \rho, \qquad \text{yy2} \tag{36}$$

$$\frac{dU}{dy} = 1 - T. \qquad \text{yy3} \tag{37}$$

Integration of (35) yields $y - C$ as a function of ρ, where C is an integration constant which could be determined by fitting a known value of y with its known value of ρ. Inverting that function gives ρ as a function of $y - C$. Integrating (36) and then (37) finally provides T and U. It turns out (Sec. 5.4.1) that the resulting function U is logarithmic if and only if $A =$ constant.

5.2 Alternative expressions for $A(y)$

We denote by $A(y)$ the function $A(\rho(y))$, where $\rho(y)$ is the value of ρ such that $y_m(\rho) = y$. The equations (26) and (29) provide an expression for $A(y)$ in which the parameter ρ does not appear, and therefore which may be useful in computing $A(y)$. Here primes denote derivatives with respect to y.

$$A(y) = -T''(y)\,(T'(y))^{-3/2}.$$

Similarly in terms of U from (8),

$$A(y) = U'''(y)\left(-U''(y)\right)^{-3/2}.$$

5.3 Properties of $A(\rho)$

sA

In view of the connection shown in Sec. 5.1, it is essential to discuss the salient properties of the function A.

There is an analogy between the invariances associated with the inner layer when ϵ changes, on the one hand, and those associated with the hierarchy of layers when ρ changes, on the other. In the former case, the inner scaling is such that the resulting DE and boundary conditions are independent of ϵ, and we therefore surmise the following invariance principle, called the law of the wall: the mean velocity and Reynolds stress profiles in the inner layer are, to lowest order in ϵ, functions only of the inner scaled coordinate y.

In the latter case, the scalings in the various layers L_ρ similarly produce a DE (22) exactly (no approximation) which is independent of ρ, i.e. independent of the layer. The same is true of the numerical values, at $\hat{y} = 0$, of the functions $\hat{U}$ and $\hat{T}^\rho$ and certain of their derivatives with respect to the scale variable $\hat{y}$. We are referring to the derivatives discussed in Sec. 4.1. This invariance suggests that in each scaling patch the functions $\hat{U}(\hat{y})$ and $\hat{T}^\rho(\hat{y})$, of the ρ-dependent variable $\hat{y}$ would be invariant (approximately) when ρ changes, i.e. would enjoy some ρ-independence when evaluated at the same value of $\hat{y}$ within the various different scaling patches. This would hold as well for their derivatives. This conclusion is given more credence, in fact, by the observation that at the point $\hat{y} = 0$, each term appearing in (22) has a value $(-1, 0, 1)$, respectively, independent of ρ, and the undifferentiated quantities $\hat{T}^\rho = \hat{U} = 0$ do as well. The function $-A(\rho)$ is one of these derivatives (29), so should not depend in a major way on ρ, except where the invariance feature is disrupted. Moreover since $-A$ is the second derivative at a local maximum, necessarily $A \geq 0$, and the typical case will be $A > 0$. (At the beginning of the hierarchy, where T^ρ has an incipient maximum in the form of an inflection point, $A = 0$; but $A > 0$ for ρ less than that value.) This, together with the approximate invariance with respect to ρ, serves to indicate (not a rigorous indication) that $A(\rho)$ has constant order of magnitude $A(\rho) = O(1)$ for intervals with parameter ρ (or equivalently the location $y_m(\rho)$) bounded away from where the onset of the hierarchy occurs.

Accurate empirical data for the function $A(y)$ are not available. Some plots of the function $A(y)$ based on a finite difference approximation to the second derivative in (30), using data for moderate size Reynolds numbers, are shown in Fig. 1. These results are probably inaccurate, but indicate that A is $O(1)$, and may be approximately constant in certain interior intervals.

5.4 The question of logarithmic-type growth

s3.4.3

A central issue ([10, 6, 9, 12, 1, 2], etc.) in the history of turbulent channel flow investigations is whether and where the mean velocity exhibits a logarithmic profile. There have been a great many papers utilizing classical overlap arguments which suggest such behavior. In addition, many studies of this question conclude with other scaling laws; recent studies are best represented in several papers by Barenblatt, Chorin, and Prostokishin, e.g. [1, 2]. Experimental data bearing on this question are shown in Fig. 2 The approach adopted in this paper provides new insight into this issue. The first conclusion to be reached is that logarithmic natures of profiles of U depend crucially on $A(\rho)$ being constant. This was already shown in Sec. 5.1. If it is constant, then exact logarithmic

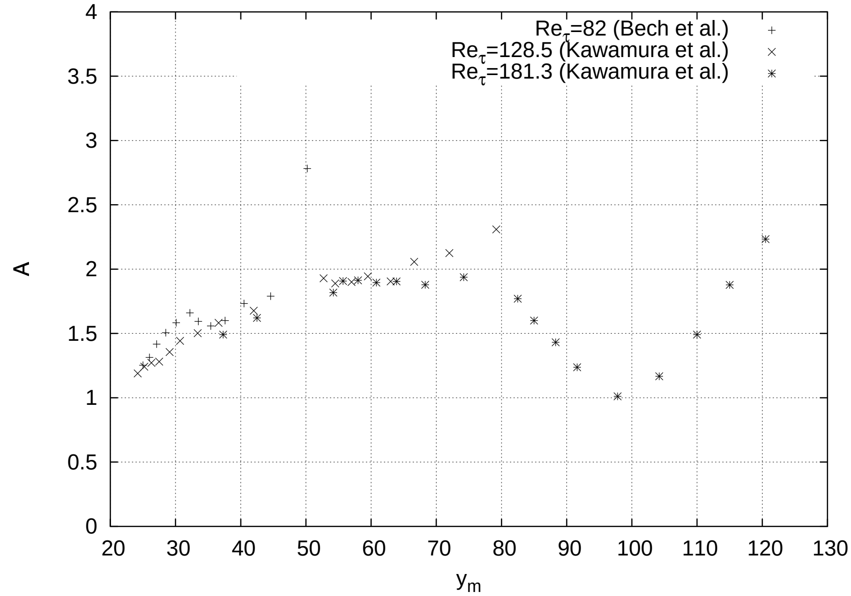

Figure 1: $A(y)$ for different Reynolds number Couette flow as estimated by finite difference of $T(y)$. Data are from Bech et al. [3] and Kawamura's group [7, 11].

growth in the sense of (42) follows easily from those calculations. If it is not constant, then the growth is not logarithmic. Finally if A is almost constant (and reasons for supposing that it is so under certain circumstances will be given), then the profile of U is bounded between two nearby logarithmic functions. Finally in Sec. 5.5, a nonrigorous argument is presented leading to the conclusion that as $Re \to \infty$, A approaches a constant in certain moving, yet explicitly characterized, ranges of y values.

5.4.1 The connection with the constancy of $A(\rho)$

const The reasoning below in Sec. 5.5 indicates that A may be approximately constant for values of y_m far from

the limits of its allowed range (23). For now, suppose that $A =$ constant in some interval. From (35), one finds

$$y_m = C + \frac{2}{A}\rho^{-1/2}, \quad \rho = \frac{4}{A^2}(y_m - C)^{-2}, \qquad \text{yy4} \tag{38}$$

and hence from (36),

$$\frac{dT}{dy}(y_m) = (2/A)^2(y_m - C)^{-2}. \qquad \text{v9} \tag{39}$$

Replacing y_m by the general variable y and integrating, we get

$$T(y) = C' - (2/A)^2(y - C)^{-1}. \qquad \text{v10} \tag{40}$$

Since $\frac{dU}{dy} \to 0$ as $y \to \infty$ and hence (37) $T \to 1$, the constant $C' = 1$.

Putting this into (37) yields

$$\frac{dU}{dy} = 1 - T = (2/A)^2(y - C)^{-1}. \qquad \text{v11} \tag{41}$$

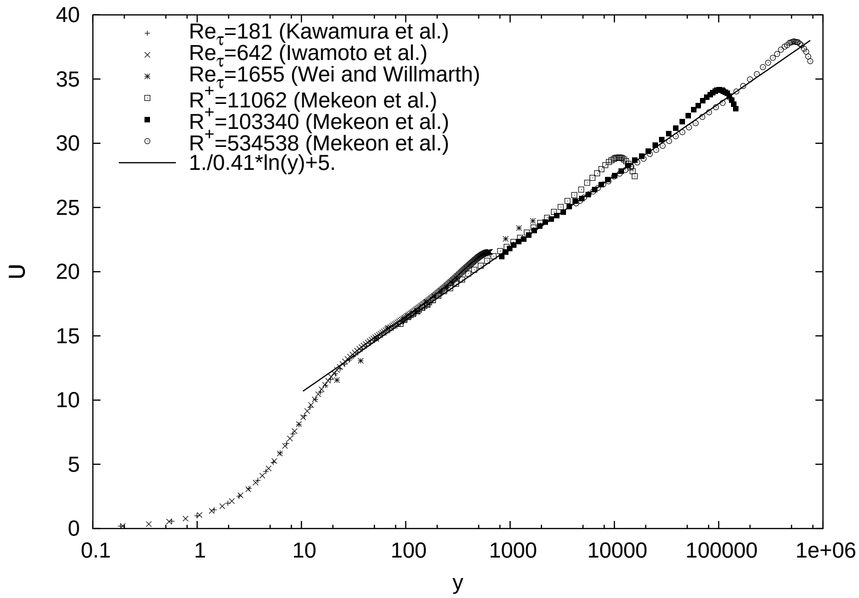

Figure 2: Inner normalized mean streamwise velocity in Couette flow, pressure-driven channel flow and pipe flow. Couette flow data are from DNS of Kawamura's group [7, 11]. Pressure-driven channel flow DNS data are from Iwamoto et al. [5] and experimental data are from Wei and Willmarth [13]. Pipe flow data are from superpipe data of McKeon et al. [8].

Integrating again,

$$U(y) = (2/A)^2 \ln (y - C) + C'', \qquad \boxed{\text{v13}} \tag{42}$$

providing logarithmic growth with a "von Karman–like constant" $\kappa = \frac{1}{4}A^2$, although the usual empirical law lacks the constant C. This latter constant may seriously affect the value of the pre-logarithmic coefficient. Estimates for C, C'' and κ could be found by fitting (42) to empirical data.

The conclusions (42) and (40) were under the assumption that $A =$ constant, and under that very restrictive assumption are valid in the region (given explicitly) where the hierarchy was constructed. That assumption of constancy is unlikely ever to be exactly true, although we have given reasons above and in Sec. 5.5 to believe its approximate constancy in some cases.

The effect of an approximate constancy of A on the validity of (40) and (42) can be easily seen. Write the dependence of A on ρ as a dependence on $y_m = y_m(\rho)$, i.e. $A = A(y_m(\rho))$. Suppose that the function $A(y_m)$ has range lying in the interval $A_0 - \sigma \le A(y_m) \le A_0 + \sigma$ for some constant A_0 and some small positive number σ. Then (31) becomes a pair of inequalities which bound the left side inside an interval depending on σ. The integration steps analogous to (39)–(42) then result in inequalities of the form

$$1 - (c_0 + \sigma c_1)(y - C)^{-1} \le T \le 1 - (c_0 - \sigma c_1)(y - C)^{-1}, \qquad \boxed{\text{v70}} \tag{43}$$

$$(c_2 - \sigma c_3) \ln (y - C) \le U - C'' \le (c_2 + \sigma c_3) \ln (y - C). \qquad \boxed{\text{v71}} \tag{44}$$

5.5 A limiting situation

lim

In the hierarchy, each y can be identified as being a point $y_m(\rho)$ for some ρ. The corresponding ρ will be called $\rho(y)$. In this way, each y has a layer $L_{\rho(y)}$ containing y, such that $-A(\rho)$ is the scaled second derivative of T^ρ at its peak. What mechanism will cause $A(\rho)$ to vary? Certainly not the mean momentum balance partial differential equation (22) in that vicinity, nor the values of the scaled derivatives $\frac{d\hat{T}^\rho}{d\hat{y}} = 0$ or $\frac{d^2\hat{U}}{d\hat{y}^2} = -1$ (from (22)) at that peak location, because these things do not change with ρ. The only source for such a variation would be influence from neighboring layers. Extending that chain of influence, one could speak, on the one hand, of the influence due to layers $L_{\rho'}$ lower in the hierarchy with $\rho' > \rho$, stretching down to those values of y at or near the lower limit y_m' of the hierarchy, i.e. the smallest values of y which accommodate a layer, $y \sim 10$. It stands to reason that this influence of the lower part of the hierarchy will diminish as it becomes more remote, i.e. as the original y becomes large.

The similar chain of influence extends toward higher values of y, i.e. $\rho' < \rho$, capped only by the upper bound $y = \epsilon^{-2}$, at or near the centerline. The centerline, however, becomes further, as $\epsilon \to 0$, from the original point y if the latter is fixed or moves outward as $\epsilon \to 0$ more slowly than ϵ^{-2}.

Consider, then, a band of values of y, depending on ϵ, which migrate away from the wall (measured in the wall coordinate y) as $\epsilon \to 0$, but more slowly than ϵ^{-2}. An example would be the intermediate band $\{\epsilon^{-1/2} < y < \epsilon^{-3/2}\}$. In that interior band, the above argument suggests that the values of A will become more and more independent of any influence from the upper and lower limits of the hierarchy, and therefore would tend to become constant. In the limit as $\epsilon \to 0$, therefore, the analysis relating to the case $A =$ constant would apply so that (40) and (42) would be approached in that band.

5.6 Velocity increments across the layers

velinc For the purpose of this section, we consider the spatial extent of each layer L_ρ to be $O(1)$ in the local scaled variable $\hat{y}$, hence $O(\rho^{-1/2})$ in the inner variable y. In L_ρ, our local scaling implies $\frac{dU}{d\hat{y}} = O(1)$, so that by integrating, we get the increment ΔU in U to be $O(1)$. Thus **U changes by an amount $O(1)$ across each layer L_ρ.**

6 Pressure-driven channel flow

s4 The purpose of this section is to illustrate that scale hierarchies similar to those in Couette flow exist in pressure driven flow. Well accepted lore has the mean velocity profile divided into several zones, one of which is dominated by logarithmic growth of the profile. But in fact, the evidence below indicates why these hierarchies not only comprise the "stress gradient balance layer", where the dynamics is similar to that governed by (8), but the entire flow domain of the traditionally defined logarithmic layer.

6.1 Comparative description of channel flow

comp

Flow through a channel with fixed walls is the 2D version of flow through a pipe. We now consider flow that is driven by an imposed pressure gradient P_x rather than a mobile upper wall, as in the case of Couette flow.

The equation of momentum balance analogous to (1) is

$$\mu \frac{d^2U^*}{dy^{*2}} - \rho_m \frac{d}{dy^*}\langle u'v' \rangle - P_x = 0. \qquad \text{p1c} \qquad (45)$$

The inner normalized dimensionless form is, in place of (8),

$$\frac{d^2U}{dy^2} + \frac{dT}{dy} + \epsilon^2 = 0. \qquad \boxed{\text{a1c}} \qquad (46)$$

Here ϵ is the same (6) as before; the dimensionless pressure forcing term ϵ^2 arises because there is a well-known relation between P_x and u_τ, namely $u_\tau^2 = -\frac{h}{\rho_m} P_x$.

From the point of view of the flow structure, the most significant difference between Couette and pressure-driven channel flow is that whereas turbulent Couette flow has T rising monotonically from the wall to the centerline, in pressurized flow with fixed walls it vanishes at the walls as well as at the centerline, attaining a maximum somewhere between the lower wall and the centerline. The location of the maximum is the site of a third primary layer, called the mesolayer and denoted in [4] by Layer III. This is in addition to the inner and outer primary layers.

The main point of structural similarity, however, is that in both cases there exists a continuum of scaling patches (secondary layers) forming a hierarchy between the traditional inner and outer scales. In the case of pressure-driven channel flow, the mesoscale is embedded within this continuum. The point of similarity revolves around a simple transformation (47) of the Reynolds stress. This transformation reduces the Couette scaling problem to one which is essentially the channel flow.

6.2 Hierarchy

s4.2 To exhibit a hierarchy of layers in the channel flow profile, all that is needed is to revise slightly the definition of the adjusted Reynolds stresses (13). The new one is defined by

$$T^\rho(y) = T(y) + \epsilon^2 y - \rho y. \qquad \boxed{\text{q60}} \qquad (47)$$

This transforms the basic momentum balance equation (46) into

$$\frac{d^2U}{dy^2} + \frac{dT^\rho}{dy} + \rho = 0, \qquad \boxed{\text{d61}} \qquad (48)$$

which is of the same form as (15).

Therefore, with the newly adjusted Reynolds stresses, the channel flow context is amenable to the scaling arguments and balance exchange described in Section 4, hence the construction of a continuum of scalings with associated layers L_ρ, and (under some assumptions) the derivation of logarithmic-like profiles in Section 5.4.

6.3 Profiles when $A =$ const

The mean profile calculations are given here only for the simplest case $A =$ constant, although analogs of (35)–(37) can be derived. As before, the expressions (31) and (30) are obtained in the present setting as well. But the integration of (30) yields a different integration constant. It is required that $\frac{dT}{dy} = 0$ at $y = y_m$, the location of the maximum of the original unadjusted T. Therefore (39) is replaced, under the same supposition that $A =$ constant, by

$$\frac{dT}{dy}(y) = (2/A)^2 \left[(y - C)^{-2} - (y_m - C)^{-2} \right], \qquad \boxed{\text{v9'}} \qquad (49)$$

where now the variable y is the same variable as in (40) and y_m was just defined. Note that this derivative changes sign as y passes through y_m, as it should. Integrating once again, one obtains

$$T(y) = C' - (2/A)^2(y - C)^{-1} - (2/A)^2(y_m - C)^{-2}y. \quad \boxed{\text{v10'}} \tag{50}$$

But there is now a known boundary condition, $T = 0$ at $y = 1/\epsilon^2$; this serves to determine the constant C'.

Similar to the previous procedure, one may now use the integrated form analogous to (10) to determine $\frac{dU}{dy}$ and integrate it with the boundary conditions that the derivatives of U vanish as $y \to \infty$ to obtain the same log dependence as in (42):

$$U(y) = (2/A)^2 \ln(y - C) + C'', \quad \boxed{\text{v13'}} \tag{51}$$

Again, this is all under the (doubtful) assumption that A is constant. In the case that it is almost constant, one gets a pair of bounds like (44), valid now for the mean velocity in channel flow for the range of y constructed as before. Note that in the case $\rho = \epsilon^2$, by (47) $T^\rho = T$.

6.4 The mesolayer

When $\rho = \epsilon^2$, the adjusted Reynolds stress T^ρ (47) coincides with the actual Reynolds stress T, so that the corresponding layer $L_{\rho=\epsilon^2}$ will be located near the location of the maximum of T. As mentioned in Section 6.1, this is in part how the mesolayer III was identified in [14].

Each of the layers L_ρ can be thought of as an adjusted mesolayer, constructed by replacing the actual T by T^ρ. In this sense, the actual mesolayer $L_{\rho=\epsilon^2}$ = III is just one among many. It is distinguished, however, on the one hand as the location where the actual Reynolds stress reaches its maximum and its gradient changes sign, and on the other hand as the location where an important force balance exchange takes place.

7 Discussion

The problem of determining analytically the mean velocity profile of steady turbulent flow in a channel, of either Couette type or pressure-driven type, can be reduced, by averaging, to a second order ODE for two unknown functions U and T. The underdetermined nature of this problem precludes any exact analytic solution. However, scaling tools based on large Reynolds number and a second artificial small parameter ρ, together with physical and mathematical assumptions about conditions for the existence of scaling patches (layers), provide remarkable detail about the structure of the mean velocity and Reynolds stress profiles. In particular, it has been shown here that these profiles enjoy a whole continuum of layers, of which the traditional outer layer is an extreme case, and such that the inner layer is near to the lower extreme of the continuum. In addition to these structural results, information about the profile functions themselves is obtained. Portions of the velocity profile which are logarithmic are associated with intervals of constancy of an $O(1)$ characteristic function $A(\rho)$ associated with the layer continuum. It is argued, again based on asymptotic considerations, that strict logarithmic behavior, while never the case for finite Reynolds numbers, may be seen in certain intervals in the limit as $Re \to \infty$.

These results are totally independent of any overlap argument, which is heavily used in classical treatments, and are practically independent of empirical data.

We deal entirely with orders of magnitude, characteristic lengths, and layers. This lack of precision is the penalty imposed by the underdetermined nature of the problem. Rather than giving explicit formulas for turbulence quantities, our purposes here are (a) to derive the qualitative and approximate quantitative structure of profiles, especially relative to scaling considerations, and (b) to present a new approach in an attempt to elucidate the connection between the flow profile features and the governing basic momentum balance equation.

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