# Numerical Integration Errors in the Material Point Method

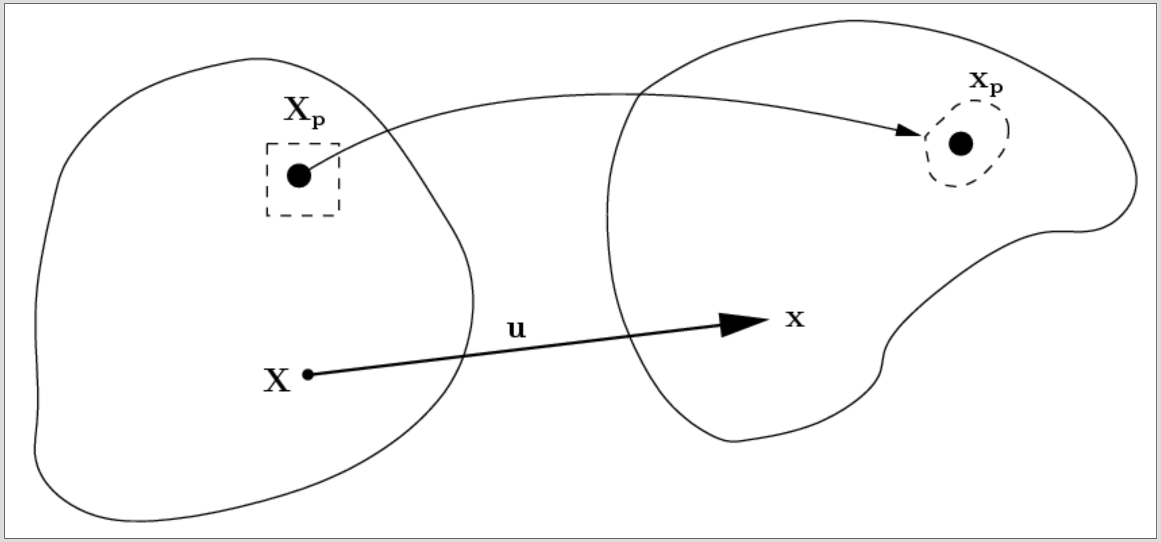
Mike Steffen
University of Utah
Scientific Computing and Imaging Institute

Fourth Material Point Method Workshop
University of Utah
March 17, 2008

# Outline

- *“Analysis and Reduction of Quadrature Errors in MPM”*
  M. Steffen, R.M. Kirby, and M. Berzins, IJNME, 2008
  - Interpretation of particle volume
  - Midpoint integration across discontinuities
  - Internal force errors
  - 1D simulation convergence (basis functions)
- GIMP as Implemented in UINTAH (UGIMP)
  M. Steffen, P. Wallstedt, J. Guilkey, R.M. Kirby, M. Berzins
  - Basis function framework/similarities
  - Smoothing length tests
  - Stability tests
  - Revisit GIMP bar under gravity

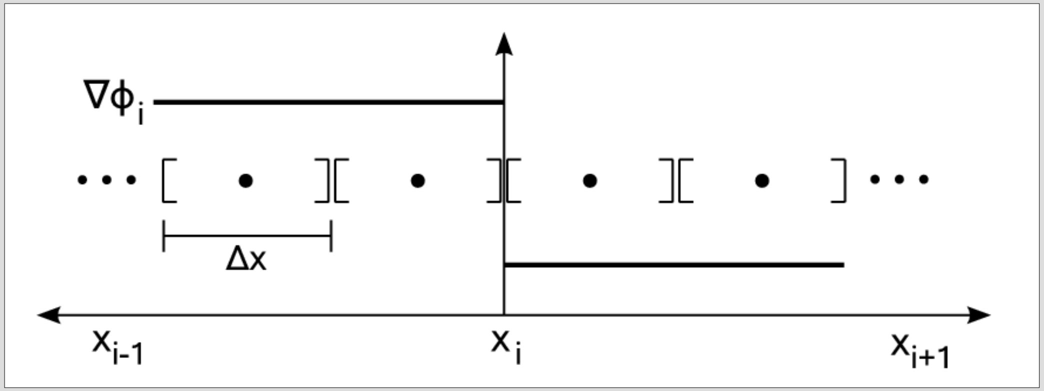
# Particle Volume



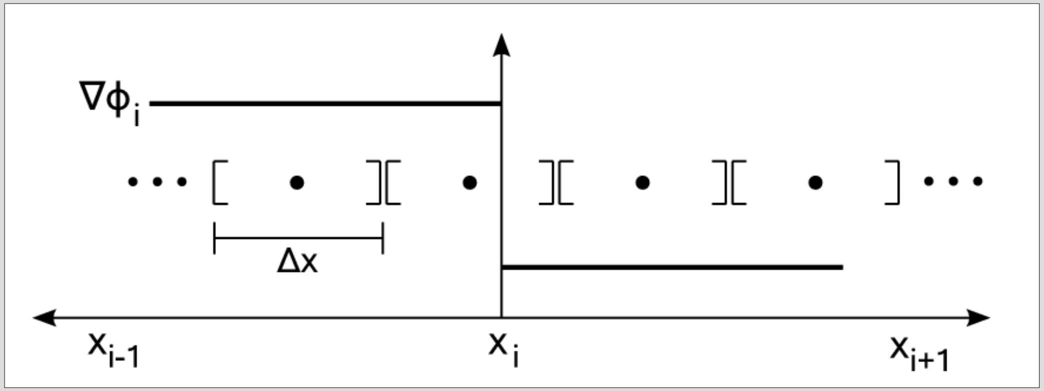

- Deformed voxel properties approximated with information at one sample point
- Can try to improve representation of the deformed voxel (GIMP, Ma *et al.)*
- Can acknowledge error – understand how different basis functions affect this error

# Integration Across Discontinuities

Particle arrangement respects discontinuity:




Particle arrangement does not respect discontinuity:



# Integration Across Discontinuities

- Nodal integration errors in MPM related to midpoint integration across discontinuities
  - Midpoint integration term is $2^{nd}$ order
  - Jump term depends on smoothness of function

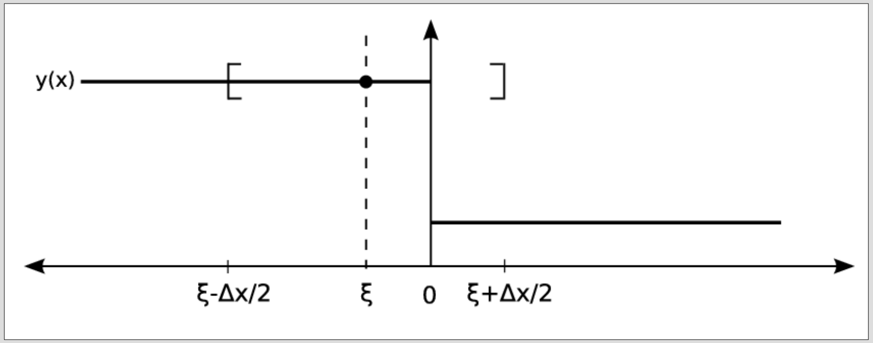
$$\text{Error} = \underbrace{O(\Delta x^2)}_{\text{Midpoint}} + \underbrace{O(\Delta x^{p+2})}_{\text{Jump}}$$

$p$ - Continuity of the function being integrated

# Integration Across Discontinuities

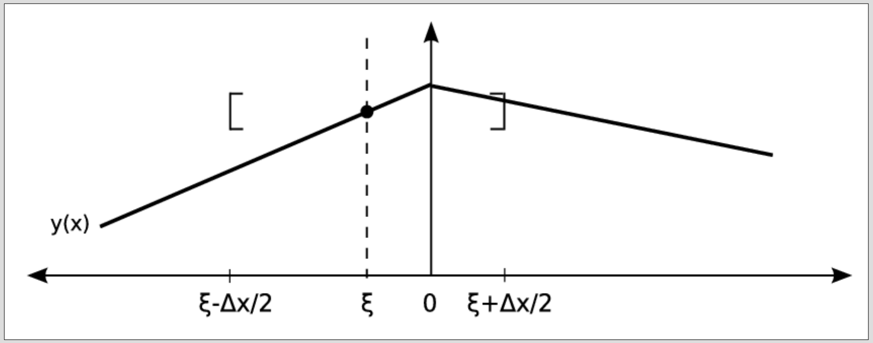
Piecewise-Linear:

$$E = \underbrace{O(\Delta x^2)}_{\text{Midpoint}} + \underbrace{O(\Delta x)}_{\text{Jump}}$$



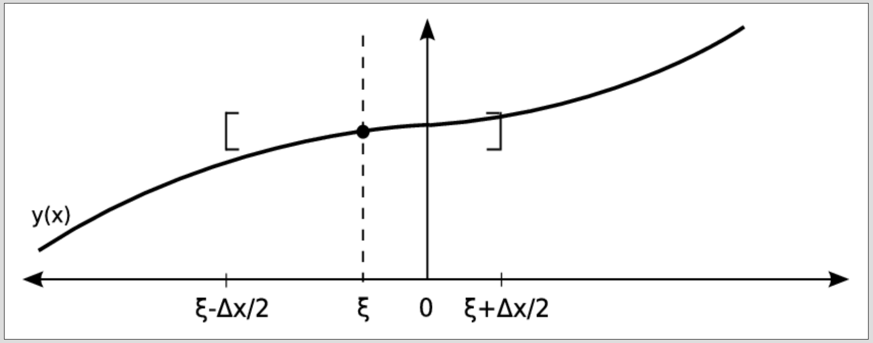

Quadratic B-spline:

$$E = \underbrace{O(\Delta x^2)}_{\text{Midpoint}} + \underbrace{O(\Delta x^2)}_{\text{Jump}}$$



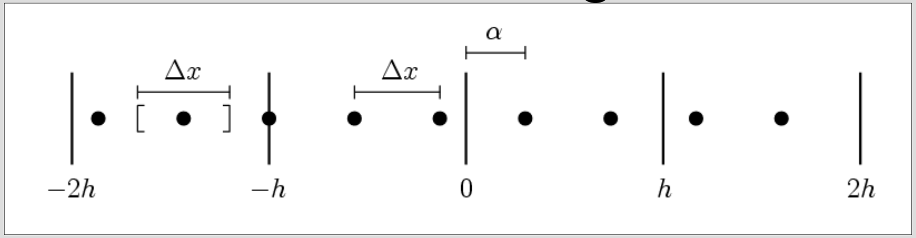

Cubic B-spline:

$$E = \underbrace{O(\Delta x^2)}_{\text{Midpoint}} + \underbrace{O(\Delta x^3)}_{\text{Jump}}$$

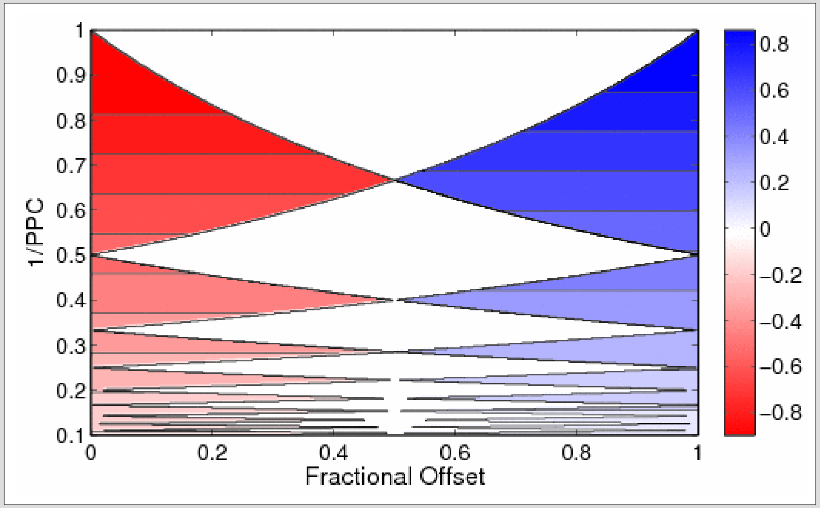


# Internal Force Errors
## Evenly Spaced Particles – Constant Stress

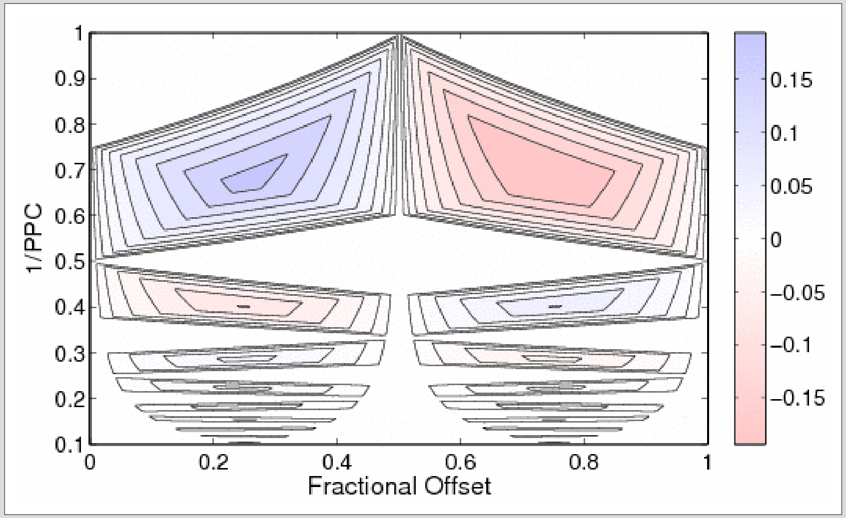
Particle Arrangement



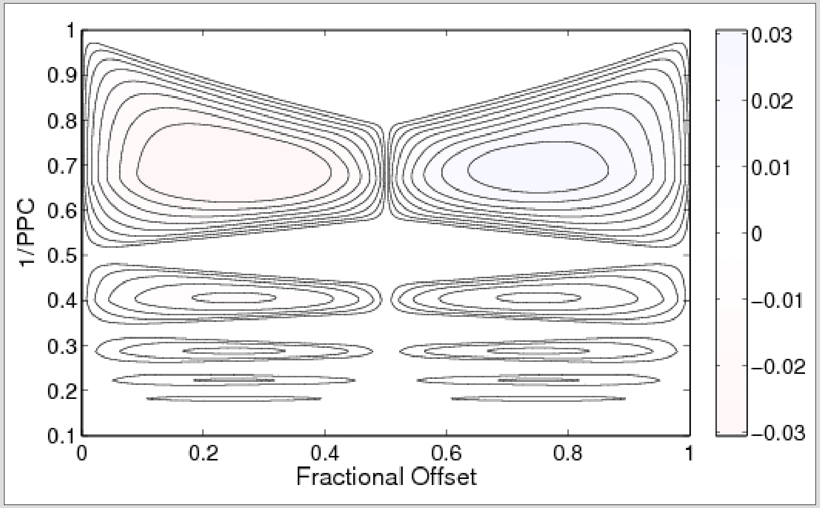

Piecewise-Linear



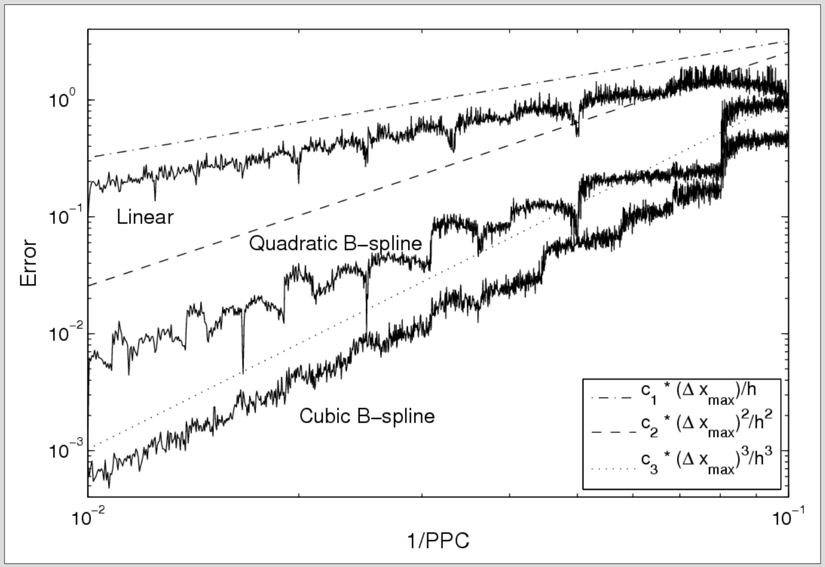

Quadratic B-splines

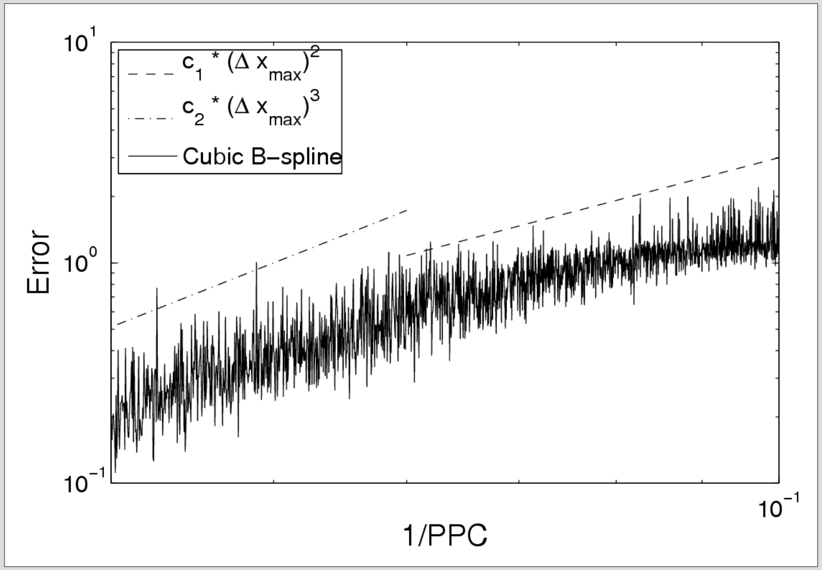



Cubic B-splines



# Internal Force Errors
## Non Uniform Particle Spacing

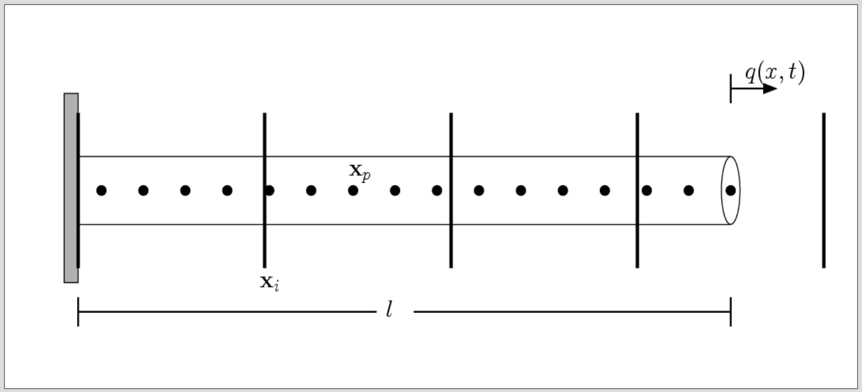



- Locally non-uniform:
  - Piecewise Linear
    - 1st Order
  - Quadratic B-spline
    - 2nd Order
  - Cubic B-spline
    - 3rd Order

- Globally non-uniform:
  - Cubic B-spline
    - 2nd Order

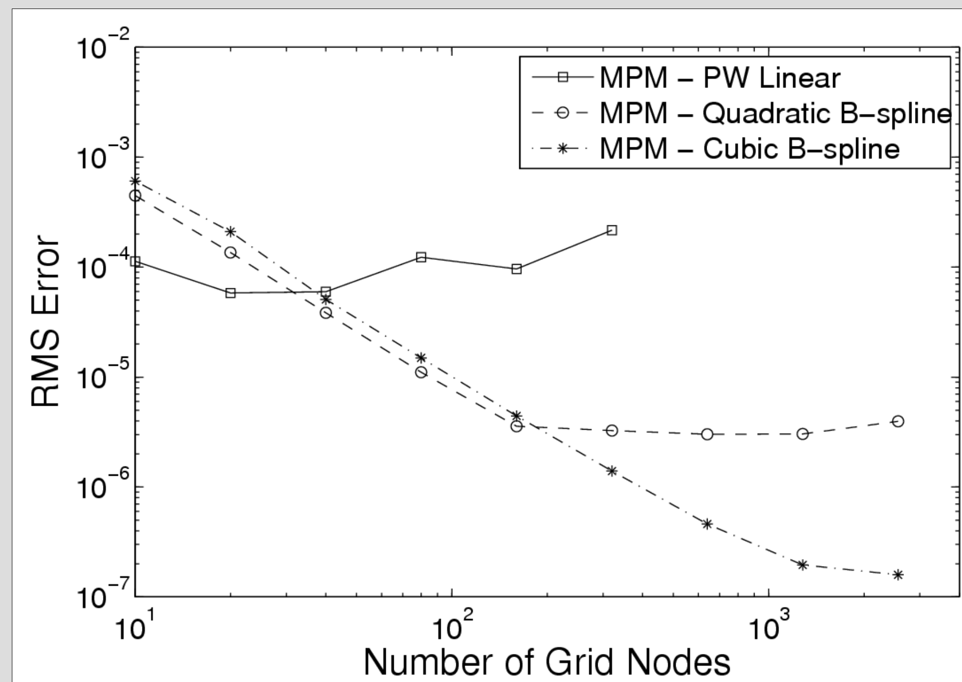
# 1D Bar with Traction Force




$$q(x,t) = \delta(x-l)H(t)\tau\sin(xt/l),$$

$$u(x,t) = \begin{cases} 0 & : \quad t \in [0, l-x) \\ \alpha[1+\cos(\omega(t+x))] & : \quad t \in [l-x, l+x) \\ \alpha[\cos(\omega(t+x)) - \cos(\omega(t-x))] & : \quad t \in [l+x, 3l-x) \\ \alpha[-1-\cos(\omega(t-x))] & : \quad t \in [3l-x, 3l+x) \\ 0 & : \quad t \in [3l+x, 4l] \end{cases}$$

- Piecewise-Linear
  - lack of convergence > 20 Grid cells
- B-spline
  - $2^{nd}$ order spatial convergence
  - Cubic B-splines have lower error plateau



# Quadrature Error Review

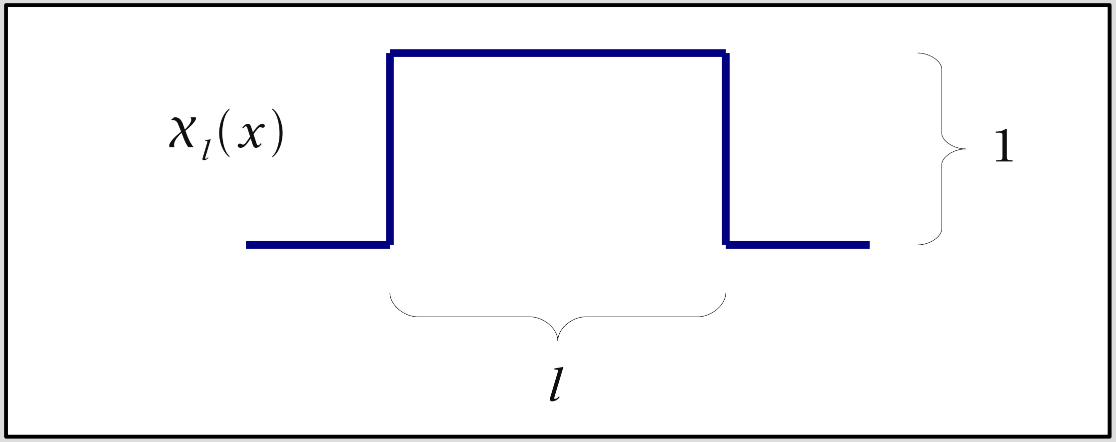
- Nodal integration errors in MPM related to midpoint integration across discontinuities
- Smoother basis functions provide much less quadrature error
  - Piecewise-Linear, 1st order w.r.t. PPC
  - Quadratic B-splines, 2nd order
  - Cubic B-splines, 2nd / 3rd order
- Suggest using $C_1$ or smoother basis functions

M. Steffen, R.M. Kirby, M. Berzins, “Analysis and Reduction of Quadrature Errors in MPM”, IJNME, 2008 (accepted)

# UGIMP

- Full GIMP: Smoothing length determined by particle width at time t
- Contiguous Particle GIMP: Smoothing length determined by particle width at time t=0
  - Results in particle-specific smoother basis functions, but constant for all time
- UGIMP: Disassociate smoothing lengths in GIMP from particle widths
  - Results in a single set of smoother basis functions for all time

# Basis Function Framework




Full GIMP: $\phi_p^t(x) = \chi_h * \chi_h * \chi_p^t / (|\chi_h||\chi_p^t|)$

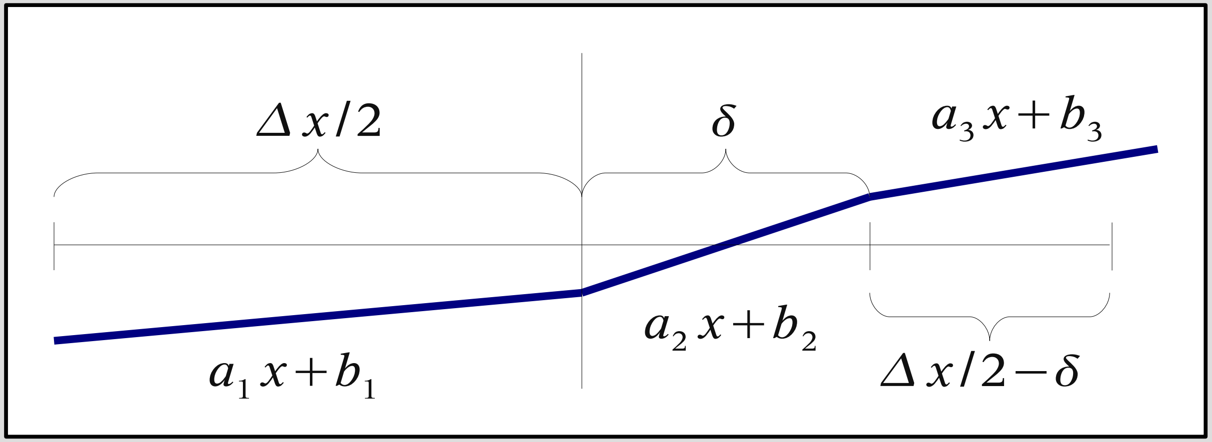
Contiguous Particle GIMP: $\phi_p(x) = \chi_h * \chi_h * \chi_p^0 / (|\chi_h||\chi_p^0|)$

UGIMP: $\phi(x) = \chi_h * \chi_h * \chi_l / (|\chi_h||\chi_l|)$

Quadratic B-spline: $\phi(x) = \chi_h * \chi_h * \chi_h / (|\chi_h|)^2$

Piecewise Linear: $\phi(x) = \chi_h * \chi_h * \delta / (|\chi_h||\delta|)$

# Midpoint Error - Integrating Across Two Discontinuities

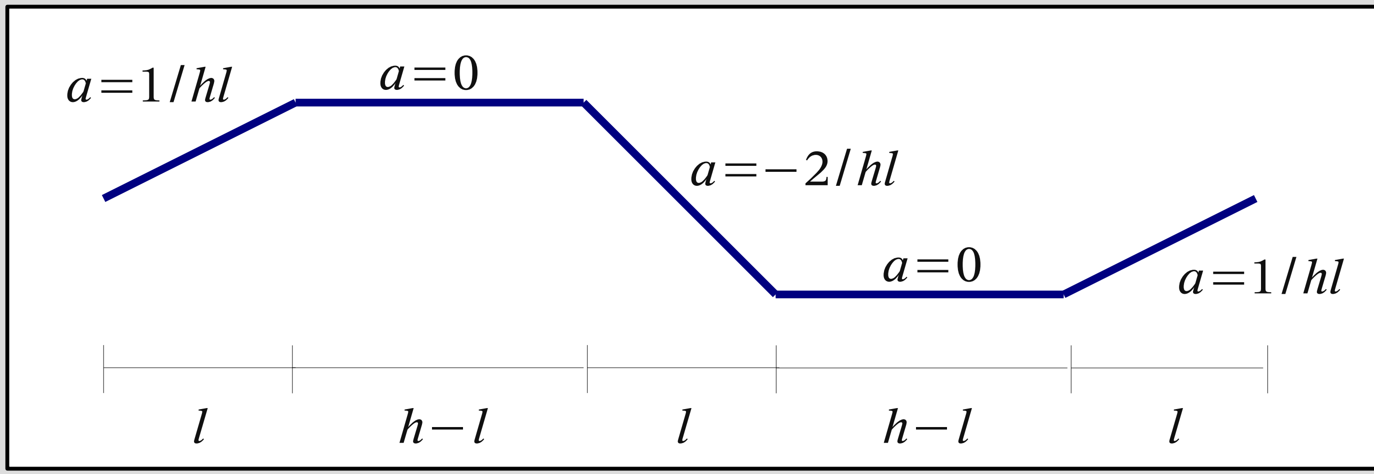


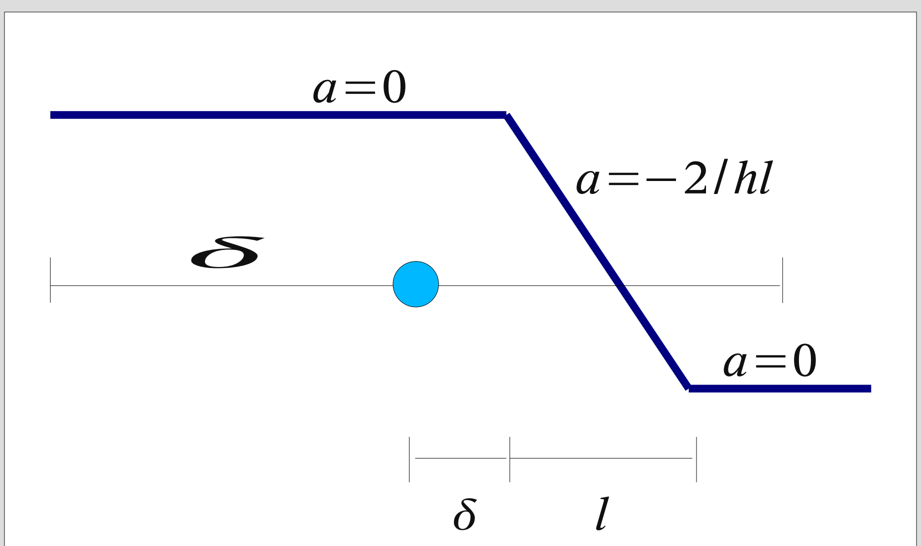

$$E=\int f(x)-\text{Midpoint Approximation}$$

$$E=-\frac{1}{8}(a_3-a_1)\Delta x^2+\frac{1}{2}(a_2-a_3)\Delta x\,\delta-\frac{1}{2}(a_2-a_3)\delta^2$$

# Integration Errors in UGIMP

UGIMP $\nabla \phi_i$ :




Spans two jumps: $E=\frac{l-\Delta x}{h}$

$l<\Delta x/2$

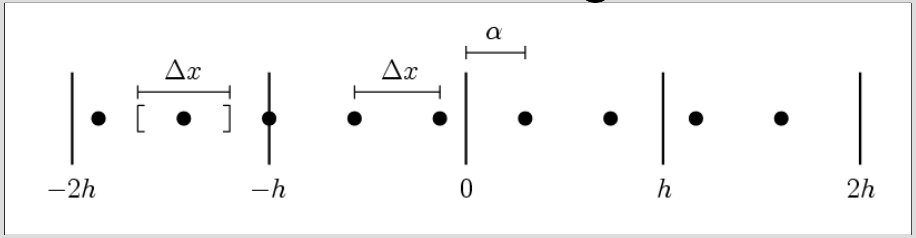
Spans one jump: $E=-\frac{1}{4lh}\Delta x^2$

$l>\Delta x/2$

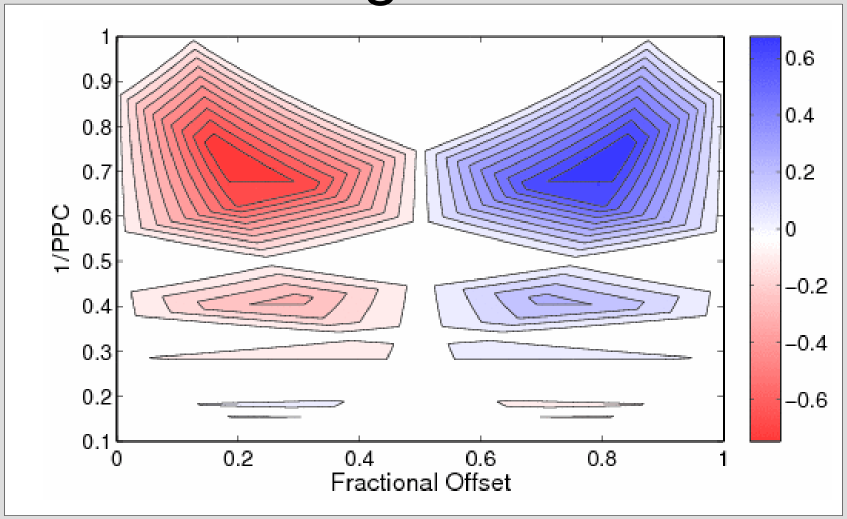
# UGIMP Internal Force Errors

## Evenly Spaced Particles – Constant Stress

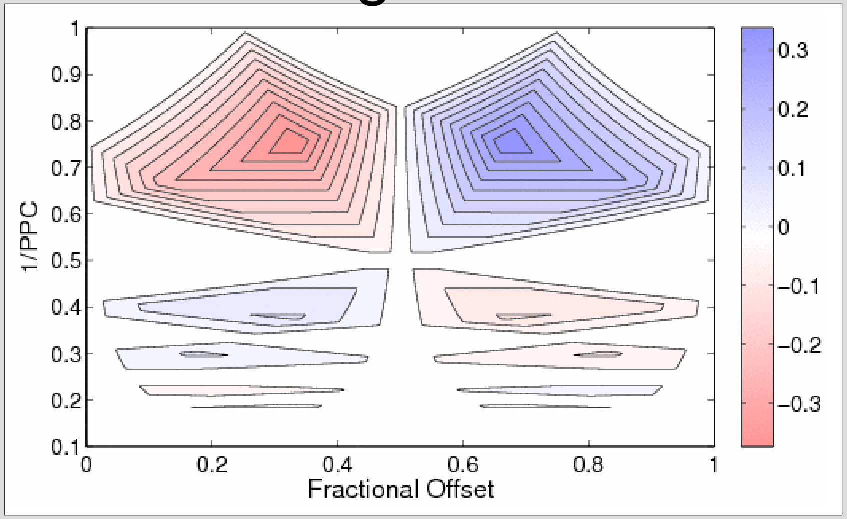
Particle Arrangement



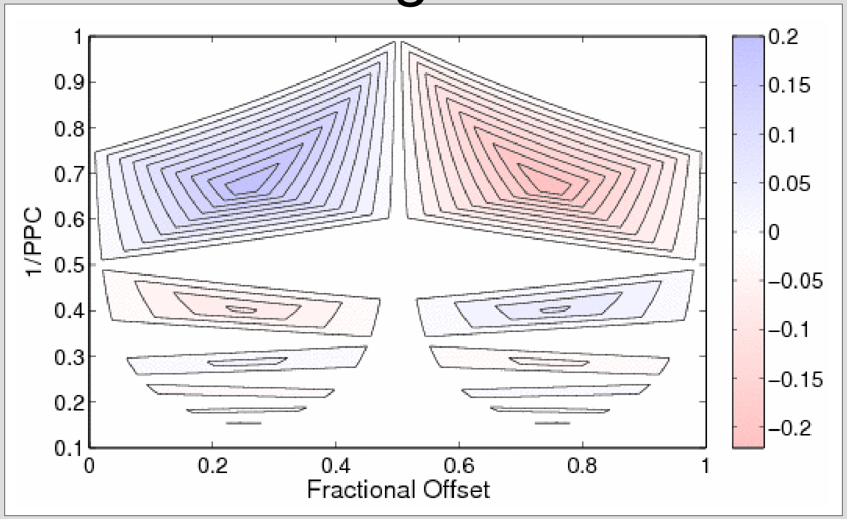

Smoothing Fraction = 1/4




Smoothing Fraction = 1/2




Smoothing Fraction = 1



# Periodic Bar

- Manufactured Solution-Wallstedt and Guilkey
- Periodicity allows us to neglect boundary conditions – focus on other numerics

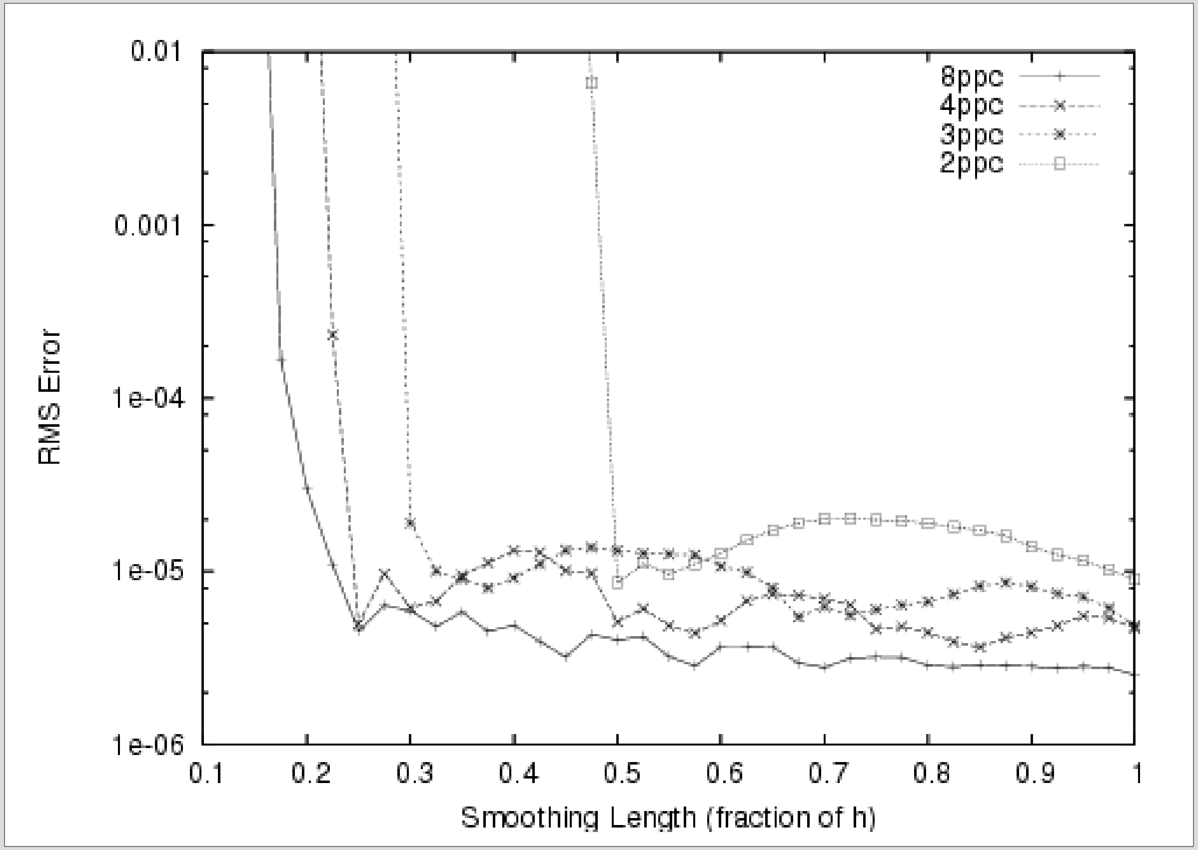
Exact Solution:

$$u(X,t)=A\sin(2\pi x)\cos(C\pi t)$$
$$v(X,t)=-AC\pi\sin(2\pi x)\sin(C\pi t)$$
$$F(X,t)=1+2\mathrm{A}\pi\cos(2\pi x)\cos(C\pi t)$$
$$b(X,t)=E\pi^2 u(X,t)(1+2F(X,t)^{-2})$$

Initialize MPM: $x_p=x_p^0+u(x_p^0,0)$ Same with $v_p$ and $F_p$

Body Force: $b_p=b(x_p^0,t)$

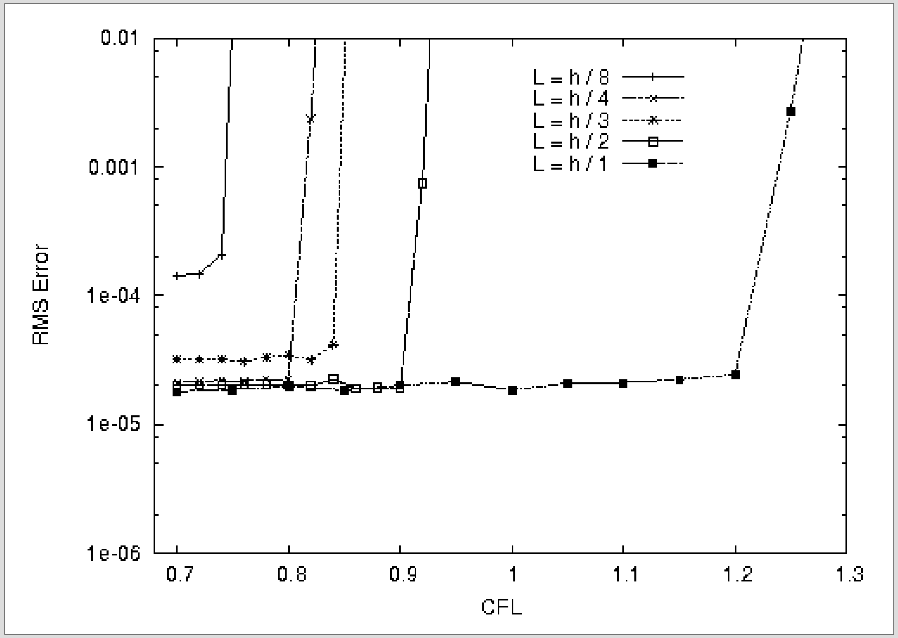
# UGIMP – Smoothing Fraction



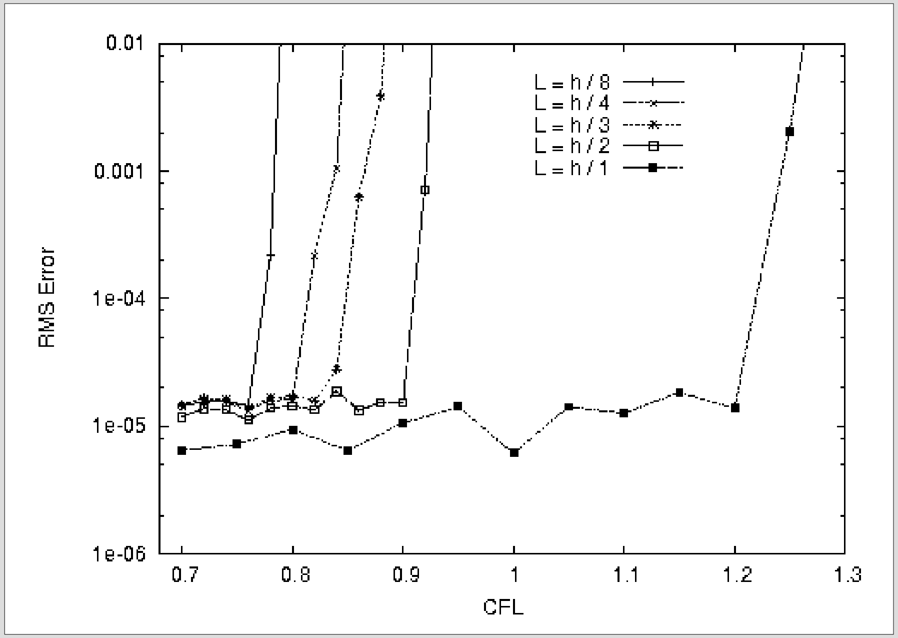

- UGIMP becomes unstable when particle widths are larger than the smoothing length

# UGIMP Stability

4 Particles Per Cell

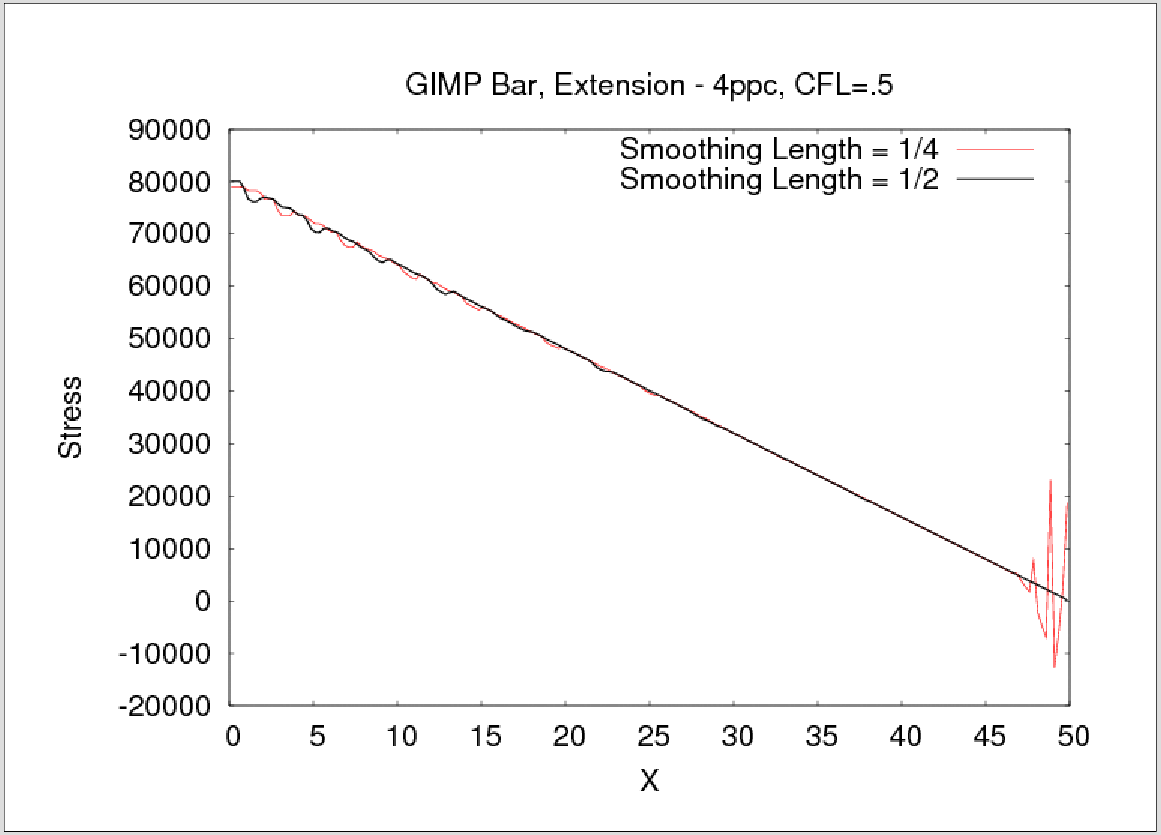



8 Particles Per Cell




- Stability of UGIMP depends more on the smoothing length than the number of particles per cell

# Original GIMP Bar* - Extension




- 1D Bar **extension** under gravity
- Slowly ramp up body force over 40 periods
- Hold at full gravity for 40 periods
- UGIMP shows instabilities when L = 1/PPC
- Stable when L > 1/PPC

*S.G. Bardenhagen and E.M. Kober, *The Generalized Interpolation Material Point Method*, CMES, 5(6):477-495, 2004

# Conclusions

- Benefits to using $C_1$ or smoother basis functions
  - Midpoint Approximation limits us to 2nd order
- Quadratic B-splines and GIMP are closely related
- Smoothing length is more important for stability than number of particles per cell
- Smoothing lengths larger than the particle width provide extra stability

# Acknowledgments


- Utah MPM group. Especially:
  - Mike Kirby, Phillip Wallstedt, Martin Berzins, and Jim Guilkey

- Center for the Simulation of Accidental Fires and Explosions (CSAFE) DOE Grant W-7405-ENG-48


SCI INSTITUTE

THE UNIVERSITY OF UTAH

C-SAFE UNIVERSITY OF UTAH