Manufactured Solutions Suitable for Verification of MPM and GIMP Codes

Philip Wallstedt – philip.wallstedt@utah.edu
Jim Guilkey – james.guilkey@utah.edu

The Problem

MPM and GIMP don't have known orders of accuracy

Few MPM/GIMP publications compare to exact solutions

MPM/GIMP "too sexy" for small strain known solutions

Known solutions are rare (non-existent?) for large-deformation, transient mechanics problems

Pseudo Verification

Eyeball Norms – Verification of Plausibility

- Not predictive: you already know the answer

Symmetry – some coding mistakes exposed

- Many mistakes are symmetric

Compare to existing code (Finite Element)

- Existing code solves different problems
- Existing code has unverified accuracy
- When differences are found, are they errors or not?

Experimental results – scattered data shows same trends

- Data availability is limited
- Differences don’t allow systematic bug finding

Known Solutions to PDE’s

- Few (no?) dynamic solutions for large deformation

A better way
The Method of Manufactured Solutions

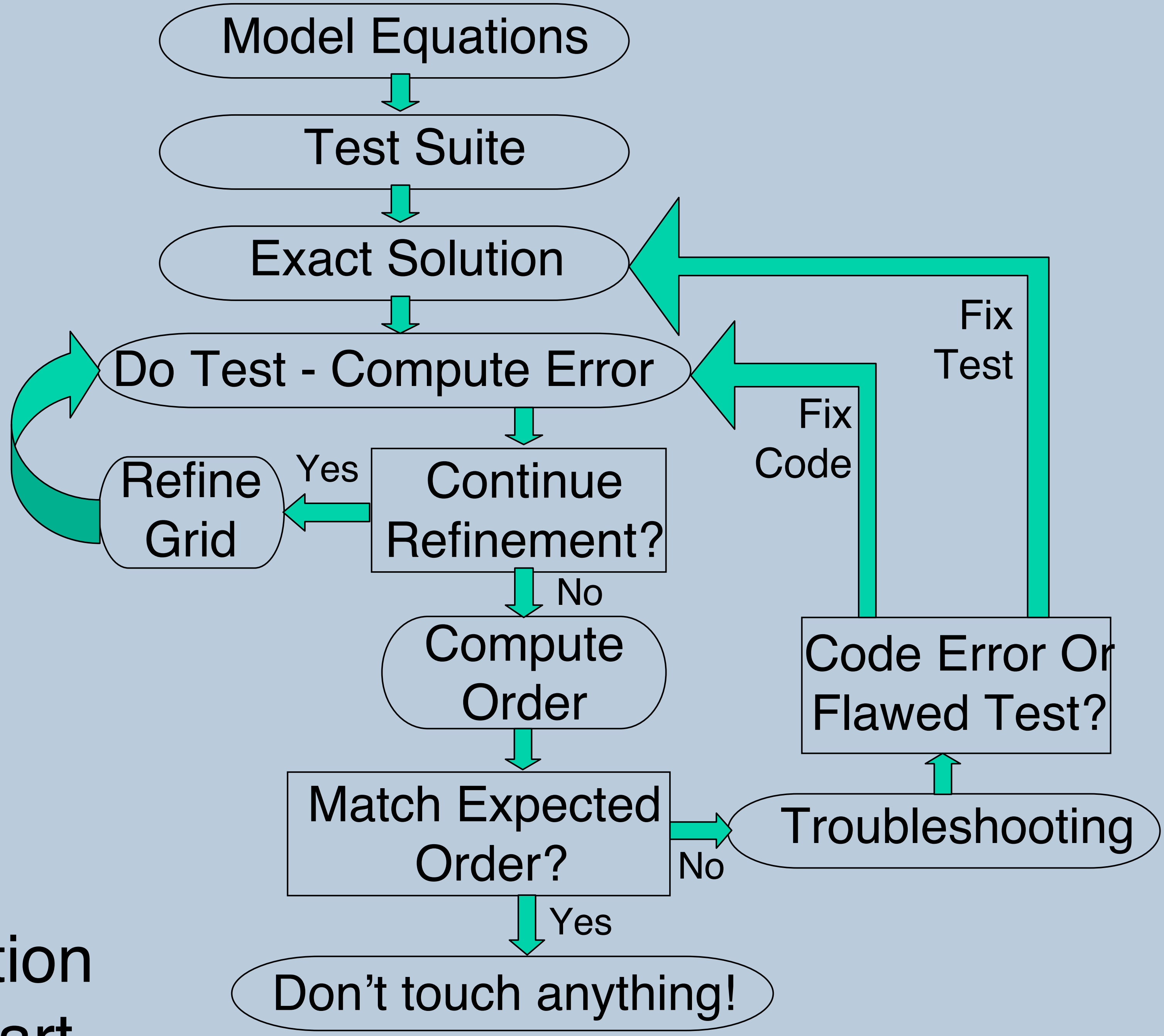
Recently proposed as ASME standard

> "V&V 10 - 2006 Guide for Verification and Validation in Computational Solid Mechanics"

Sufficient, not just necessary, if we test all modes:

- Boundary conditions
- Non-square cells and particles
- Time integration algorithms
- Shape functions

Each mode must be tested, but not all in the same test.
Once a mode has "passed", then further testing not needed.



Verification Flowchart

Rate of convergence is very sensitive to errors and can be applied to individual pieces of a method

Displacement error compares current to reference configurations.

$$\delta_u = \left(x_p - X_p\right) - u(X)_{EXACT}$$

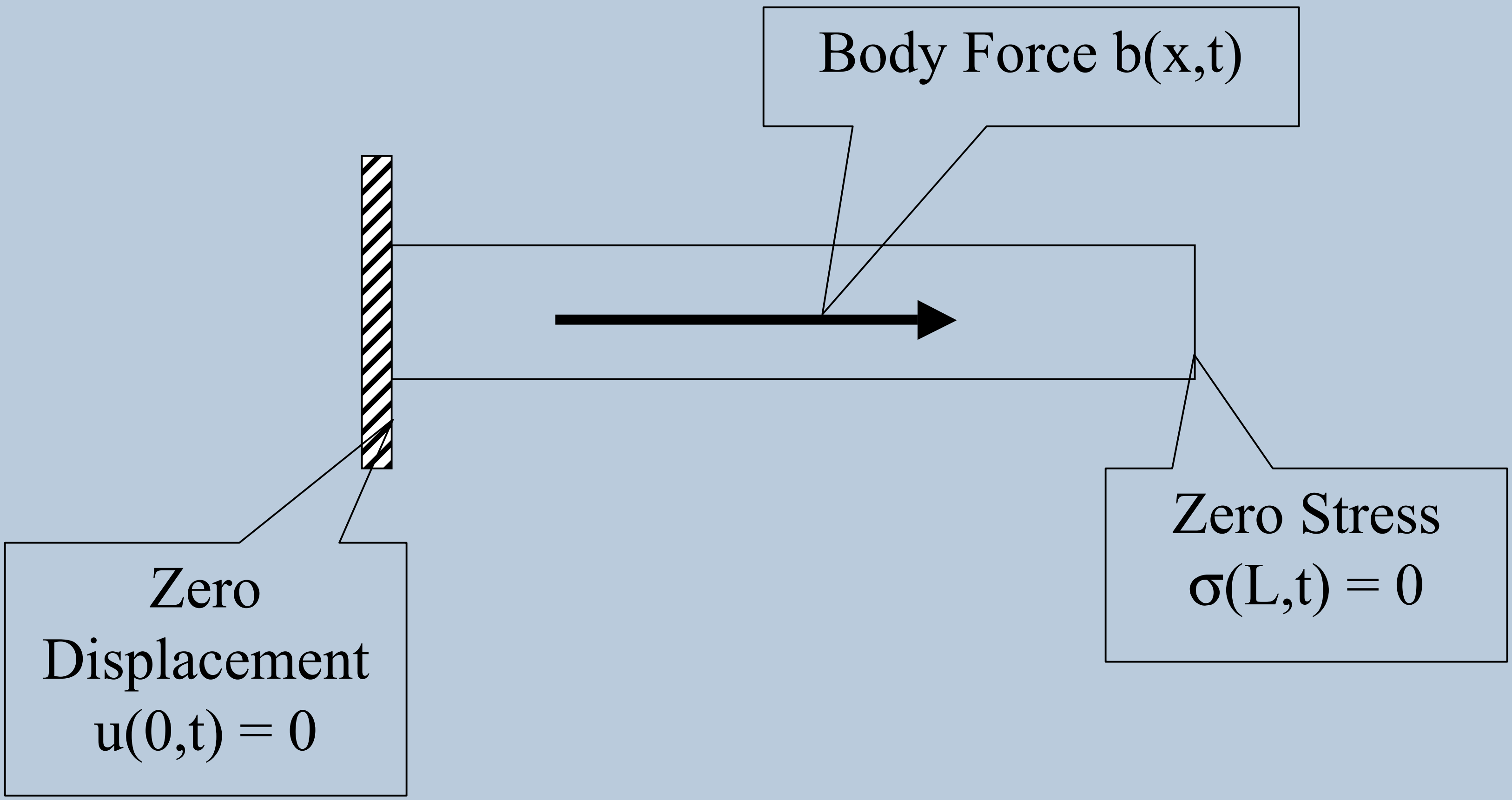
Average error

$$L_1 = \max\left(\frac{\sum_p \delta_p}{N}\right)$$

Worst Error

$$L_\infty = \max\left(\delta_p\right)$$

An Example MMS Solution: Body Force on a 1D Bar



Body Force on a 1D Bar

Given

Momentum $$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \mathbf{a}$$

Neo-Hookean Constitutive Model $$\sigma = \left(\frac{\lambda}{J} \ln J \right) \mathbf{I} + \frac{\mu}{J} \left(\mathbf{F}\mathbf{F}^{\mathrm{T}} - \mathbf{I} \right)$$

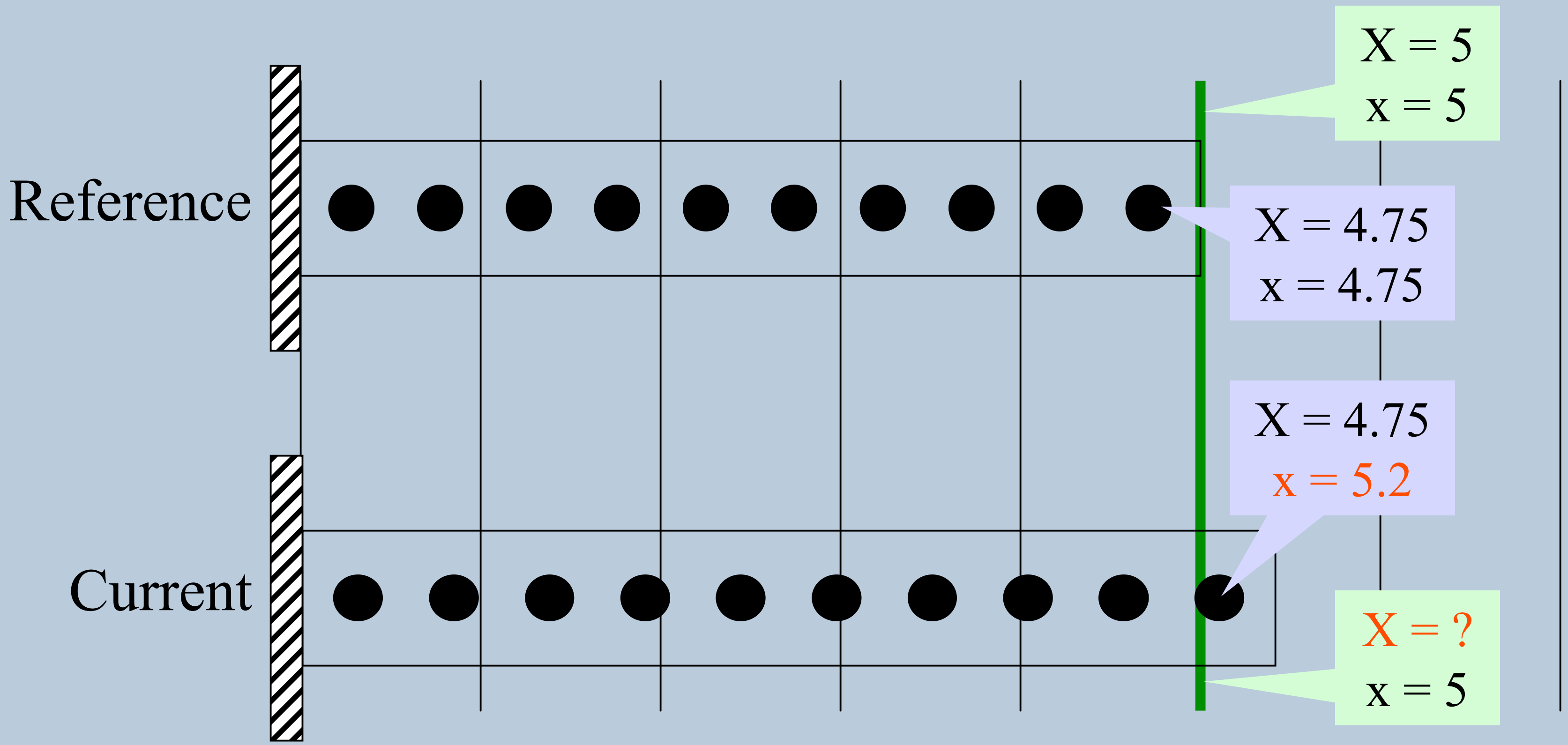
Constitutive Model with assumptions: 1D, Poisson = 0 $$\sigma = \frac{E}{2} \left(\mathbf{F} - \frac{1}{\mathbf{F}} \right)$$

Find displacement u(x) – in general this cannot be done.

A Detour and a Review: Reference versus Current Configuration

Particles stationary in reference configuration

Grid stationary in current configuration



Why manufacture solutions in the reference configuration?

– Convenience. Consider the following example:

How find the current length and apply boundary?

$$u(x) = \frac{x(2L_0 - x)}{L_0^{\ 2}} A(t)$$

$$\Delta L = u(L_0 + \Delta L) = \frac{(L_0 + \Delta L)(2L_0 - (L_0 + \Delta L))}{L_0^{\ 2}} A(t)$$

This is icky. We can avoid recursive / implicit definitions like the above by using the reference configuration.

Reference Configuration vs Current Configuration

	Reference Configuration "Total Lagrange"	Current Configuration "Updated Lagrange"
Momentum	$\nabla \cdot \mathbf{P} + \rho_0 \mathbf{b} = \rho_0 \mathbf{a}$	$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \mathbf{a}$
Deformation Gradient	$\mathbf{F}(\mathbf{X}) = \mathbf{I} + \nabla_X \mathbf{u}$	$\mathbf{F}(\mathbf{x}) = \left[\mathbf{I} - \nabla_x \mathbf{u}\right]^{-1}$
Neo-Hookean	$\mathbf{P} = (\lambda \ln J)\mathbf{F}^{-1} + \mu \mathbf{F}^{-1}\left(\mathbf{F}\mathbf{F}^{\mathrm{T}} - \mathbf{I}\right)$	$\boldsymbol{\sigma} = \left(\frac{\lambda}{J} \ln J\right)\mathbf{I} + \frac{\mu}{J}\left(\mathbf{F}\mathbf{F}^{\mathrm{T}} - \mathbf{I}\right)$
Assume 1D, Poisson = 0	$\mathbf{P} = \frac{E}{2}\left(\mathbf{F}(\mathbf{X}) - \frac{1}{\mathbf{F}(\mathbf{X})}\right)$	$\boldsymbol{\sigma} = \frac{E}{2}\left(\mathbf{F}(\mathbf{x}) - \frac{1}{\mathbf{F}(\mathbf{x})}\right)$

Stress Transformation: $\mathbf{P} = J\mathbf{F}^{-1}\boldsymbol{\sigma}$

Start with the answer and reformulate backwards

Given Displacement $\mathbf{u}(\mathbf{X})$

1D Neo-Hookean with Poisson's ratio = 0 $\mathbf{P} = \frac{E}{2}\left(\mathbf{F} - \frac{1}{\mathbf{F}}\right)$

Momentum $\nabla \cdot \mathbf{P} + \rho_0 \mathbf{b} = \rho_0 \mathbf{a}$

Solve for Gravity $\mathbf{b} = \mathbf{a} - \frac{1}{\rho_0} \nabla \cdot \mathbf{P}$

Now we just take derivatives . . .

What answer (displacement field) do we start with?

The chosen displacement field(s) must:

• exercise all features of the code; large deformation, translation, rotation, Dirichlet and Neumann boundaries

• be "smooth enough" – sufficiently differentiable in time and space

• Conform to assumptions made by the method. For GIMP this means zero normal stress at free boundaries.

For the 1D rod assume a displacement of the form:

$$u = \left(c_0 + c_1 X + c_2 X^2\right) A(t)$$

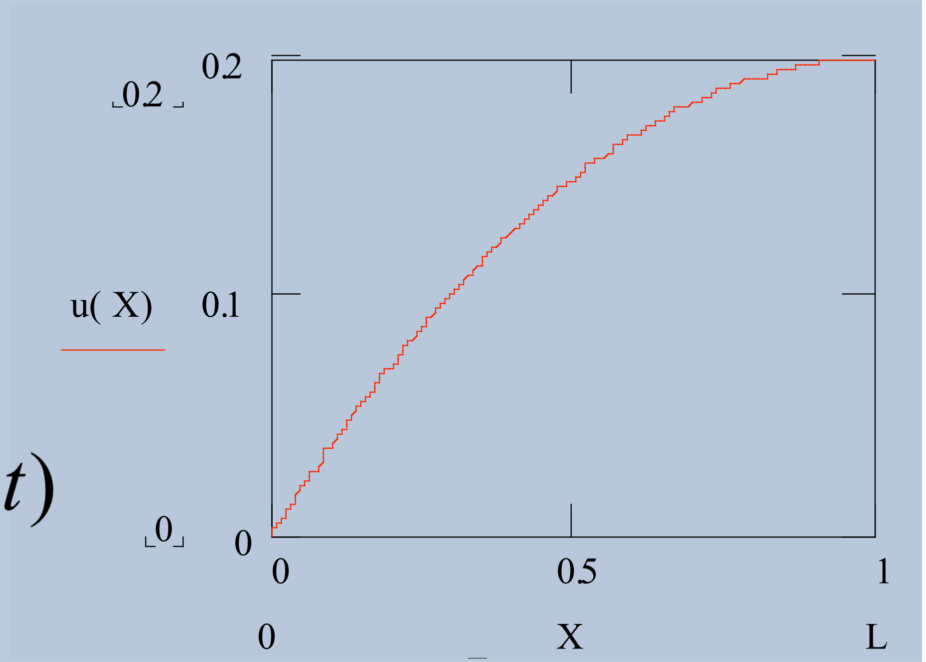
Constants for the 1D bar

Zero displacement at X = 0

$$0 = \left(c_0 + c_1 0 + c_2 0^2\right) A(t)$$

Scale displacement at L

$$A(t) = \left(c_0 + c_1 L + c_2 L^2\right) A(t)$$



Zero stress at X = L

$$P(L) = 0 = \frac{E}{2}\left(F - \frac{1}{F}\right) = \frac{E}{2}\left(1 + \left(c_1 + 2c_2 L\right)A(t) - \frac{1}{1 + \left(c_1 + 2c_2 L\right)A(t)}\right)$$

$$\Rightarrow \quad u(X) = \frac{2LX - X^2}{L^2} A(t)$$

Return to the 1D Bar: Take Derivatives

Given

Displacement $$u(X)=\frac{2LX-X^2}{L^2}A(t)$$

Deformation Gradient $$F=1+\frac{2(L-X)}{L^2}A$$

Divergence of Stress $$\nabla\cdot P=-\frac{E}{L^2}\left(1+\left[1+\frac{2(L-X)}{L^2}A\right]^{-2}\right)A$$

Solve for b(X) $$b=\frac{1}{L^2}\left[X(2L-X)\ddot{A}-\frac{E}{\rho_0}\left(1+\left[1+\frac{2(L-X)}{L^2}A\right]^{-2}\right)A\right]$$

Choose a convenient time function A(t)

Trigonometric functions have nice properties:

- Easy to differentiate
- Amount of deformation is bounded
- Tests ability to stay in phase
- Can be made self-similar in time

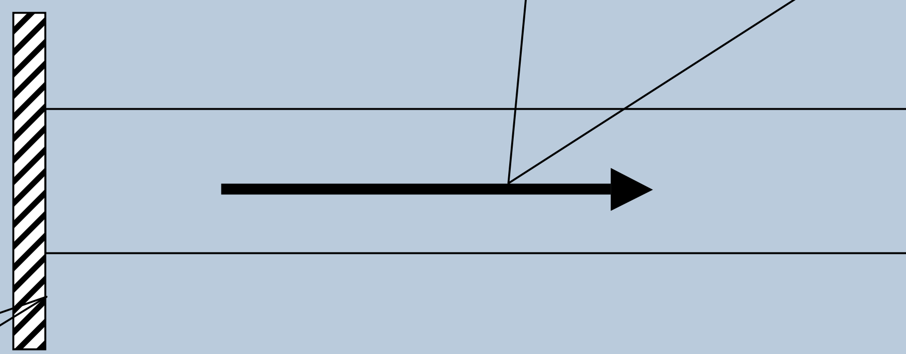
  i.e. – same number of time steps per period,
  regardless of material stiffness.

$$u = \frac{X(2L - X)}{L^2} 0.2 \cos\left(\sqrt{\frac{E}{\rho_0}}\, \pi\ \mathrm{t} \right)$$

1D Bar: Restate the Problem

Body Force

$$b = \frac{1}{L^2}\left[X(2L - X)\ddot{A} - \frac{E}{\rho_0}\left(1 + \left[1 + \frac{2(L-X)}{L^2} A \right]^{-2} \right) A \right]$$



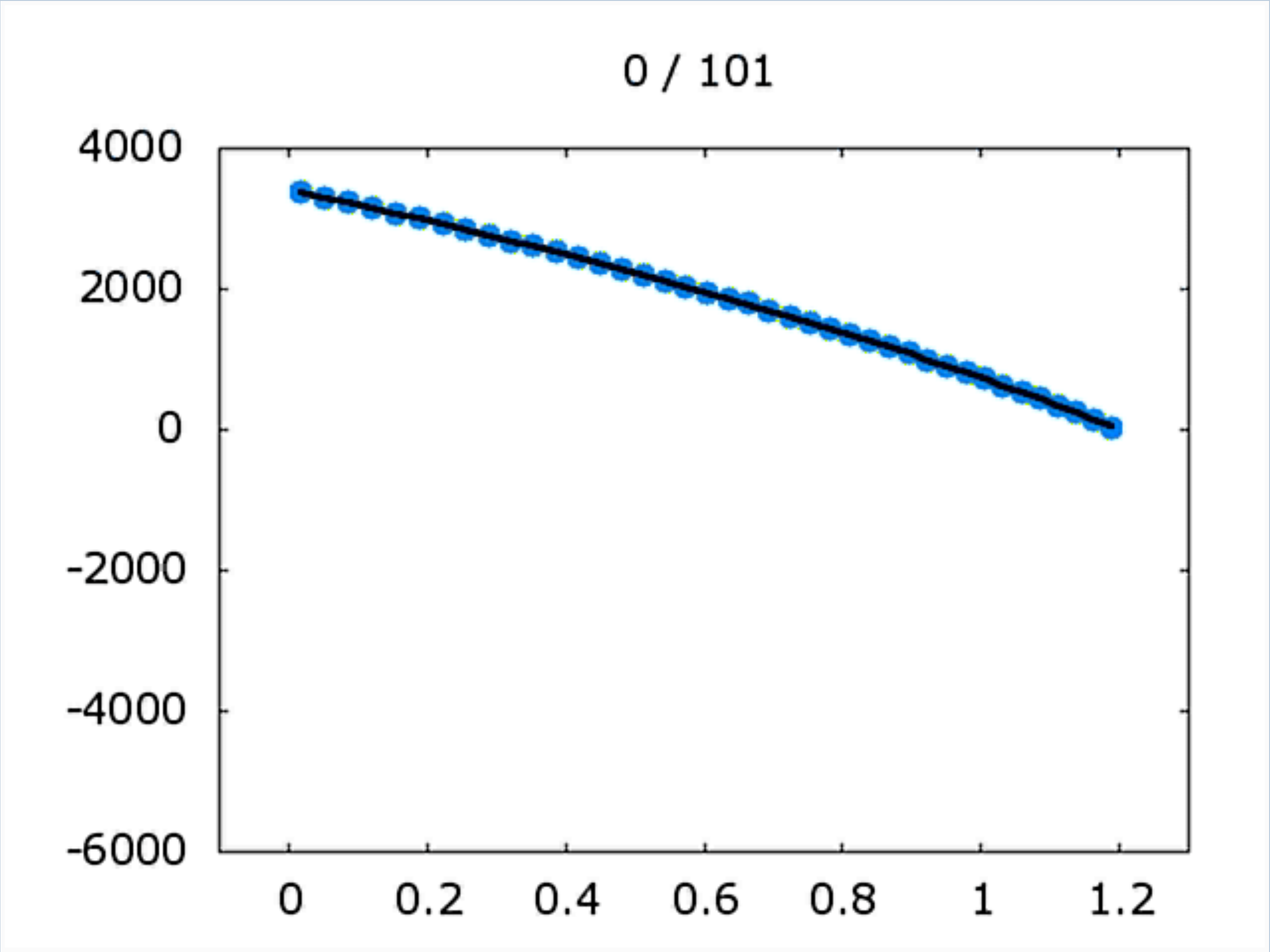
Zero Displacement
$u(0,t) = 0$

Zero Stress
$\sigma(L,t) = 0$

The answer is (should be):

$$u(X) = \frac{2LX - X^2}{L^2} A(t)$$

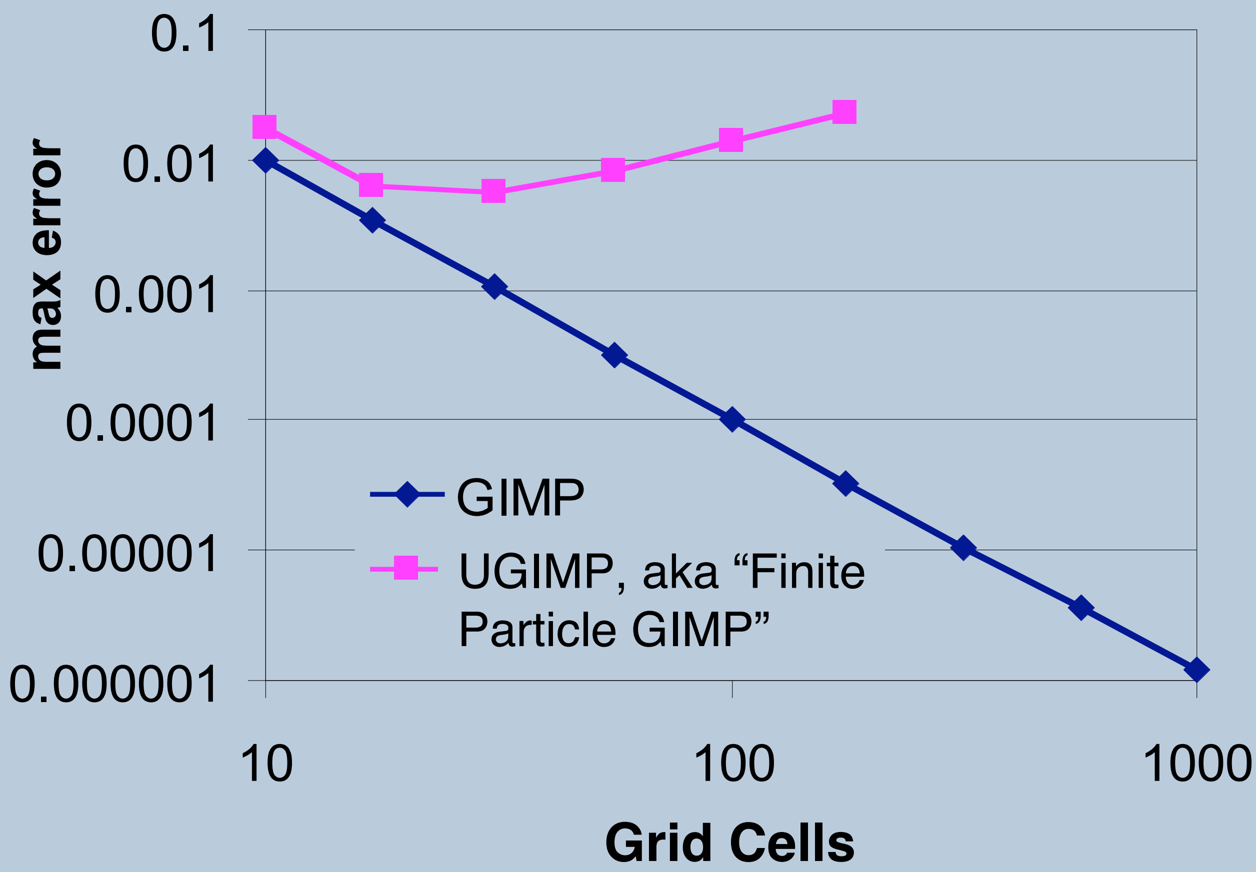
Solve with GIMP where $A(t) = 0.2\cos\left(\sqrt{\frac{E}{\rho_0}}\pi \mathrm{t}\right)$



Stress

Position (Little x)

Now we can measure convergence under large deformation – the kind of problem MPM/GIMP is designed to solve



Conclusions

- Manufactured Solutions Also Generated in 2D and 3D
- MMS Provides a Tool for:
    - Better Understanding MPM and GIMP algorithms
    - Isolating Error Sources
    - Finding Bugs
- No Excuse Left for not Showing Convergence Behavior

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