Medical Image Analysis

CS 593 / 791

Computer Science and Electrical Engineering Dept.
West Virginia University

23rd January 2006

Outline

Outline

Edge Enhancement

Inhomogeneous diffusion may actually enhance edges, for a certain choice of $g()$.

1D example:

Let $s(x) = \frac{\partial I}{\partial x}$, and $\phi(s) = g(s)s = g(I_x)I_x$.

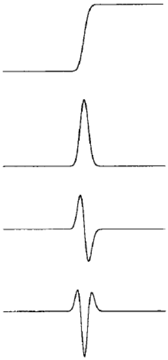
The 1D heat equation becomes

$$\begin{aligned} I_t = \frac{\partial}{\partial x}(g(I_x)I_x) &= \frac{\partial}{\partial x}\phi(s) \\ &= \phi'(s)I_{xx} \end{aligned}$$

$\frac{\partial}{\partial t}(I_x)$ is the rate of change of edge slope with respect to time.

$$\begin{aligned} \frac{\partial}{\partial t}(I_x) &= \frac{\partial}{\partial x}(I_t) \\ &= \phi''(s)I_{xx}^2 + \phi'(s)I_{xxx} \end{aligned}$$

Edge Enhancement



I, I_x, I_{xx}, I_{xxx}

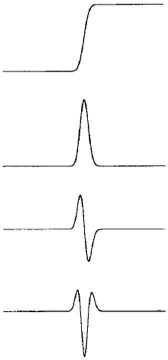
$$\frac{\partial}{\partial t}(I_x) = \phi''(s)I_{xx}^2 + \phi'(s)I_{xxx}$$

For a step edge with $I_x > 0$ look at the inflection point, p.
Observe that $I_{xx}(p) = 0$, and $I_{xxx}(p) < 0$.

$$\frac{\partial}{\partial t}(I_x)(p) = \phi'(s)I_{xxx}(p)$$

The sign of this quantity depends only on $\phi'(s)$.

Edge Enhancement



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The sign of this quantity depends only on $\phi'(s)$.

Edge Enhancement

At the inflection point:

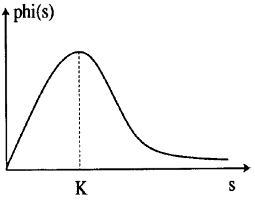
$$\frac{\partial}{\partial t}(I_x)(p) = \phi'(s) I_{xxx}(p)$$

- If $\phi'(s) > 0$, then $\frac{\partial}{\partial t}(I_x)(p) < 0$ (slope is decreasing).
- If $\phi'(s) < 0$, then $\frac{\partial}{\partial t}(I_x)(p) > 0$ (slope is increasing).

Since $\phi(s) = g(s)s$, selecting the function $g(s)$ determines which edges are smoothed and which are sharpened.

Outline

The function $\phi(s) = g(s)s$



- $\phi(0) = 0$
- $\phi'(s) > 0$ for $s < K$
- $\phi'(s) < 0$ for $s > K$
- $\lim_{s\to\infty} \phi(s) \to 0$

The function $g(s)$

Perona and Malik suggest two possible functions ($s = ||\nabla I||$):

$$g(|\nabla I|) = e^{-(\frac{||\nabla I||}{K})^2}$$

$$g(|\nabla I|) = \frac{1}{1 + (\frac{||\nabla I||}{K})^{1+\alpha}} \quad (\alpha > 0)$$

Effect of varying K on $g()$

$$g(|\nabla I|) = \frac{1}{1 + (\frac{||\nabla I||}{K})^{1+\alpha}} \quad (\alpha > 0)$$

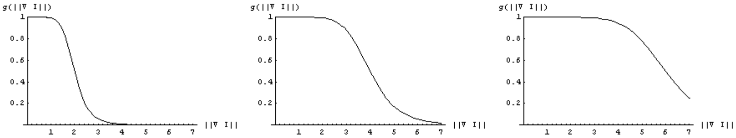


Figure: K = 2, 4, 6

Effect of varying α on $g()$

$$g(|\nabla I|) = \frac{1}{1 + (\frac{||\nabla I||}{K})^{1+\alpha}} \quad (\alpha > 0)$$

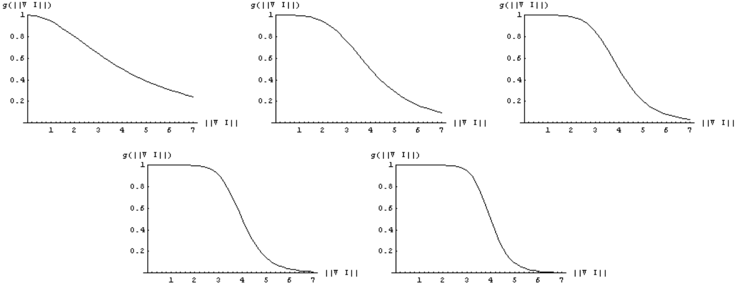
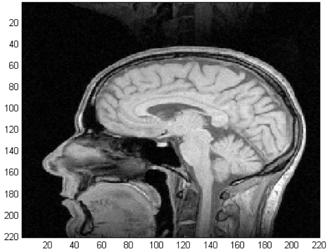


Figure: α = 1, 3, 5, 7, 9

Effect of varying K and α on $c()$



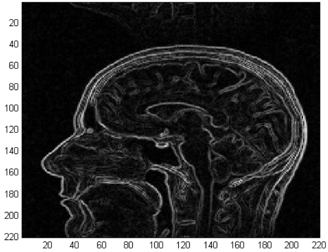


Figure: I and $||\nabla I||$.

Effect of varying K on $c()$

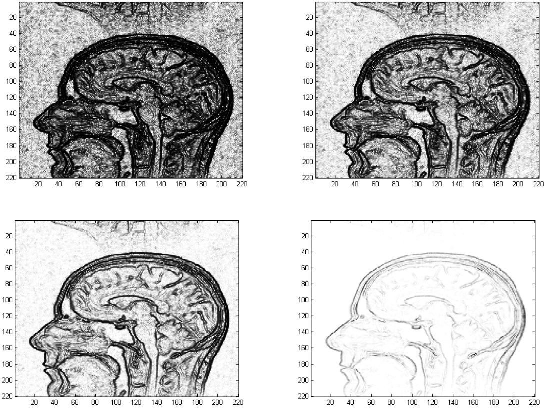


Figure: $K = 3, 5, 10, 100.$

Effect of varying α on $c()$

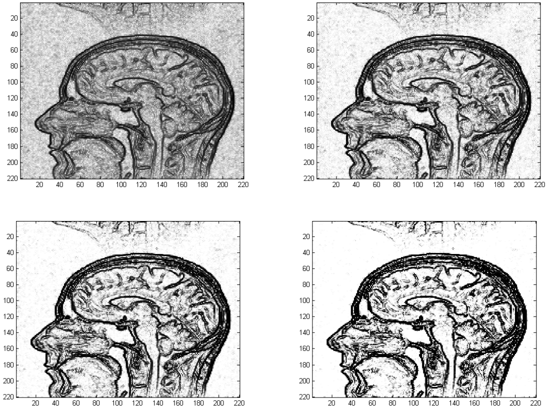


Figure: $\alpha = 1, 2, 3, 5.$

Outline

Explicit Formulation

$$\frac{\partial I}{\partial t} = c(x,y,t)\nabla^2 I + \nabla c \cdot \nabla I$$

Using centered differences for the Laplacian and gradients:

$$\begin{aligned} \frac{I_{x,y}^{t+1} - I_{x,y}^{t}}{\lambda} &= c_{x,y}(I_{x-1,y}^{t} + I_{x+1,y}^{t} + I_{x,y-1}^{t} + I_{x,y+1}^{t} - 4I_{x,y}^{t}) \\ &+ (\frac{c_{x+1,y} - c_{x-1,y}}{2})(\frac{I_{x+1,y} - I_{x-1,y}}{2}) \\ &+ (\frac{c_{x,y+1} - c_{x,y-1}}{2})(\frac{I_{x,y+1} - I_{x,y-1}}{2}) \end{aligned}$$

Implicit Formulation

- Same diagonal structure as homogeneous heat equation? Yes.
- Symmetric? No.
- Diagonal dominance? Data dependent.

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Explicit Formulation

$$\frac{\partial I}{\partial t} = c(x,y,t)\nabla^2 I + \nabla c \cdot \nabla I$$

By splitting the Laplacian and averaging the forward and backward differences in the gradient:

$$\begin{aligned}
\frac{I^{t+1}_{x,y} - I^{t}_{x,y}}{\lambda} &= c_{x,y}[(I^t_{x-1,y} - I^t_{x,y}) + (I^t_{x+1,y} - I^t_{x,y}) \\
&+ (I^t_{x,y-1} - I^t_{x,y}) + (I^t_{x,y+1} - I^t_{x,y})] \\
&+ \frac{\partial c}{\partial x}(\frac{I_{x+1,y} - I_{x,y}}{2} + \frac{I_{x,y} - I_{x-1,y}}{2}) \\
&+ \frac{\partial c}{\partial y}(\frac{I_{x,y+1} - I_{x,y}}{2} + \frac{I_{x,y} - I_{x,y-1}}{2})
\end{aligned}$$

Explicit Formulation

$$\frac{\partial I}{\partial t} = c(x, y, t)\nabla^2 I + \nabla c \cdot \nabla I$$

$$\begin{aligned}
\frac{I_{x,y}^{t+1} - I_{x,y}^{t}}{\lambda} &= (c_{x,y} - \frac{1}{2}\frac{\partial c}{\partial x})(I_{x-1,y}^{t} - I_{x,y}^{t}) \\
&+ (c_{x,y} + \frac{1}{2}\frac{\partial c}{\partial x})(I_{x+1,y}^{t} - I_{x,y}^{t}) \\
&+ (c_{x,y} - \frac{1}{2}\frac{\partial c}{\partial x})(I_{x,y-1}^{t} - I_{x,y}^{t}) \\
&+ (c_{x,y} + \frac{1}{2}\frac{\partial c}{\partial x})(I_{x,y+1}^{t} - I_{x,y}^{t})
\end{aligned}$$

Explicit Formulation

$$c_{x,y} + \frac{1}{2}\frac{\partial c}{\partial x} \approx c_{x+\frac{1}{2},y}$$

$$c_{x,y} - \frac{1}{2}\frac{\partial c}{\partial x} \approx c_{x-\frac{1}{2},y}$$

$$c_{x+\frac{1}{2},y} \approx g(\frac{s_{x,y} + s_{x+1,y}}{2})$$

$$c_{x-\frac{1}{2},y} \approx g(\frac{s_{x,y} + s_{x-1,y}}{2})$$

Where $s_{x,y} = ||\nabla I(x,y)||$.

Explicit Formulation

$$
\begin{aligned}
\frac{I^{t+1}_{x,y} - I^{t}_{x,y}}{\lambda} &= g\left(\frac{s_{x,y} + s_{x-1,y}}{2}\right)(I^{t}_{x-1,y} - I^{t}_{x,y}) \\
&+ g\left(\frac{s_{x,y} + s_{x+1,y}}{2}\right)(I^{t}_{x+1,y} - I^{t}_{x,y}) \\
&+ g\left(\frac{s_{x,y} + s_{x,y-1}}{2}\right)(I^{t}_{x,y-1} - I^{t}_{x,y}) \\
&+ g\left(\frac{s_{x,y} + s_{x,y+1}}{2}\right)(I^{t}_{x,y+1} - I^{t}_{x,y})
\end{aligned}
$$

Anisotropic Implementation

Compute $g()$ using the projection of the gradient along one direction.
For example, in $g(\frac{s_{x,y}+s_{x+1,y}}{2})$, let

$$\begin{aligned} s_{x,y} &= \left|\frac{\partial I}{\partial x}(x,y)\right| \\ s_{x+1,y} &= \left|\frac{\partial I}{\partial x}(x+1,y)\right| \end{aligned}$$

Computing $s_{x,y}$ using forward differences, and $s_{x+1,y}$ using backward differences

$$\begin{aligned} s_{x,y} &= |I_{x+1,y} - I_{x,y}| \\ s_{x+1,y} &= |I_{x+1,y} - I_{x,y}|, \end{aligned}$$

so $g(\frac{s_{x,y}+s_{x+1,y}}{2}) = g(|I(x+1,y) - I(x,y)|)$.

Explicit Formulation

$$\begin{aligned}\frac{I_{x,y}^{t+1}-I_{x,y}^{t}}{\lambda} &= g(|I_{x-1,y}-I_{x,y}|)(I_{x-1,y}^{t}-I_{x,y}^{t})\\ &+ g(|I_{x+1,y}-I_{x,y}|)(I_{x+1,y}^{t}-I_{x,y}^{t})\\ &+ g(|I_{x,y-1}-I_{x,y}|)(I_{x,y-1}^{t}-I_{x,y}^{t})\\ &+ g(|I_{x,y+1}-I_{x,y}|)(I_{x,y+1}^{t}-I_{x,y}^{t})\end{aligned}$$

Notation:

The authors use $\bigtriangledown$ to denote finite differences. This is not the gradient operator (∇).

Perona-Malik Implementation

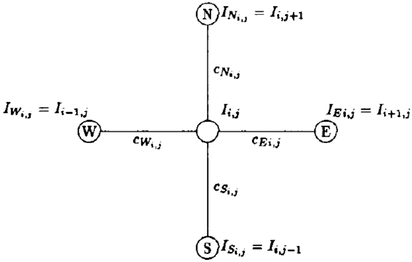

Image neighborhood system

$$
\begin{aligned}
\nabla_N I_{i,j} &\equiv I_{i-1,j} - I_{i,j}\\
\nabla_S I_{i,j} &\equiv I_{i+1,j} - I_{i,j}\\
\nabla_E I_{i,j} &\equiv I_{i,j+1} - I_{i,j}\\
\nabla_W I_{i,j} &\equiv I_{i,j-1} - I_{i,j}
\end{aligned}
$$

$$
\begin{aligned}
c_{N_{i,j}} &= g(|\nabla_N I_{i,j}|)\\
c_{S_{i,j}} &= g(|\nabla_S I_{i,j}|)\\
c_{E_{i,j}} &= g(|\nabla_E I_{i,j}|)\\
c_{W_{i,j}} &= g(|\nabla_W I_{i,j}|)
\end{aligned}
$$

Explicit Formulation

The previous explicit formulation

$$
\begin{aligned}
\frac{I^{t+1}_{x,y} - I^{t}_{x,y}}{\lambda} &= g(|I_{x-1,y} - I_{x,y}|)(I^{t}_{x-1,y} - I^{t}_{x,y}) \\
&+ g(|I_{x+1,y} - I_{x,y}|)(I^{t}_{x+1,y} - I^{t}_{x,y}) \\
&+ g(|I_{x,y-1} - I_{x,y}|)(I^{t}_{x,y-1} - I^{t}_{x,y}) \\
&+ g(|I_{x,y+1} - I_{x,y}|)(I^{t}_{x,y+1} - I^{t}_{x,y})
\end{aligned}
$$

can be rewritten as

$$
\begin{aligned}
I^{t+1}_{x,y} &= I^{t}_{x,y} + \lambda(c_{N_{i,j}} \bigtriangledown_N I_{i,j} + c_{S_{i,j}} \bigtriangledown_S I_{i,j} \\
&+ c_{E_{i,j}} \bigtriangledown_E I_{i,j} + c_{W_{i,j}} \bigtriangledown_W I_{i,j})^{t}
\end{aligned}
$$

Let $I_{M_{ij}} = \max(I, I_N, I_S, I_E, I_W)$

$$\begin{aligned} I_{x,y}^{t+1} &= I_{x,y}^{t} + \lambda(c_{N_{i,j}} \nabla_N I_{i,j} + c_{S_{i,j}} \nabla_S I_{i,j} \\ &+ c_{E_{i,j}} \nabla_E I_{i,j} + c_{W_{i,j}} \nabla_W I_{i,j}) \end{aligned}$$

$$\begin{aligned} I_{x,y}^{t+1} &= I_{x,y}^{t}(1 - \lambda(c_{N_{i,j}} + c_{S_{i,j}} + c_{E_{i,j}} + c_{W_{i,j}})) \\ &+ \lambda(c_{N_{i,j}} I_{N_{i,j}} + c_{S_{i,j}} I_{S_{i,j}} + c_{E_{i,j}} I_{E_{i,j}} + c_{W_{i,j}} I_{W_{i,j}}) \end{aligned}$$

Since all c's and λ are positive and between 0,1:

$$\begin{aligned} I_{x,y}^{t+1} &\leq I_{M_{ij}}^{t}(1 - \lambda(c_{N_{i,j}} + c_{S_{i,j}} + c_{E_{i,j}} + c_{W_{i,j}})) \\ &+ I_{M_{ij}}^{t}(\lambda(c_{N_{i,j}} + c_{S_{i,j}} + c_{E_{i,j}} + c_{W_{i,j}})) \\ &= I_{M_{ij}}^{t} \end{aligned}$$

Let $I_{M_{ij}} = \max(I, I_N, I_S, I_E, I_W)$

$$\begin{aligned} I_{x,y}^{t+1} &= I_{x,y}^{t} + \lambda(c_{N_{i,j}} \nabla_N I_{i,j} + c_{S_{i,j}} \nabla_S I_{i,j} \\ &+ c_{E_{i,j}} \nabla_E I_{i,j} + c_{W_{i,j}} \nabla_W I_{i,j}) \end{aligned}$$

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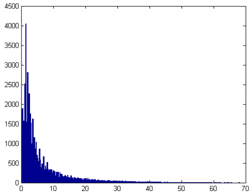
Set K every iteration

Compute a histogram, f_i, of $||\nabla I||$



Find K such that 90% of the pixels have gradient magnitude $< K$.
(If $\sum_{i=1}^{b} f_i \geq 0.9n^2$ then bin b corresponds to gradient magnitude K).

Outline

Edge Detection

Due to

- Edge enhancement
- Edge localization

edge detection algorithms will benefit from using this nonlinear diffusion process rather than using linear diffusion (Gaussian convolution).

Energy Minimization and Markov Random Field Models

- The ’anisotropic’ diffusion algorithm minimizes ’some energy function.’
- Smoothness $\rightarrow$ conditional dependence on nearest neighbors (Markovian property)

Outline

The problem with inhomogeneous diffusion

- Noise on edges will not be reduced.

Even with the anisotropic implementation of Perona-Malik, noise on diagonal edges will not be handled properly.

The problem with inhomogeneous diffusion

The inhomogeneous heat equation

$$\frac{\partial I}{\partial t} = \mathrm{div}(c(x,y,t)\nabla I)$$

only controls the magnitude of intensity (or heat) flow.
The anisotropic heat equation

$$\frac{\partial I}{\partial t} = \mathrm{div}(D(x,y,t)\nabla I),$$

where $D(x,y,t)$ is a matrix-valued function, can also control the direction of intensity (or heat) flow.

Next:

Wednesday:

- Susan Lemieux will continue the MRI acquisition lecture.

Friday:

- Weickert ”A review of nonlinear diffusion filtering”.
- The physical laws of heat flow and diffusion.
- Anisotropic diffusion filtering.