Medical Image Analysis

CS 593 / 791

Computer Science and Electrical Engineering Dept.
West Virginia University

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Outline

1. Discretizing the heat equation
2. Perona-Malik

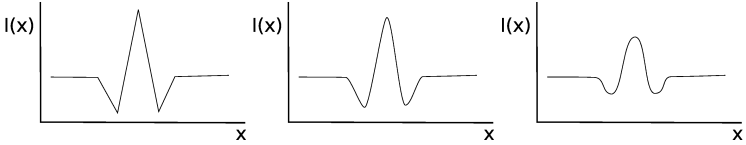
Outline

1. **Discretizing the heat equation**
   - Mapping the image to a vector
   - Boundary Conditions
   - Stability
2. **Perona-Malik**

Recall : The heat equation

In 1D

$$\frac{\partial I}{\partial t} = \frac{\partial^2 I}{\partial x^2}$$



Recall : Numerical Derivatives

First order forward difference:

$$I'(x_0) \approx I(x_0 + 1) - I(x_0)$$

First order backward difference:

$$I'(x_0) \approx I(x_0) - I(x_0 - 1)$$

Second order, second centered difference:

$$I''(x_0) \approx I(x_0 + 1) - 2I(x_0) + I(x_0 - 1)$$

Discretized 1D Heat Equation : Explicit

Using the forward difference in time we get

Explicit formulation

$$I_x^{t+\delta} = I_x^t + \delta(I_{x+1}^t - 2I_x^t + I_{x-1}^t)$$

Explicit : Update I^t using derivatives computed at time t.
Form a vector, **w** of image values, so that $\mathbf{w}_i = I(i)$

Discretized 1D Heat Equation : Explicit

We can rewrite the discretized heat equation as the system of linear equations:

$$w_i^{t+\delta} = [\delta, 1 - 2\delta, \delta] \begin{bmatrix} w_{i-1}^t \\ w_i^t \\ w_{i+1}^t \end{bmatrix}$$

This is equivalent to

$$w_i^{t+\delta} = [0, \delta, 1 - 2\delta, \delta, 0] \begin{bmatrix} w_{i-2}^t \\ w_{i-1}^t \\ w_i^t \\ w_{i+1}^t \\ w_{i+2}^t \end{bmatrix}$$

We can continue padding the row vector of coefficients with 0 entries until...

Discretized 1D Heat Equation : Explicit

$$\begin{aligned} w_i^{t+\delta} &= [0, \ldots, 0, \delta, 1-2\delta, \delta, 0, \ldots, 0]\mathbf{w}^t \\ &= \mathbf{a}_i \mathbf{w}^t \end{aligned}$$

Where $(1-2\delta)$ is in the i-th column, since it multiplies w_i^t.

We can write the whole system of equations by forming a matrix $\mathbf{A}$ whose i-th row is $\mathbf{a}_i$

$$\mathbf{w}^{t+1} = \mathbf{A}\mathbf{w}^t$$

$\mathbf{A}$ is a tridiagonal matrix.

Discretized 1D Heat Equation : Explicit

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$$\mathbf{w}^{t+1} = \mathbf{A}\mathbf{w}^t$$

A is a tridiagonal matrix.

Discretized 2D Heat Equation : Explicit

Recall the 2D heat equation

$$\frac{\partial I}{\partial t} = \frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2}$$

Using the forward difference in time we get

Explicit formulation

$$I_{x,y}^{t+\delta} = I_{x,y}^{t} + \delta(I_{x+1,y}^{t} - 4I_{x,y}^{t} + I_{x-1,y}^{t} + I_{x,y+1}^{t} + I_{x,y-1}^{t})$$

Update I^t using derivatives computed at time t.

Discretized 2D Heat Equation : Implicit

Recall the heat equation

$$\frac{\partial I}{\partial t} = \frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2}$$

Using the backward difference in time we get

Explicit formulation

$$I_{x,y}^{t+\delta} = I_{x,y}^{t} + \delta(I_{x+1,y}^{t+\delta} - 4I_{x,y}^{t+\delta} + I_{x-1,y}^{t+\delta} + I_{x,y+1}^{t+\delta} + I_{x,y-1}^{t+\delta})$$

Implicit : Update I^t using derivatives computed at time $t+\delta$.

2D image indices to 1D image index

Map 2d coordinates of $I(x, y)$

$(0,0)$	$(1,0)$	$(2,0)$
$(0,1)$	$(1,1)$	$(2,1)$
$(0,2)$	$(1,2)$	$(2,2)$

to 1d coordinates of w(i)

0	3	6
1	4	7
2	5	8

The coordinate transformation is given by

$$i(x, y) = nx + y$$

for an $n \times n$ image.

Writing central differences in 1D vector form

For the coordinate transformation function

$$i(x,y) = nx + y$$

- If $I(x,y) \to w(i)$, then $I(x,y+1) \to w(i+1)$, since $i(x,y+1) = nx + y + 1 = i(x,y) + 1$.
- If $I(x,y) \to w(i)$, then $I(x+1,y) \to w(i+n)$, since $i(x+1,y) = n(x+1) + y = i(x,y) + n$.

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Writing central differences in 1D vector form

For the coordinate transformation function

$$i(x,y) = nx + y$$

So,

$$\begin{aligned} \frac{\partial I^2}{\partial y^2}(x,y) &\approx I(x,y+1) - 2I(x,y) + I(x,y-1) \\ &\approx w(i+1) - 2w(i) + w(i-1) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial I^2}{\partial x^2}(x,y) &\approx I(x+1,y) - 2I(x,y) + I(x-1,y) \\ &\approx w(i+n) - 2w(i) + w(i-n) \end{aligned}$$

Writing central differences in 1D vector form

For the coordinate transformation function

$$i(x,y) = nx + y$$

So,

$$\begin{aligned}\frac{\partial I^2}{\partial y^2}(x,y) &\approx I(x,y+1) - 2I(x,y) + I(x,y-1)\\ &\approx w(i+1) - 2w(i) + w(i-1)\end{aligned}$$

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Writing difference equations in matrix form

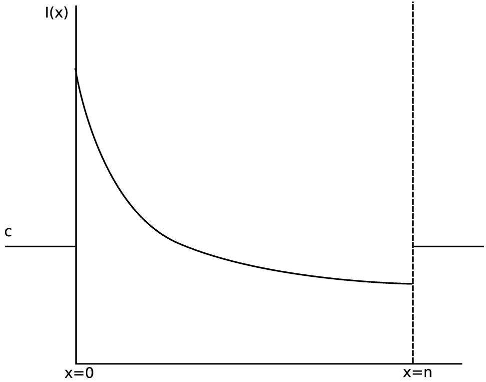
The implicit formulation of the heat equation involves solving n^2 simultaneous equations:

$$w_i^t = w_i^{t+\delta} + \delta(w_{i+1}^{t+\delta} - 4w_i^{t+\delta} + w_{i-1}^{t+\delta} + w_{i+n}^{t+\delta} + w_{i-n}^{t+\delta})$$

$$\begin{pmatrix} \vdots \\ \vdots \\ w_i^t \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \dots & \delta & \dots & \delta & 1-4\delta & \delta & \dots & \delta & \dots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ w_{i-n}^{t+\delta} \\ \vdots \\ w_{i-1}^{t+\delta} \\ w_i^{t+\delta} \\ w_{i+1}^{t+\delta} \\ \vdots \\ w_{i+n}^{t+\delta} \\ \vdots \end{pmatrix}$$

What to do when $w(i \pm 1)$ or $w(i \pm n)$ falls outside the image

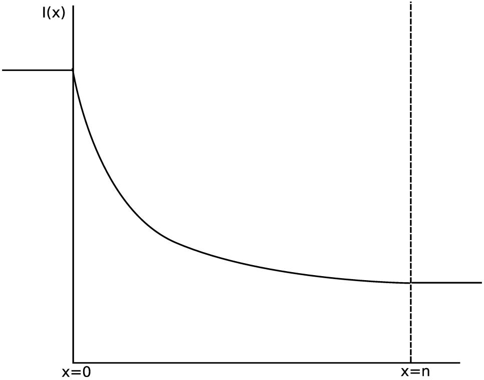
Constant Boundary Value



$(x < 0)$ or $(x > n) \rightarrow I(x) = c$
For $c = 0$:

$$I_{xx}(0) \approx -2I(0) + I(1)$$

Constant Boundary Slope



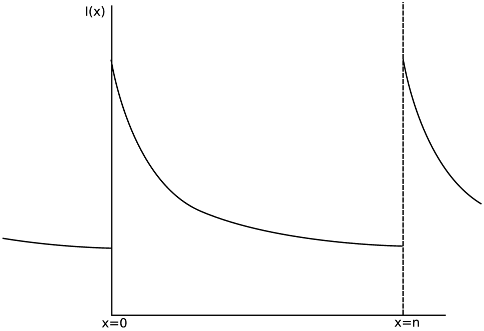
Fixing the slope at zero (adiabatic) gives

$(x < 0) \rightarrow I(x) = I(0)$

$(x > n) \rightarrow I(x) = I(n)$

$$I_{xx}(0) \approx -I(0) + I(1)$$

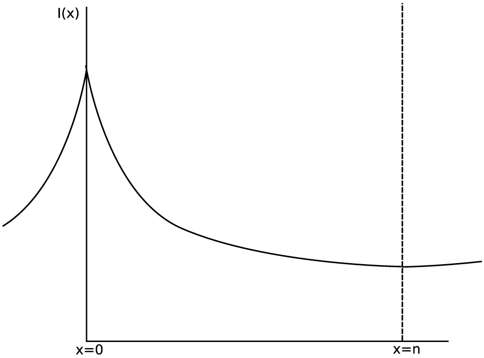
Periodic Boundary Conditions



$$(x < 0) \rightarrow I(x) = I(x+n)$$
$$(x > n) \rightarrow I(x) = I(x-n)$$

$$I_{xx}(0) \approx I(n-1) - 2I(0) + I(1)$$

Reflective Boundary Conditions



$$(x < 0) \to I(x) = I(-x)$$
$$(x > n) \to I(x) = I(2n - x)$$

$$I_{xx}(0) \approx -2I(0) + 2I(1)$$

Stability of the explicit 1D heat equation

The 1D heat equation, $I_t = I_{xx}$, has solution $I(x,t) = e^{-t}\cos(x)$. This corresponds to the problem with initial condition $I(x,0) = \cos(x)$.

Discretize only in time (forward)

Observe that $I_{xx}(x,t) = -e^{-t}\cos(x) = -I(x,t)$

$$\frac{I^{t+\delta} - I^t}{\delta} = -I^t$$

$$I^{t+\delta} = I^t - \delta I^t$$

Convergence criterion : ratio test

The sequence I^t is convergent if

$$\lim_{t\to\infty}\left|\frac{I^{t+\delta}}{I^t}\right| < 1$$

The explicit equation we formed earlier

$$I^{t+\delta} = I^t - \delta I^t$$

has convergence criterion

$$\left|\frac{I^{t+\delta}}{I^t}\right| = |1-\delta| < 1$$

This is satisfied for $0 < \delta < 2$. (Only conditionally convergent.)

Stability of the implicit 1D heat equation

Discretize only in time (backward)

$$\frac{I^{t+\delta} - I^t}{\delta} = -I^{t+\delta}$$

$$I^{t+\delta} = I^t - \delta I^{t+\delta}$$

The implicit equation has convergence criterion

$$\left|\frac{I^{t+\delta}}{I^t}\right| = \left|\frac{1}{1+\delta}\right| < 1$$

This is satisfied for $\delta > 0$.

Stability

In general, it can be shown that

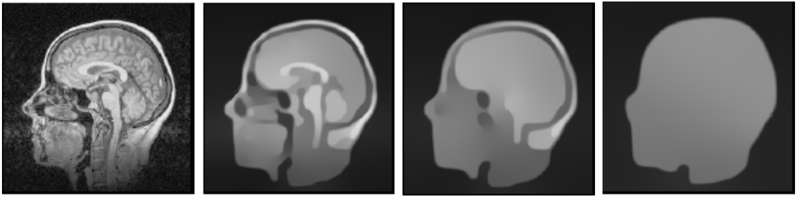
- Explicit methods are conditionally stable.
- Implicit methods are unconditionally stable.

Outline

1. **Discretizing the heat equation**
2. **Perona-Malik**
   - Introduction
   - Weaknesses of the standard scale-space paradigm
   - Inhomogeneous diffusion
   - Properties of inhomogeneous diffusion
   - Next Class

Scale-space

The need for multiscale image representations: Details in images should only exist over certain ranges of scale.



Scale-space

Definition: a family of images, $I(x, y, t)$, where

- The scale-space parameter is t.
- $I(x, y, 0)$ is the original image.
- Increasing t corresponds to coarser resolutions.

$I(x, y, t)$ can be generated by convolving with wider Gaussian kernels as t increases, or equivalently, by solving the heat equation.

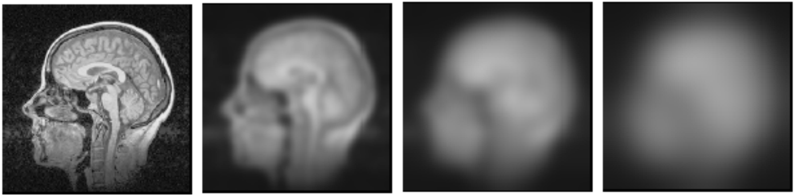
Earlier Scale-space properties

- Causality: coarse details are ”caused” by fine details.
- New details should not arise in coarse scale images.
- Smoothing should be homogeneous and isotropic.

This paper will challenge the last property, and propose a more useful scale-space definition.
The new scale-space will be shown to obey the causality property.

Lost Edge Information

- Edge location is not preserved across the scale space.
- Edge crossings may disappear.
- Region boundaries are blurred.



Gaussian blurring is a local averaging operation. It does not respect natural boundaries.

Linear Scale Space

Def: Scale spaces generated by a linear filtering operation.

- Nonlinear filters, such as the median filter, can be used to generate scale-space images.
- Many nonlinear filters violate one of the scale-space conditions.

New Criteria

- Causality.
- Immediate localization : edge locations remain fixed.
- Piecewise Smoothing : permit discontinuities at boundaries.

At all scales the image will consist of smooth **regions** separated by **boundaries** (edges).

Diffusion equation

$$\frac{\partial I}{\partial t} = \mathrm{div}(c(x,y,t)\nabla I)$$

The diffusion coefficient, $c(x,y,t)$ controls the degree of smoothing at each point in I.

The basic idea:

Setting $c(x,y,t) = 0$ at region boundaries, and $c(x,y,t) = 1$ at region interior will encourage intraregion smoothing, and discourage interregion smoothing.

Diffusion equation

By the chain rule:

$$\begin{aligned}\frac{\partial I}{\partial t} &= \operatorname{div}\begin{pmatrix} c(x,y,t)\frac{\partial I}{\partial x} \\ c(x,y,t)\frac{\partial I}{\partial y}\end{pmatrix} \\ &= \frac{\partial c}{\partial x}\frac{\partial I}{\partial x} + c(x,y,t)\frac{\partial^2 I}{\partial x^2} + \frac{\partial c}{\partial y}\frac{\partial I}{\partial y} + c(x,y,t)\frac{\partial^2 I}{\partial y^2} \\ &= c(x,y,t)\nabla^2 I + \nabla c \cdot \nabla I\end{aligned}$$

Notation

The paper uses the symbol Δ to represent the Laplacian.
$\Delta I = \nabla^2 I = \operatorname{div}(\nabla I)$

Conduction coefficient

What properties would he like $c(x, y, t)$ to have?

- c = 1 at interior of a region.
- c = 0 at boundary of a region.
- c should be nonnegative everywhere.

Since $c(x, y, t)$ depends on edge information, we need an edge descriptor, $E(x, y, t)$, to compute c.

Notation

When written as a function of the edge descriptor, the authors use the symbol g() for conduction coefficient.

Edge Estimate (or Edge Descriptor)

$E(x, y, t)$ should convey the following information:

- Location.
- Magnitude (contrast across edge).
- Direction.

and obey the following properties:

- $E(x, y, t) = \mathbf{0}$ at region interior.
- $E(x, y, t) = K\mathbf{e}(x, y, t)$ at region boundaries.

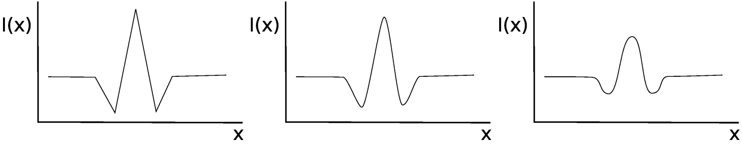
K is the contrast, $\mathbf{e}(x, y, t)$ is perpendicular to the edge.
$\nabla I(x, y, t)$ has these properties, and is a useful edge estimator.

Maximum Principle

- The maximum and minimum intensities in the scale-space image $I(x, y, t)$ occur at $t = 0$ (the finest scale image).
- Since new maxima and minima correspond to new image features, the causality requirement of scale-space can satisfied if the evolution equation obeys the maximum principle.
- We will make some less rigorous observations concerning causality...

Maximum Principle

For the 1D heat equation : $I_t = I_{xx}$.



- Solving the heat equation is equivalent to convolution.
- Convolution is a local averaging operation.
- Averaging is bounded by the values being averaged.

Maximum Principle

For the Perona-Malik equation

$$\frac{\partial I}{\partial t} = c(x, y, t)\nabla^2 I + \nabla c \cdot \nabla I$$

Note that at local minima $\nabla I = \mathbf{0}$ and we are evolving by the original heat equation.

It can be shown that this general class of PDEs obeys the maximum principle.

We will also inspect a maximum principle for the discretized equations.

Edge Enhancement

Inhomogeneous diffusion may actually enhance edges, for a certain choice of $c(x, y, t)$.

1D example:

Let $s(x) = \frac{\partial I}{\partial x}$, and $\phi(s) = g(s)s = g(I_x)I_x$.

The 1D heat equation becomes

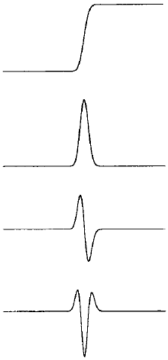
$$\begin{aligned} I_t = \frac{\partial}{\partial x}(g(I_x)I_x) &= \frac{\partial}{\partial x}\phi(s) \\ \text{by chain rule} &= \frac{\partial \phi}{\partial s}\frac{\partial s}{\partial x} \\ &= \phi'(s)I_{xx} \end{aligned}$$

Edge Enhancement

We are interested in the rate of change of edge slope with respect to time.

$$
\begin{aligned}
\frac{\partial}{\partial t}(I_x) &= \frac{\partial}{\partial x}(I_t) \quad \text{if I is smooth} \\
&= \frac{\partial}{\partial x}(\phi'(s) I_{xx}) \\
&= \phi''(s) I_{xx}^2 + \phi'(s) I_{xxx}
\end{aligned}
$$

Edge Enhancement



I, I_x, I_{xx}, I_{xxx}

$$\frac{\partial}{\partial t}(I_x) = \phi''(s) I_{xx}^2 + \phi'(s) I_{xxx}$$

For a step edge with $I_x > 0$ look at the inflection point, p.
Observe that $I_{xx}(p) = 0$, and $I_{xxx}(p) < 0$.

$$\frac{\partial}{\partial t}(I_x)(p) = \phi'(s) I_{xxx}(p)$$

The sign of this quantity depends only on $\phi'(s)$.

Edge Enhancement

At the inflection point:

$$\frac{\partial}{\partial t}(I_x)(p) = \phi'(s) I_{xxx}(p)$$

- If $\phi'(s) > 0$, then $\frac{\partial}{\partial t}(I_x)(p) < 0$ (slope is decreasing).
- If $\phi'(s) < 0$, then $\frac{\partial}{\partial t}(I_x)(p) > 0$ (slope is increasing).

Since $\phi(s) = g(s)s$, selecting the function $g(s)$ determines which edges of smoothed and which are sharpened.

The function g()

Perona and Malik suggest two possible functions

$$g(|\nabla I|) = e^{-(\frac{||\nabla I||}{K})^2}$$

$$g(|\nabla I|) = \frac{1}{1 + (\frac{||\nabla I||}{K})^{1+\alpha}} \quad (\alpha > 0)$$

Next Class

We will continue to discuss the Perona-Malik paper, looking at parameter setting and implementation details.