Spherical Harmonic Function

Advanced Image Processing

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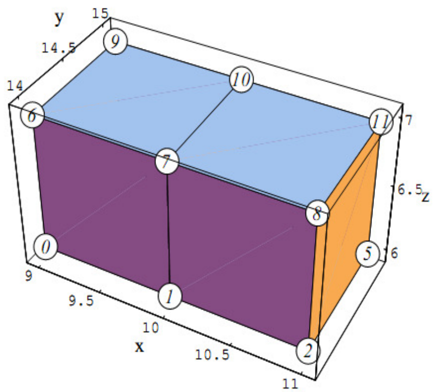
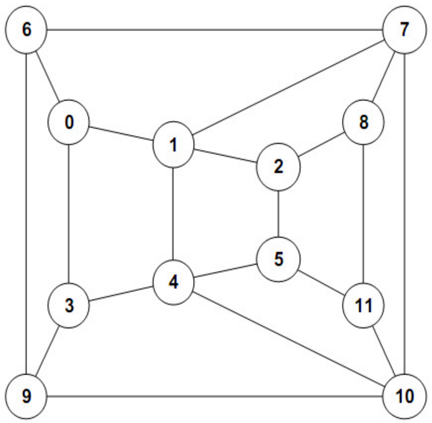
Slides from Dr.Guido Gerig

Introduction

- 3D extension to the 2D case. (pros, cons?) eg. Hand, fist
- Can not simply use chain code as 2D case, why?
- Project all the vertexes to a unit ball, why? (no necessary in 2D)
- Normalize the area (similar to 2D case,)

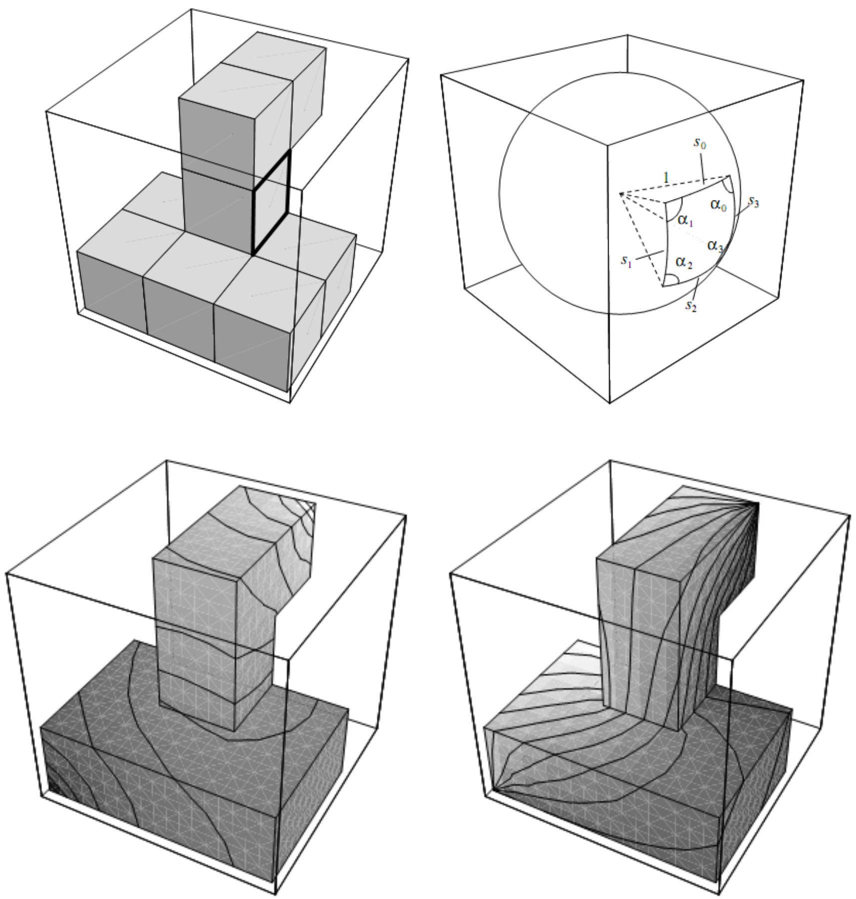
The surface data structure





node nr.	x_0 x_1 x_2	neighbors
0	{{{ 9, 14, 6},	{ 1, 7, 6, 9, 3, 4}},
1	{{ 10, 14, 6},	{ 0, 3, 4, 5, 2, 8, 7, 6}},
2	{{ 11, 14, 6},	{ 1, 4, 5, 11, 8, 7}},
3	{{ 9, 15, 6},	{ 4, 1, 0, 6, 9, 10}},
4	{{ 10, 15, 6},	{ 3, 9, 10, 11, 5, 2, 1, 0}},
5	{{ 11, 15, 6},	{ 4, 10, 11, 8, 2, 1}},
6	{{ 9, 14, 7},	{ 7, 10, 9, 3, 0, 1}},
7	{{ 10, 14, 7},	{ 6, 0, 1, 2, 8, 11, 10, 9}},
8	{{ 11, 14, 7},	{ 7, 1, 2, 5, 11, 10}},
9	{{ 9, 15, 7},	{ 10, 4, 3, 0, 6, 7}},
10	{{ 10, 15, 7},	{ 9, 6, 7, 8, 11, 5, 4, 3}},
11	{{ 11, 15, 7},	{ 10, 7, 8, 2, 5, 4}}}

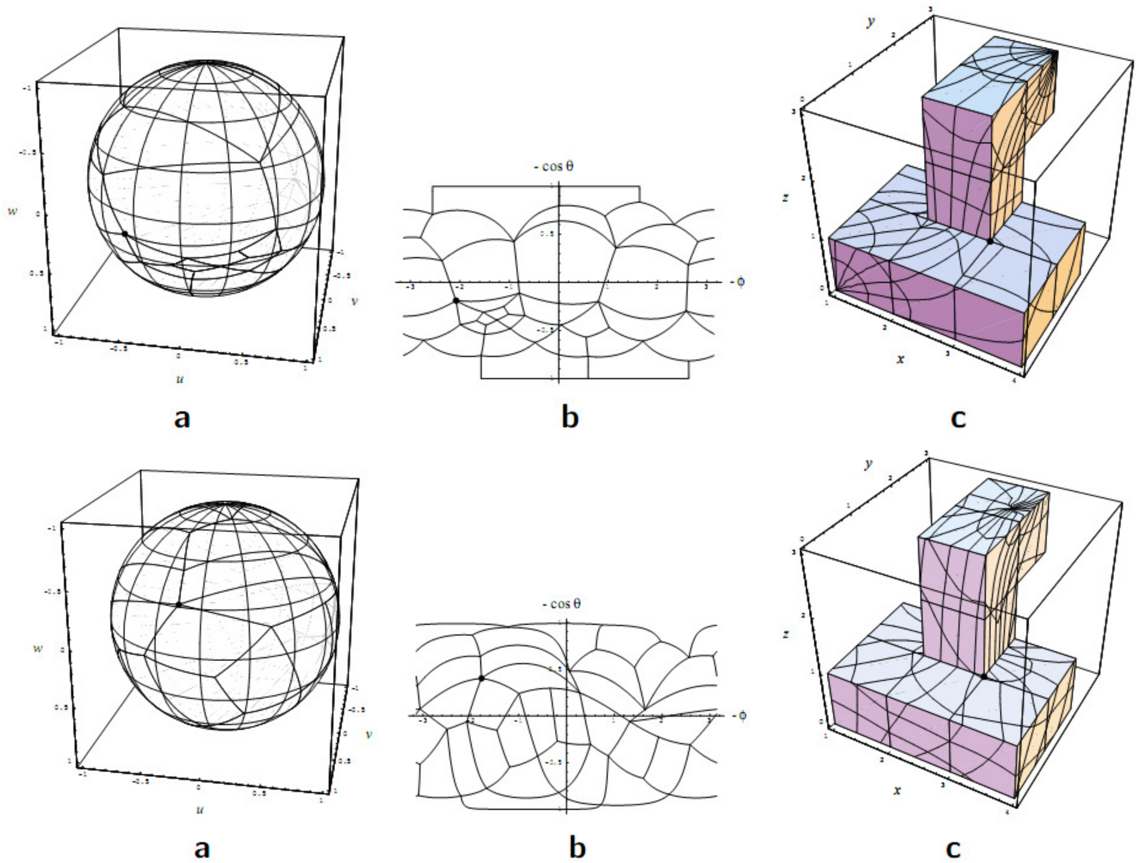
Parameterization of closed surface



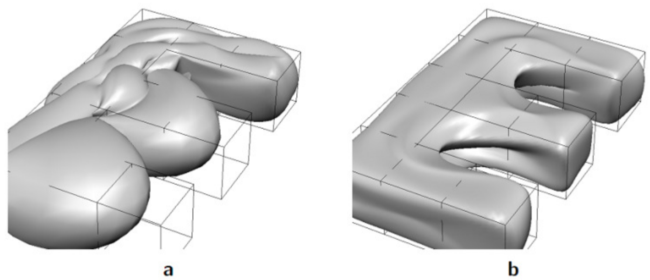
$$\begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \sin\theta\ \cos\phi \\ \sin\theta\ \sin\phi \\ \cos\theta \end{pmatrix}$$

- Starting point
  - North pole: lower left
  - South pole: upper right
- Assign latitude and longitude

Optimization



- Newton-Lagrange



Parameterization by spherical harmonics basis functions

$$\boldsymbol{v}(\theta,\phi) = \begin{pmatrix} x(\theta,\phi) \\ y(\theta,\phi) \\ z(\theta,\phi) \end{pmatrix} \qquad \boldsymbol{v}(\theta,\phi,\boldsymbol{p}) = \sum_{k=0}^{K}\sum_{m=-k}^{k} \boldsymbol{c}_k^m \, Y_k^m(\theta,\phi)$$

$$\boldsymbol{c}_k^m = \begin{pmatrix} c_{x\,k}^{\,m} \\ c_{y\,k}^{\,m} \\ c_{z\,k}^{\,m} \end{pmatrix} \qquad \boldsymbol{p} = (c_{x\,0}^{\,0}, c_{y\,0}^{\,0}, c_{z\,0}^{\,0}, c_{x\,1}^{\,-1}, c_{x\,1}^{\,0}, c_{x\,1}^{\,1}, c_{y\,1}^{\,-1}, c_{y\,1}^{\,0}, c_{y\,1}^{\,1}, c_{z\,1}^{\,-1}, c_{z\,1}^{\,0}, c_{z\,1}^{\,1}, \ldots c_{x\,K}^{\,-K} \ldots c_{z\,K}^{\,K})^\top$$

Legendre polynomials

$$P_l(w) = \frac{1}{2^l l!} \frac{d^l}{dw^l} (w^2-1)^l$$

Parameterization by spherical harmonics basis functions

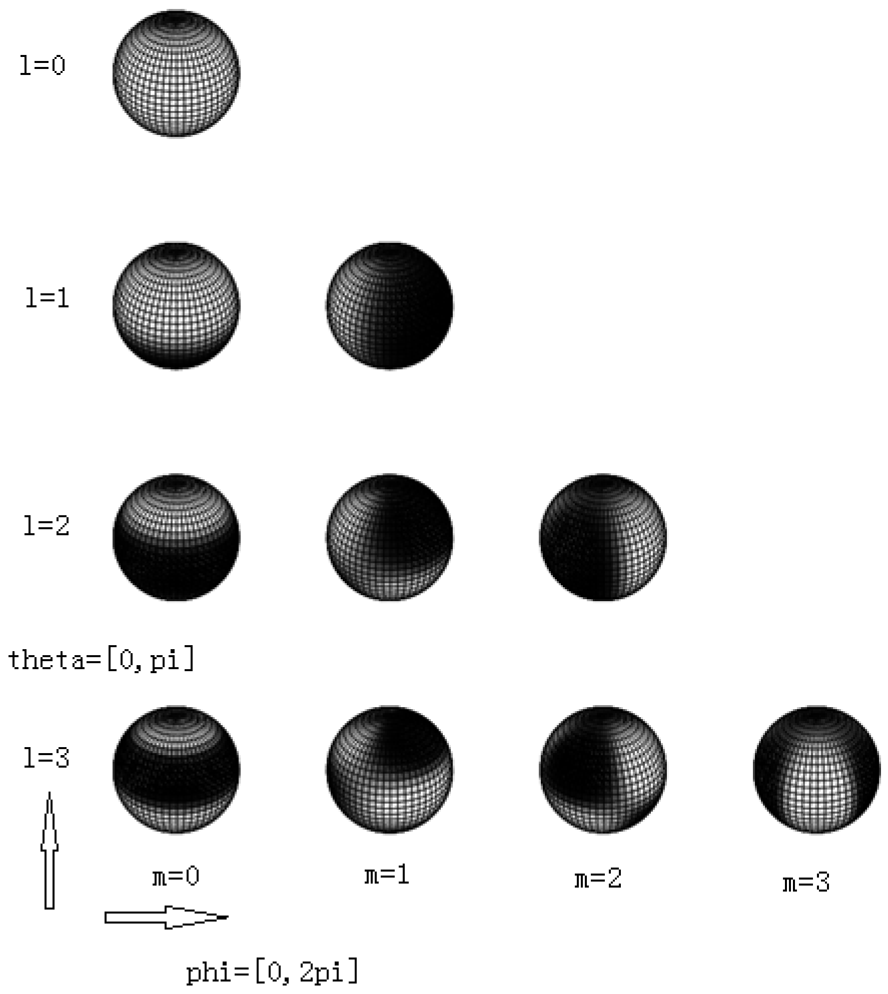
Associated Legendre polynomials

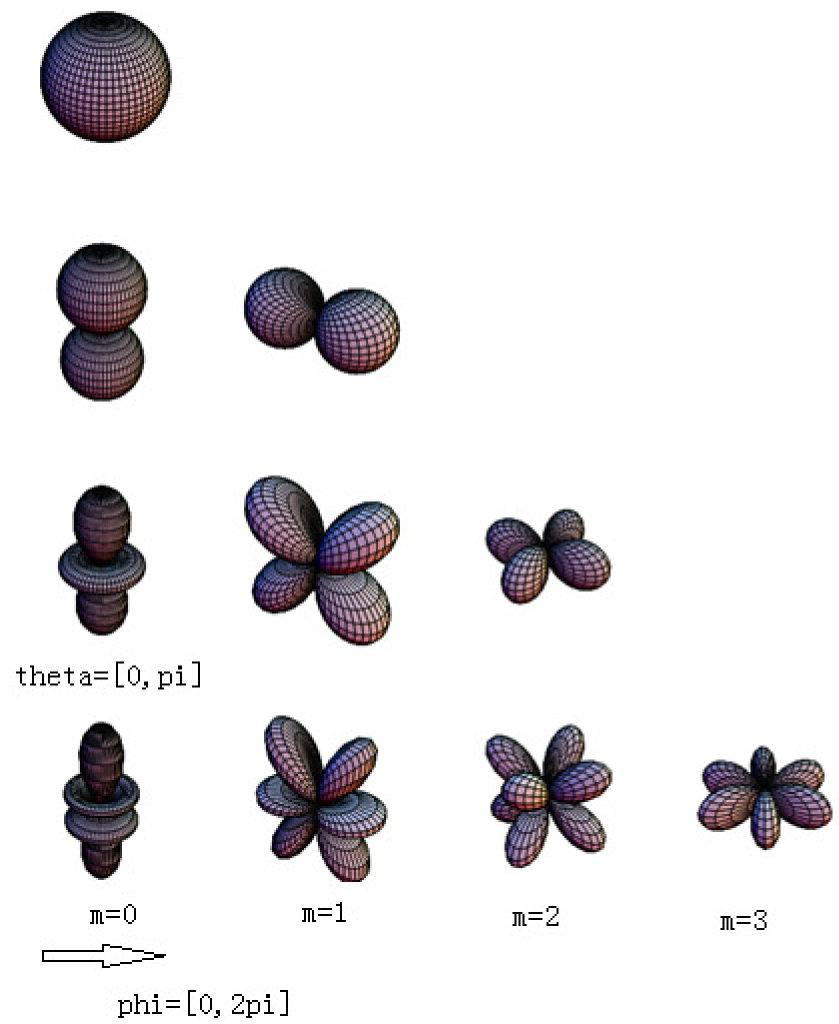
$$P_l^m(w) = (-1)^m (1-w^2)^{\frac{m}{2}} \frac{d^m}{dw^m} P_l(w)$$
$$= \frac{(-1)^m}{2^l l!} (1-w^2)^{\frac{m}{2}} \frac{d^{m+l}}{dw^{m+l}} (w^2-1)^l$$

Spherical harmonic functions

$$Y_l^m(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} \; P_l^m(\cos\theta) \; e^{im\phi}$$
$$Y_l^{-m}(\theta, \phi) = (-1)^m Y_l^{m*}(\theta, \phi)$$

Y_l^m	$m=0$	$m=1$	$m=2$
$l=0$	$\frac{1}{2\sqrt{\pi}}$		
$l=1$	$\sqrt{\frac{3}{4\pi}} \cos(\theta)$	$-\sqrt{\frac{3}{8\pi}} e^{i\phi} \sin(\theta)$	
$l=2$	$\sqrt{\frac{5}{16\pi}} \left(-1+3\cos(\theta)^2\right)$	$-\sqrt{\frac{15}{8\pi}} e^{i\phi} \cos(\theta) \sin(\theta)$	$\sqrt{\frac{15}{2\pi}} e^{2i\phi} \sin(\theta)^2$


l=0
l=1
l=2
theta=[0, pi]
l=3
m=0
m=1
m=2
m=3
phi=[0, 2pi]


theta=[0, pi]
m=0
m=1
m=2
m=3
phi=[0, 2pi]

Invariant descriptors

- Rotation independent descriptors

$$\boldsymbol{v}_1(\theta, \phi, \boldsymbol{p}) = \sum_{m=-1}^{1} \boldsymbol{c}_1^m \, Y_1^m(\theta, \phi)$$

Substituting the basis functions $Y_1^{-1} = \frac{\sqrt{3}}{2\sqrt{2\pi}}(u_0 - iu_1)$, $Y_1^0 = \frac{\sqrt{3}}{2\sqrt{\pi}}u_2$ and $Y_1^1 = -\frac{\sqrt{3}}{2\sqrt{2\pi}}(u_0 + iu_1)$ this sum can be written as

$$\boldsymbol{v}_1(\boldsymbol{u}) = \boldsymbol{A}\boldsymbol{u} = \boldsymbol{A}\begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} = \boldsymbol{a}_1 \, u_0 + \boldsymbol{a}_2 \, u_1 + \boldsymbol{a}_3 \, u_2$$

$$\boldsymbol{A} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) = \frac{\sqrt{3}}{2\sqrt{2\pi}}\left(\boldsymbol{c}_1^{-1} - \boldsymbol{c}_1^1, i(\boldsymbol{c}_1^{-1} + \boldsymbol{c}_1^1), \sqrt{2}c_1^0\right)$$

$$\boldsymbol{u}_i \mid^V = \boldsymbol{R}_u^T veu_i \qquad \boldsymbol{c}_l^m \mid^R = \boldsymbol{R}_x \; \boldsymbol{c}_l^m \mid^V$$

where the rotation matrix is defined as $\boldsymbol{R}_x = \text{diagonal}(\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_2})\boldsymbol{A}^T\boldsymbol{R}_u^T$.

Invariant descriptors

Scale independence

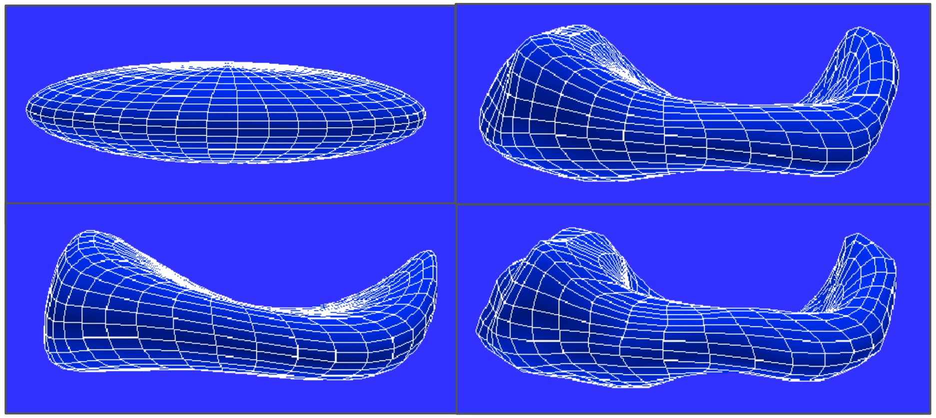
Scaling invariance can be achieved by dividing all descriptors by l_1, the length of the longest main axis

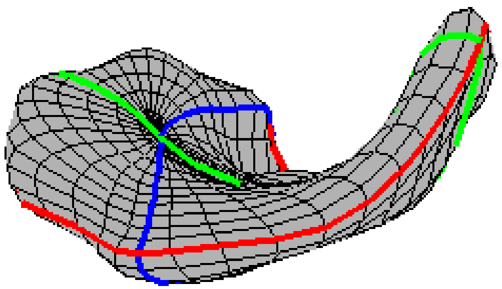
$$\boldsymbol{c}_l^m \mid^S = \boldsymbol{R}_x \; \boldsymbol{c}_l^m \mid^R , \tag{4.34}$$

Invariant spherical harmonic descriptors

Ignoring the coefficients of degree $l = 0$, that is setting $\boldsymbol{c}_0^0 \mid^T = (0, 0, 0)^T$ achieves translation invariance.

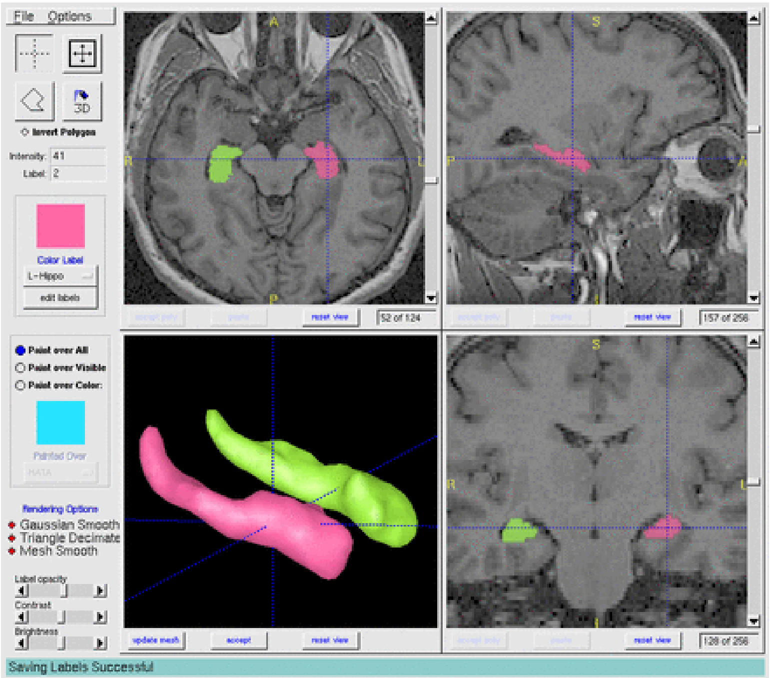
Parameterization with spherical harmonics

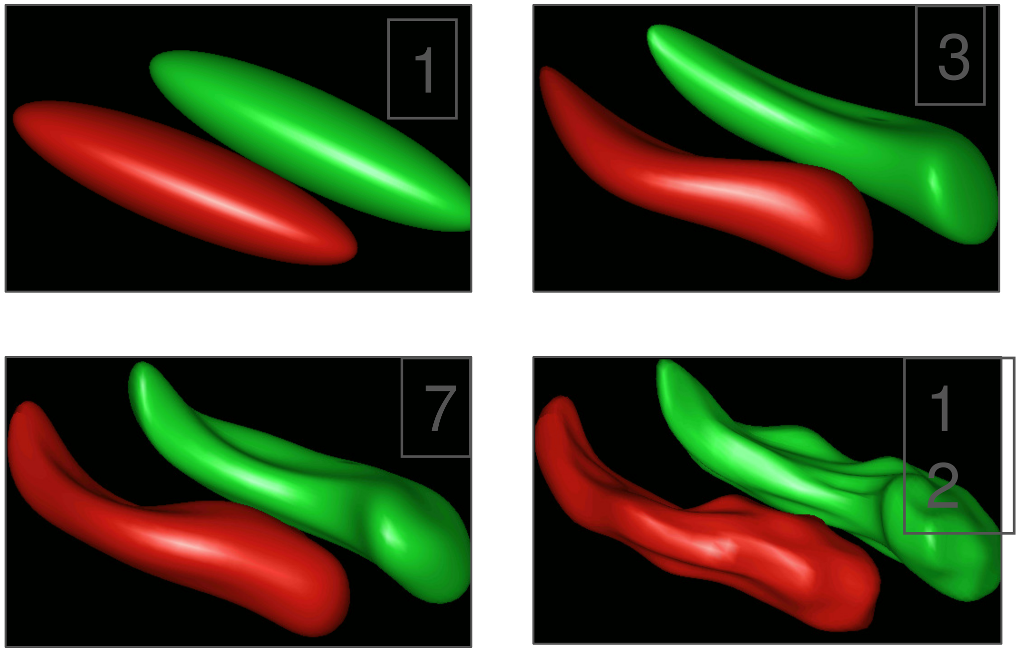




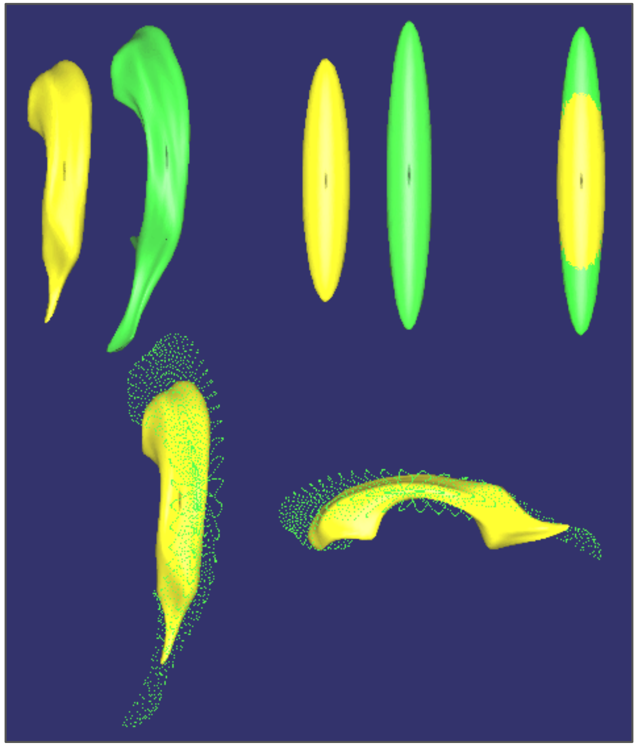
- Surface Parameterization & Expansion into spherical harmonics.
- Normalization of surface mesh (alignment to first ellipsoid).
- Correspondence: Homology of 3D mesh points.

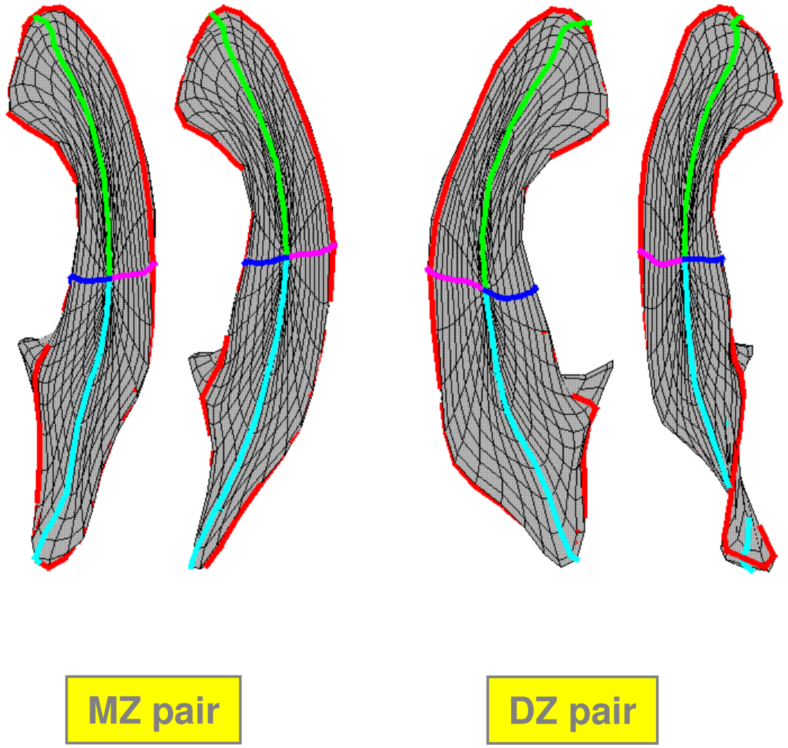
Parameterization with spherical harmonics





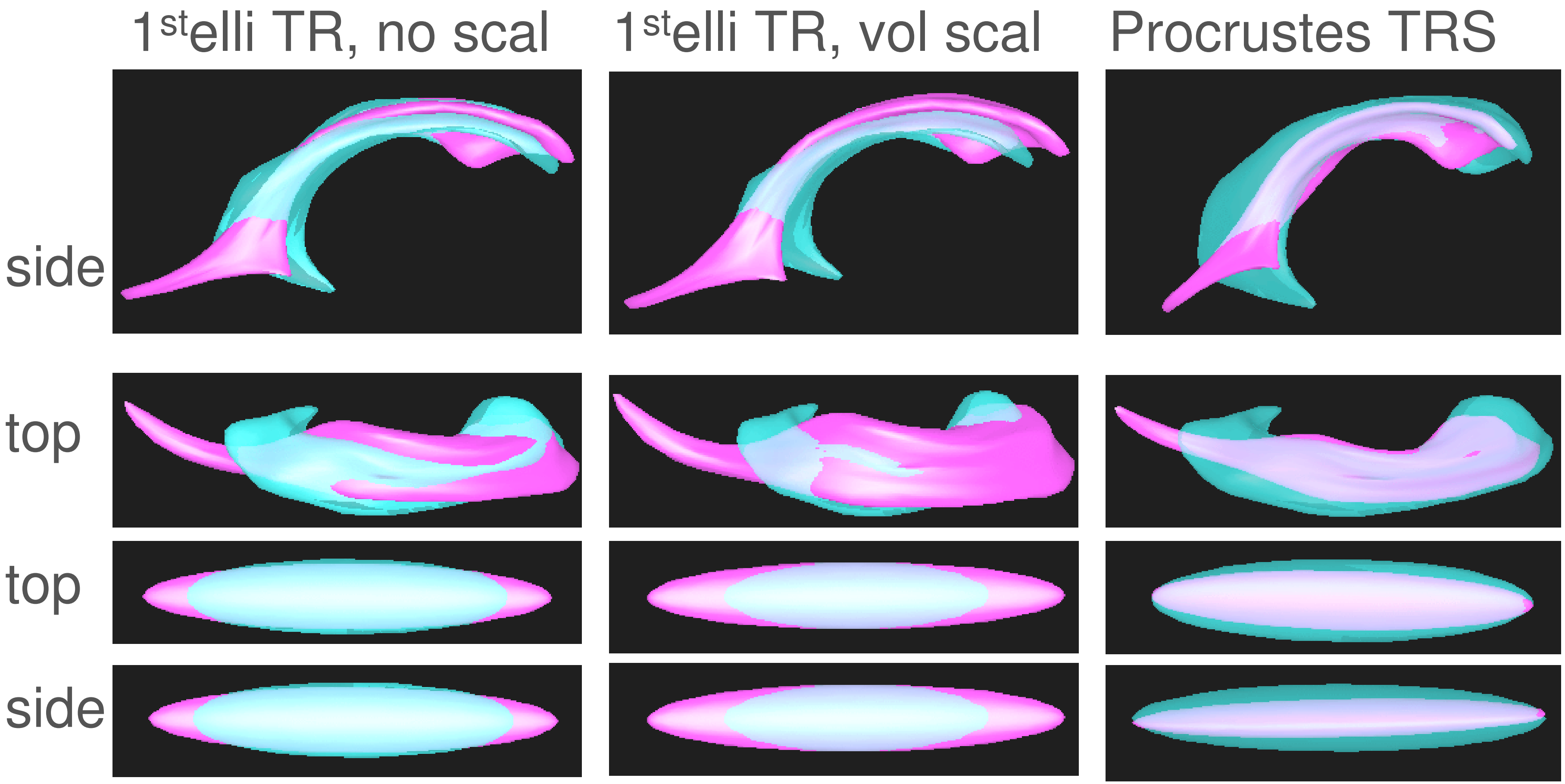
Object Alignment / Surface Homology



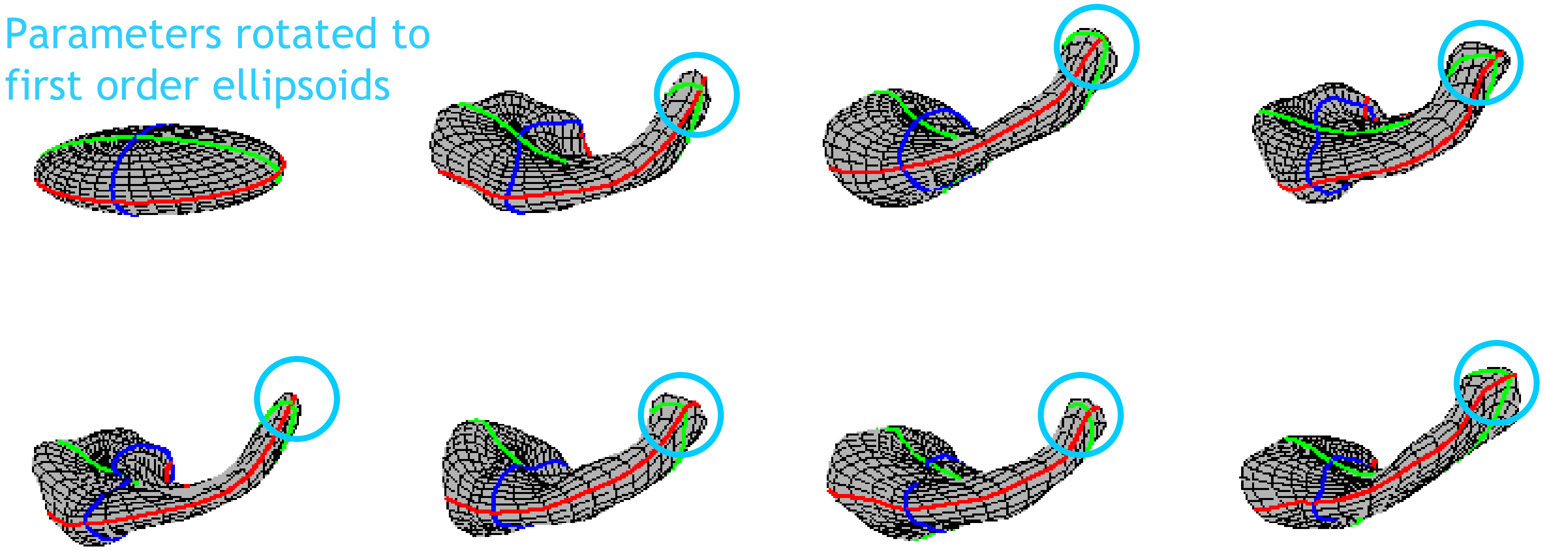


Surface Correspondence

Object Alignment prior to Shape Analysis



Correspondence through parameter space rotation



- Normalization using first order ellipsoid:
- Rotation of parameter space to align major axis
- Spatial alignment to major axes

Reference

- [1]. Parametrization of closed surfaces for 3-D shape description, Ch. Brechbuhler, G.Gerig and O.Kubler, Communication Technology Laboratory Image Science, ETH, March 29, 1996
- [2]. Description and analysis of 3-D shapes by parametrization of closed surfaces, Christian Michael Brechbuhler-Miskuv, Ph.D thesis,Diss. ETH No. 10979
- [3] Elastic Model-Based Segmentation of 2-D and 3-D Neuroradiological Data Sets, Andras Kelemen, Ph.D thesis
- [4] Slides from Dr. Guido Gerig