Spherical Harmonic Function

Advanced Image Processing
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Slides from Dr. Guido Gerig
Introduction

- 3D extension to the 2D case. (pros, cons? ) eg. Hand, fist
- Can not simply use chain code as 2D case, why?
- Project all the vertexes to a unit ball, why? (no necessary in 2D)
- Normalize the area (similar to 2D case, )
The surface data structure

<table>
<thead>
<tr>
<th>nr.</th>
<th>node</th>
<th>$x_0$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>neighbors</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>{9, 14, 6}</td>
<td>{1, 7, 6, 9, 3, 4}</td>
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<td></td>
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<tr>
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<td>{11, 14, 6}</td>
<td>{1, 4, 5, 11, 8, 7}</td>
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<td>3</td>
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<td>{4, 1, 0, 6, 9, 10}</td>
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<tr>
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<td>4</td>
<td>{10, 15, 6}</td>
<td>{3, 9, 10, 11, 5, 2, 1, 0}</td>
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<tr>
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<td>{4, 10, 11, 8, 2, 1}</td>
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<tr>
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<td>6</td>
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<td>{7, 10, 9, 3, 0, 1}</td>
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<td>{11, 15, 7}</td>
<td>{10, 7, 8, 2, 5, 4}</td>
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<td></td>
</tr>
</tbody>
</table>
Parameterization of closed surface

\[
\begin{pmatrix}
  u_0 \\
  u_1 \\
  u_2
\end{pmatrix} =
\begin{pmatrix}
  \sin \theta \cos \phi \\
  \sin \theta \sin \phi \\
  \cos \theta
\end{pmatrix}
\]

- Starting point
  - North pole: lower left
  - South pole: upper right
- Assign latitude and longitude
Optimization

- Newton-Lagrange
Parameterization by spherical harmonics basis functions

\[ \mathbf{v}(\theta, \phi) = \begin{pmatrix} x(\theta, \phi) \\ y(\theta, \phi) \\ z(\theta, \phi) \end{pmatrix} \]

\[ \mathbf{v}(\theta, \phi, \mathbf{p}) = \sum_{k=0}^{K} \sum_{m=-k}^{k} \mathbf{c}_k^m Y_k^m(\theta, \phi) \]

\[ \mathbf{c}_k^m = \begin{pmatrix} c_{x_k^m} \\ c_{y_k^m} \\ c_{z_k^m} \end{pmatrix} \]

\[ \mathbf{p} = \begin{pmatrix} c_{x_0^0}, c_{y_0^0}, c_{z_0^0}, c_{x_1^{-1}}, c_{x_1^0}, c_{x_1^1}, c_{y_1^{-1}}, c_{y_1^0}, c_{y_1^1}, c_{z_1^{-1}}, c_{z_1^0}, c_{z_1^1}, \ldots, c_{x_K^{-K}}, \ldots, c_{z_K^{-K}} \end{pmatrix}^T \]

Legendre polynomials

\[ P_l(w) = \frac{1}{2^l l!} \frac{d^l}{dw^l} (w^2 - 1)^l \]
Parameterization by spherical harmonics basis functions

Associated Legendre polynomials

\[ P_l^m(w) = (-1)^m (1 - w^2)^{m/2} \frac{d^m}{dw^m} P_l(w) \]
\[ = \frac{(-1)^m}{2^l l!} (1 - w^2)^{m/2} \frac{d^{m+l}}{dw^{m+l}} (w^2 - 1)^l \]

Spherical harmonic functions

\[ Y_l^m(\theta, \phi) = \sqrt{\frac{2l + 1}{4\pi} \frac{(l - m)!}{(l + m)!}} \ P_l^m(\cos \theta) \ e^{im\phi} \]
\[ Y_l^{-m}(\theta, \phi) = (-1)^m Y_l^{m*}(\theta, \phi) \]

<table>
<thead>
<tr>
<th>( Y_l^m )</th>
<th>( m = 0 )</th>
<th>( m = 1 )</th>
<th>( m = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l = 0 )</td>
<td>( \frac{1}{2\sqrt{\pi}} )</td>
<td>( )</td>
<td>( ) )</td>
</tr>
<tr>
<td>( l = 1 )</td>
<td>( \sqrt{\frac{3}{4\pi}} \cos(\theta) )</td>
<td>( -\sqrt{\frac{3}{8\pi}} e^{i\phi} \sin(\theta) )</td>
<td>( ) )</td>
</tr>
<tr>
<td>( l = 2 )</td>
<td>( \sqrt{\frac{5}{16\pi}} \left(-1 + 3 \cos(\theta)^2\right) )</td>
<td>( -\sqrt{\frac{15}{8\pi}} e^{i\phi} \cos(\theta) \sin(\theta) )</td>
<td>( \sqrt{\frac{15}{2\pi}} e^{2i\phi} \sin(\theta)^2 )</td>
</tr>
</tbody>
</table>
Invariant descriptors

- Rotation independent descriptors

\[ v_1(\theta, \phi, p) = \sum_{m=-1}^{1} c_1^m Y_1^m(\theta, \phi) \]

Substituting the basis functions \( Y_1^{-1} = \frac{\sqrt{3}}{2\sqrt{2\pi}} (u_0 - iu_1), Y_1^0 = \frac{\sqrt{3}}{2\sqrt{2\pi}} u_2 \) and \( Y_1^1 = -\frac{\sqrt{3}}{2\sqrt{2\pi}} (u_0 + iu_1) \) this sum can be written as

\[ v_1(u) = Au = A \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} = a_1 u_0 + a_2 u_1 + a_3 u_2 \]

\[ A = (a_1, a_2, a_3) = \frac{\sqrt{3}}{2\sqrt{2\pi}} \left( c_1^{-1} - c_1^1, i(c_1^{-1} + c_1^1), \sqrt{2} c_1^0 \right) \]

\[ u_i^V = R_u^T v e u_i \]

\[ c_i^m |^V_R = R_x \cdot c_i^m |^V \]

where the rotation matrix is defined as \( R_x = \text{diagonal}(\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_2}) A^T R_u^T \).
Invariant descriptors

Scale independence

Scaling invariance can be achieved by dividing all descriptors by $l_1$, the length of the longest main axis

$$c^m_l \mid S = R_x c^m_l \mid R,$$

(4.34)

Invariant spherical harmonic descriptors

Ignoring the coefficients of degree $l = 0$, that is setting $c^0_l \mid T = (0, 0, 0)^T$ achieves translation invariance.
Parameterization with spherical harmonics

- Surface Parameterization & Expansion into spherical harmonics.
- Normalization of surface mesh (alignment to first ellipsoid).
- Correspondence: Homology of 3D mesh points.
Parameterization with spherical harmonics
Object Alignment / Surface Homology
Object Alignment prior to Shape Analysis

1stelli TR, no scal  1stelli TR, vol scal  Procrustes TRS

side

top

top

side
Correspondence through parameter space rotation

Parameters rotated to first order ellipsoids

- Normalization using first order ellipsoid:
- Rotation of parameter space to align major axis
- Spatial alignment to major axes
Reference


• [2]. Description and analysis of 3-D shapes by parametrization of closed surfaces, Christian Michael Brechbuhler-Miskuv, Ph.D thesis, Diss. ETH No. 10979

• [3] Elastic Model-Based Segmentation of 2-D and 3-D Neuroradiological Data Sets, Andras Kelemen, Ph.D thesis

• [4] Slides from Dr. Guido Gerig