Hamilton-Jacobi Skeleton and Shock Graphs

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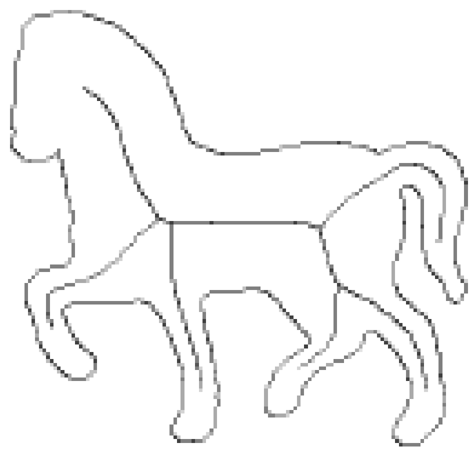
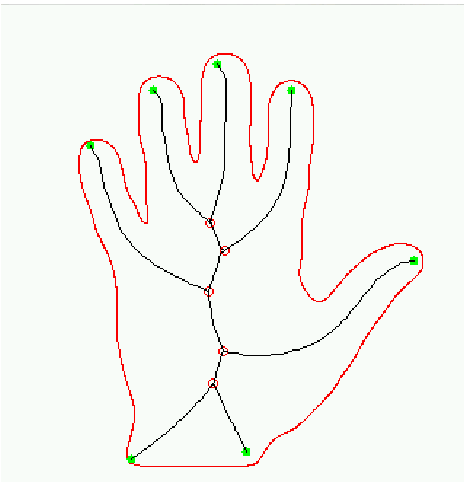
Papers:
Hamilton-Jacobi Skeleton (Siddiqi et al.)
Shock Grammar (Kimia, Siddiqi)

Introduction

- Skeleton (medial axis)
  - A thin representation of shape.





- good skeleton:
  - Thin set
  - Homotopic to the original shape
  - Invariant under Euclidean transformations
  - Given the radius, the object can be reconstructed exactly

Curve Evolution Equation

Eikonal Equation:

$$\frac{\partial C}{\partial t} = F\mathcal{N}$$

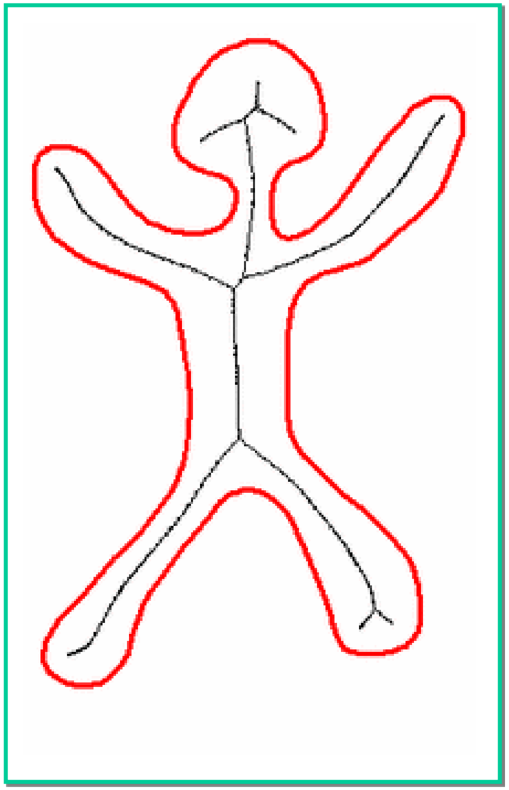
C --vector of curve coordinates

$\mathcal{N}$-- inward normal

F -- speed of the front

Shocks (skeletal points):

Where the curves collapse



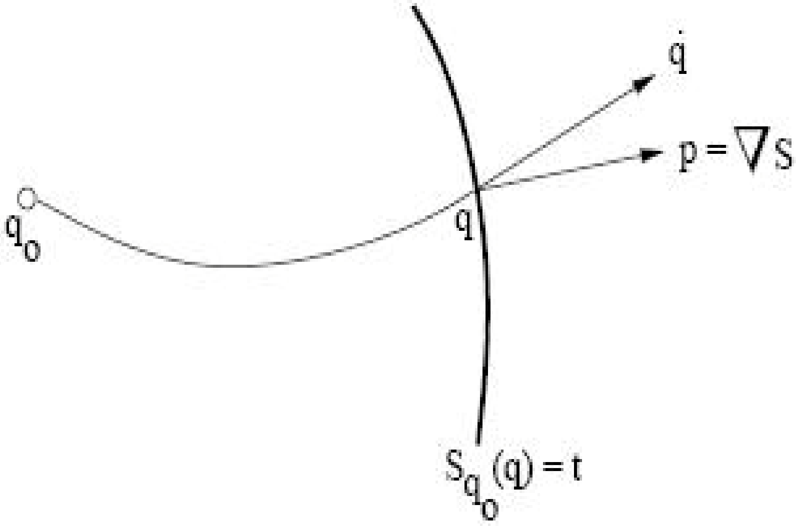
From: PhD thesis Hui Sun, U-Penn

Lagrangian Formulation

Action function:

$$S_{q_0,t_0}(q,t) = \int \mathcal{L}(q,\dot{q})dt$$

q --coordinates $\dot{q}$ --velocities



By minimizing S, we got:

$$L = \frac{1}{F(x,y)}||\frac{\partial\gamma}{\partial t}|| = \frac{1}{F(x,y)}||\dot{q}||$$

In the special case of $F(x,y) = 1$

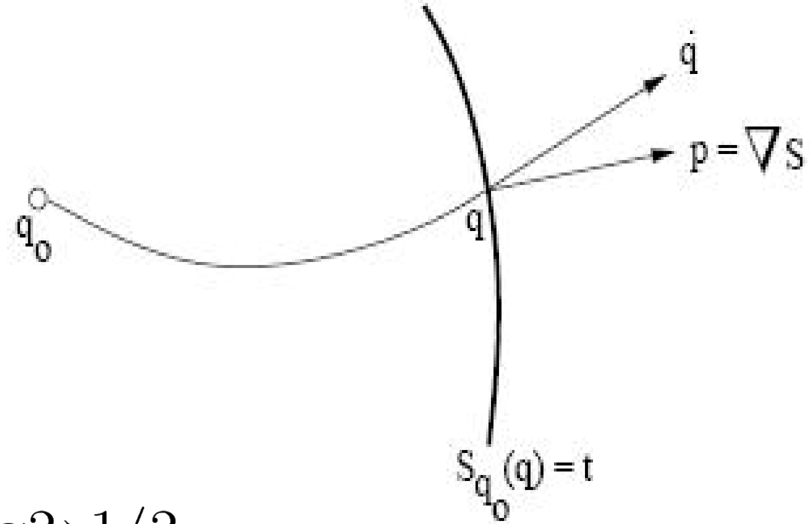
$$L = ||\dot{q}||$$

Hamilton-Jacobi Skeleton FLow

Legendre transformation:

$$H(q,p) = p\dot{q} - \mathcal{L}$$

$$p = \frac{\partial S}{\partial q} = (S_x, S_y)$$



Huygen's principle:

$$p.\dot{q} = 1 \quad \Rightarrow \quad ||\dot{q}|| = (S_x^2 + S_y^2)^{1/2}$$

$$\Rightarrow \; H = 1 - (S_x^2 + S_y^2)^1/2$$

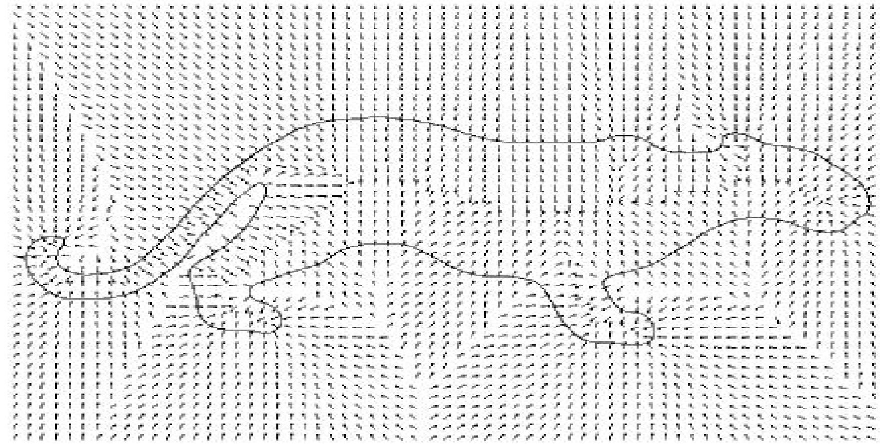
Hamilton-Jacobi skeleton flow formalism:

$$\dot{p} = -\frac{\partial H}{\partial q} = (0,0) \qquad \dot{q} = \frac{\partial H}{\partial p} = -(S_x, S_y)$$

Shock Detection

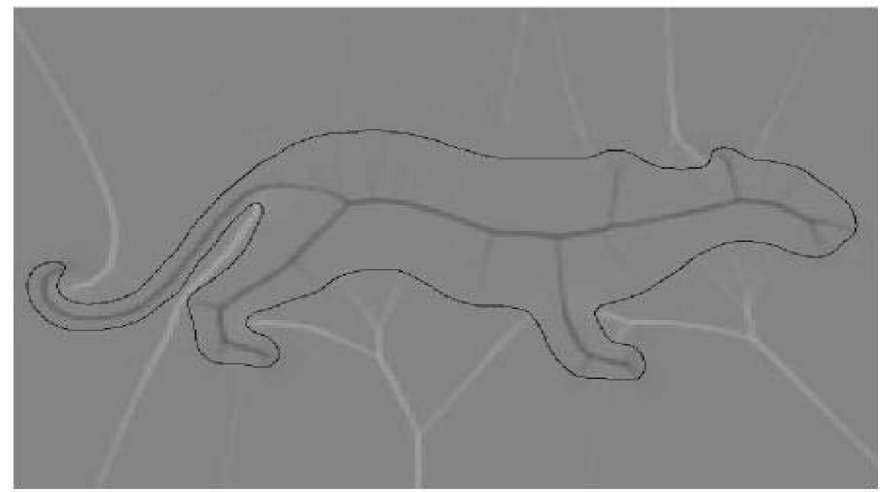
Average outward flux of :

$$\frac{\int_{\delta R} < \dot{q}, N > ds}{length(\delta R)}$$



Via the divergence theorem:

$$\int_R div(\dot{q}) da = \int_{\delta R} < \dot{q}, N > ds$$

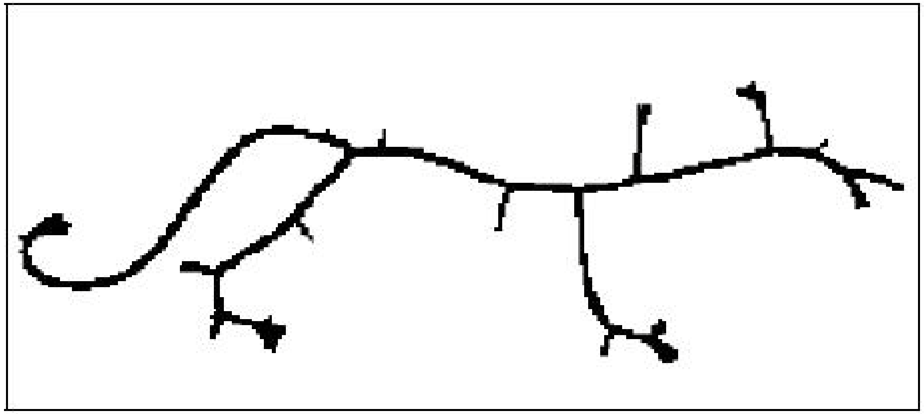


- Conclusion:

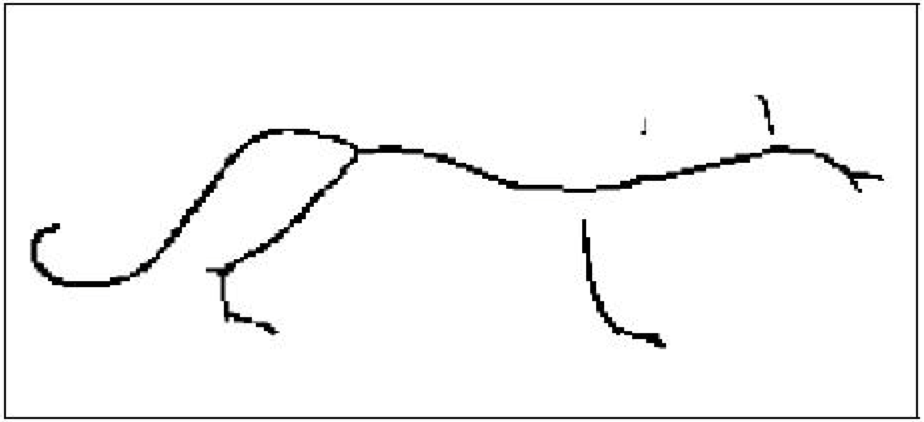
  Non-medial points give values close to zero;
  while medial points(shocks) which corresponding to a strong singularities give large values.

Thresholding

High threshold:



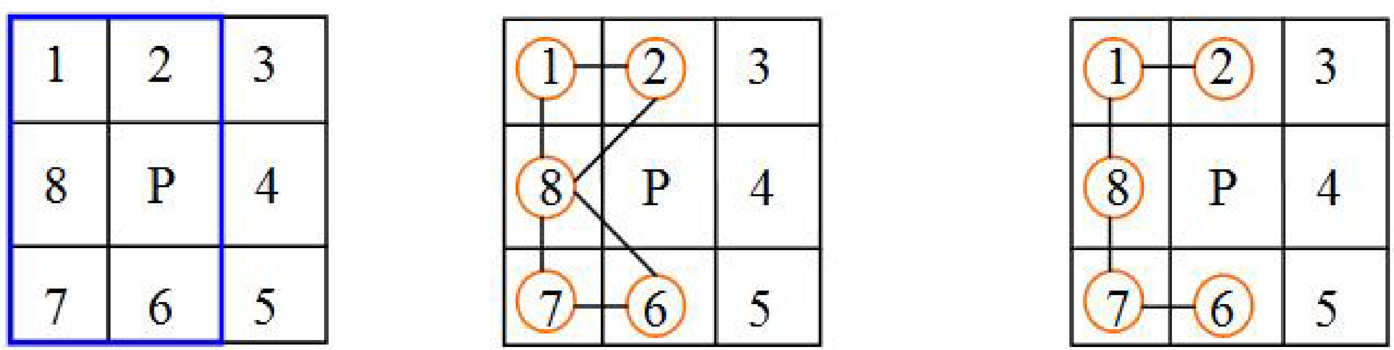
Low threshold:



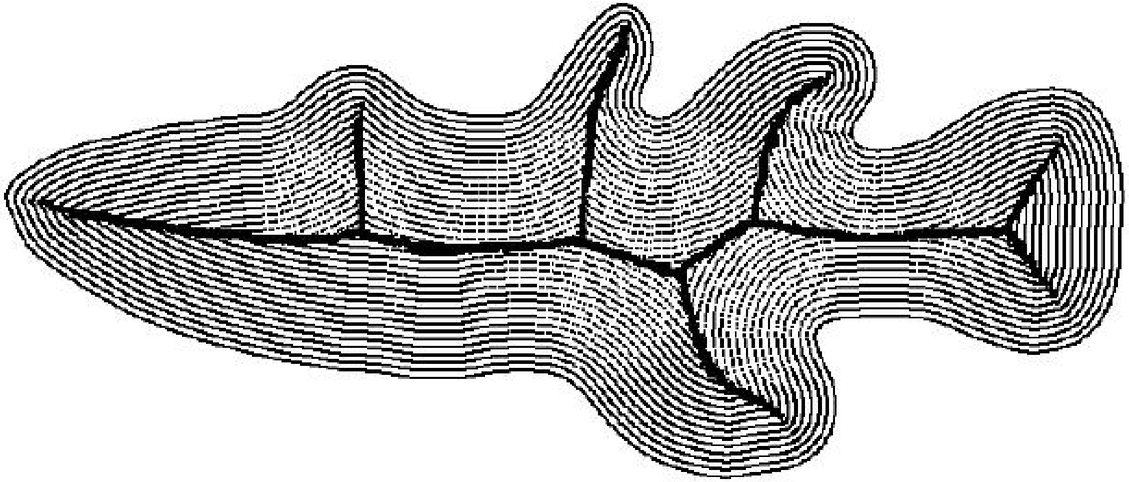
Homotopy Preserving Skeletons

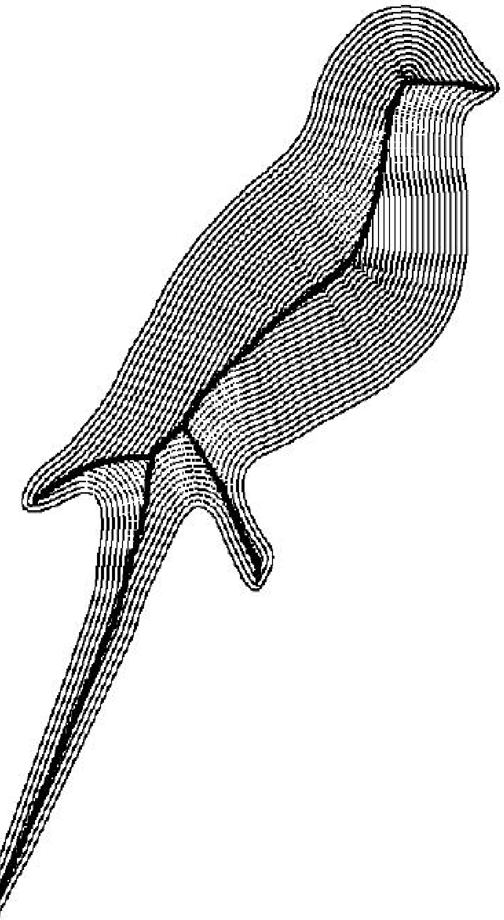
- 'simple' point:
  Its removal does not change the topology of the object.

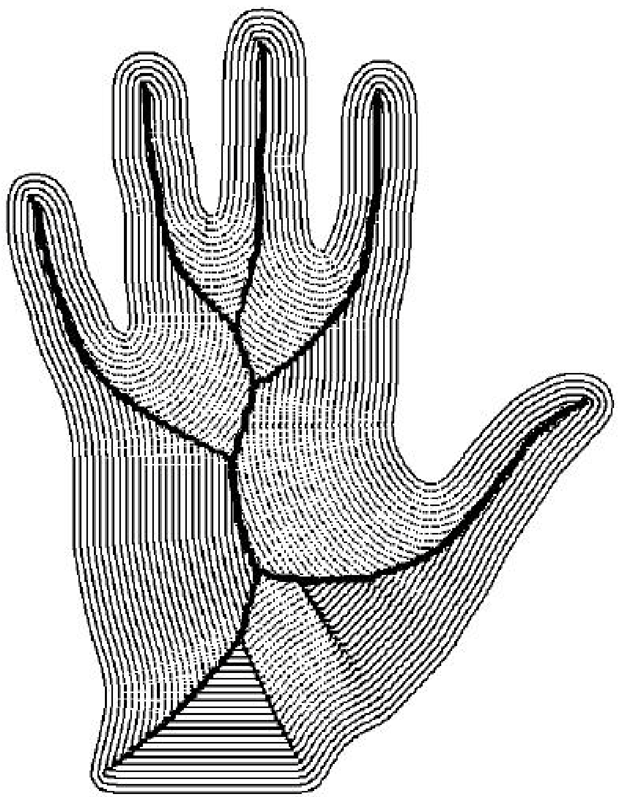
- Goal:
  To move the simple points as many as possible and get a thin skeleton.



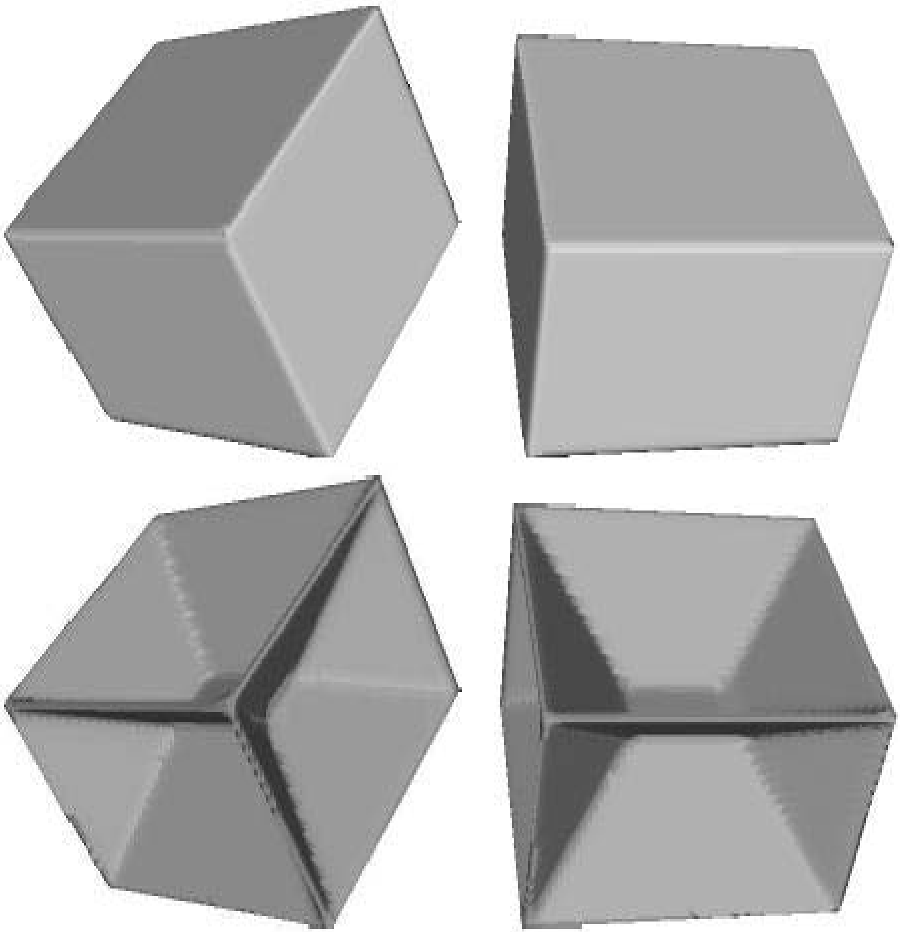
Shock Detection Results (2D)

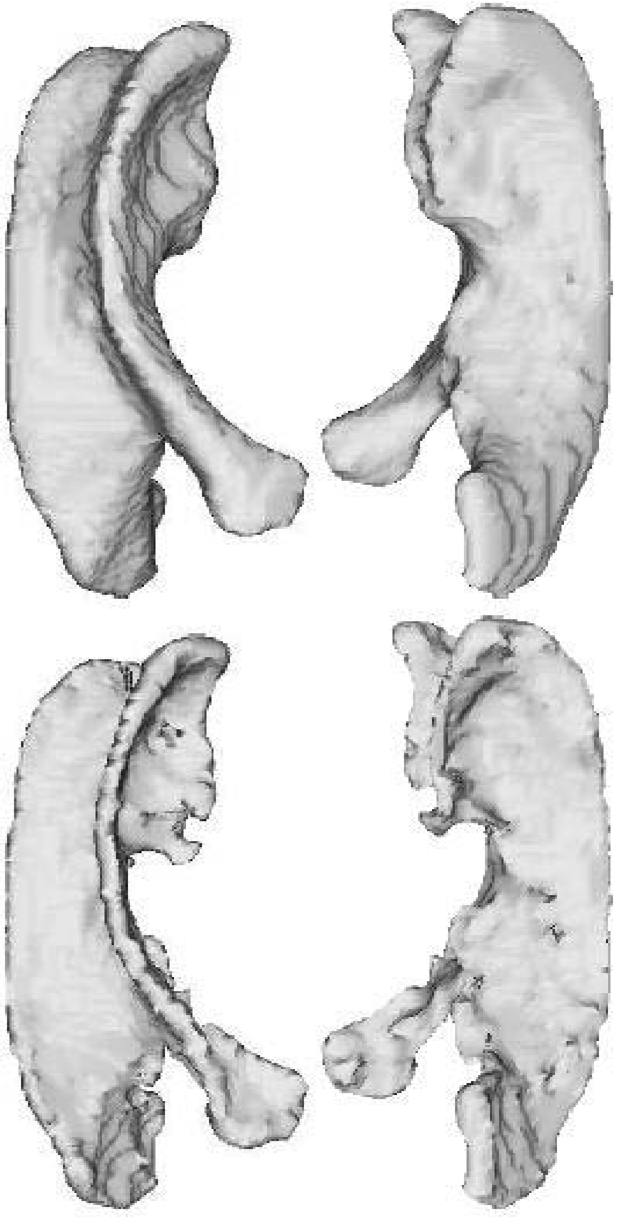






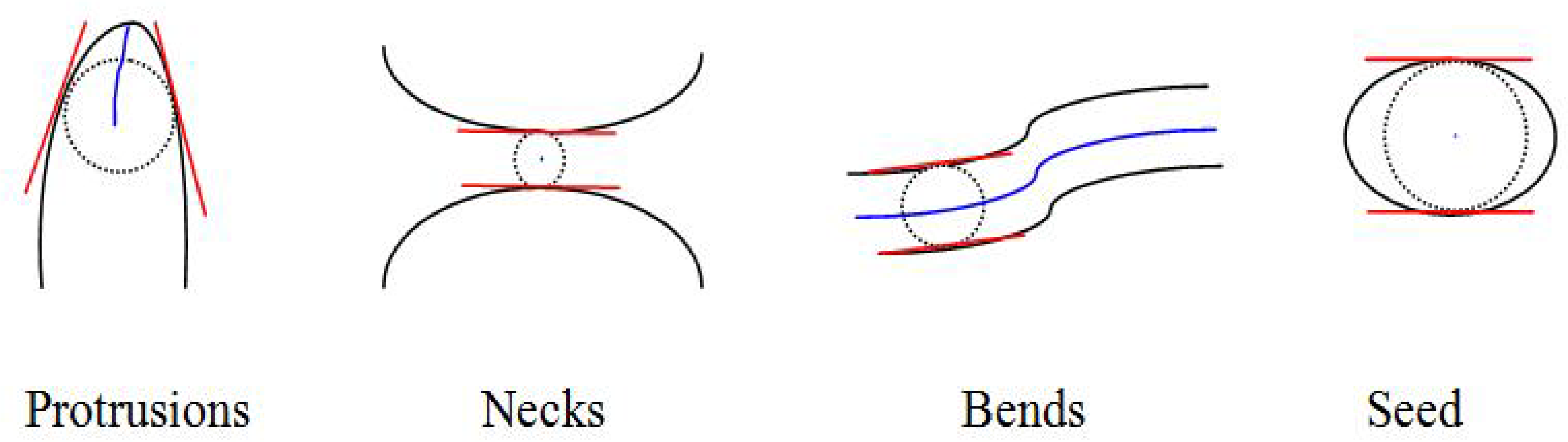
Shock Detection Results (3D)





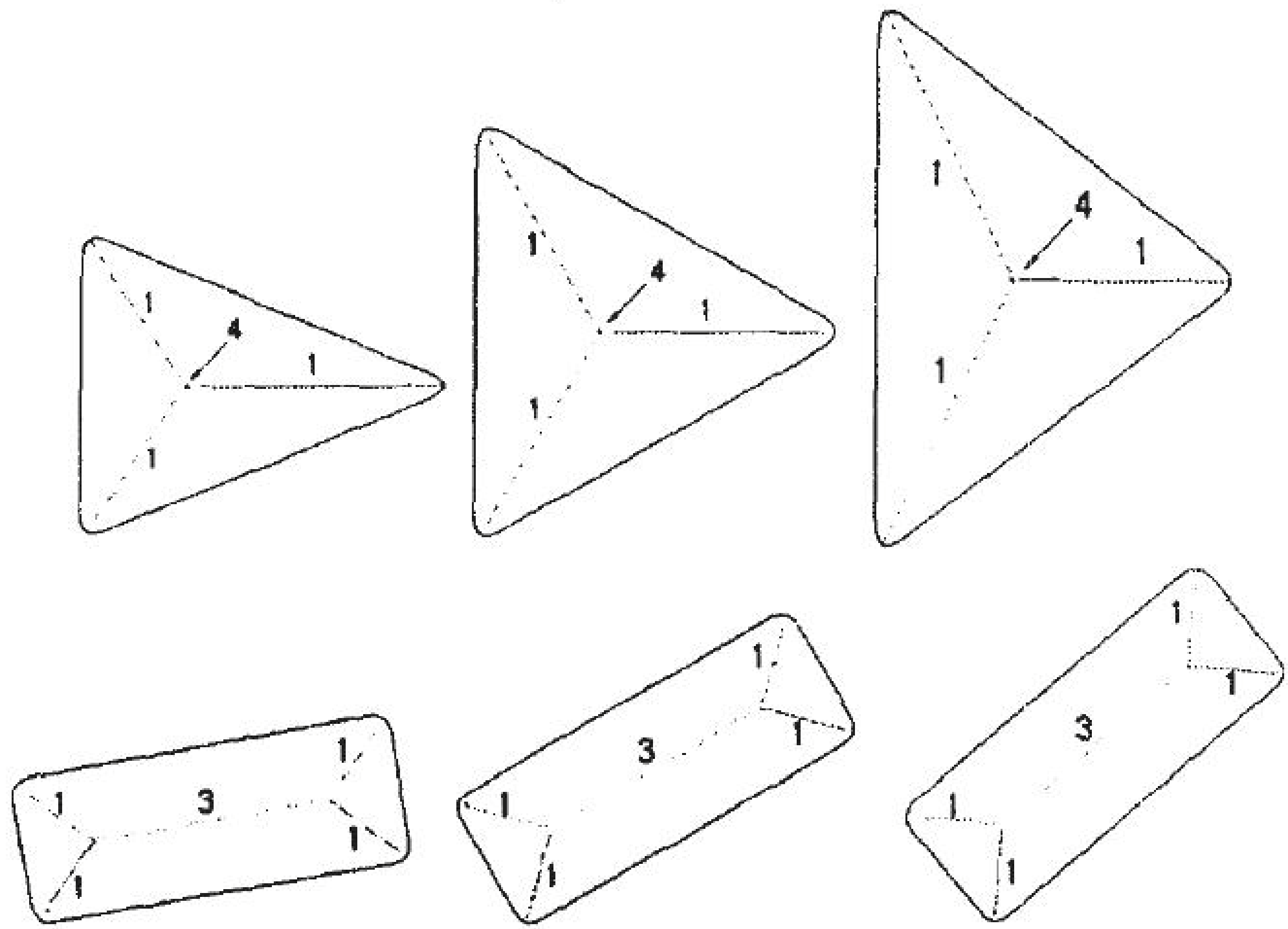
Shock Grammar

- Four types of shocks:



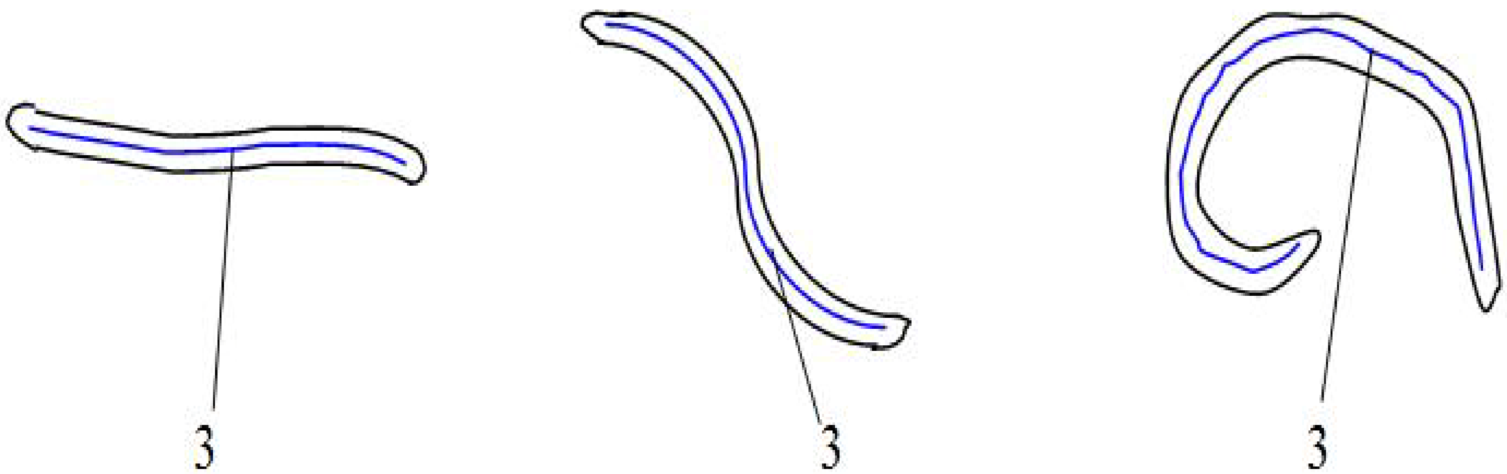
Examples of shock graph

Size and rotation invariant



Worm Example

Allow deformation:



straight bended spiral

Shock grammar definition

Grammar $G = (V, \Sigma, R, S)$

- V -- alphabet
- Σ -- terminal symbols
- S -- start symbols
- R -- rules

example:

$$V = \{S_1, S_2, S_{\bar{3}}, S_4, S_I, S_T, E\}$$
$$\Sigma = \{S_T\}$$
$$R = \{S_I \to S_1 E, S_I \to S_2 E, S_I \to S_{\bar{3}} E, S_I \to S_4, S_1 E \to S_1 S_1 E, S_1 E \to S_1 S_{\bar{3}} E, S_1 E \to S_4, S_2 E \to S_2 S_1 E, S_{\bar{3}} E \to S_{\bar{3}} S_1 E, S_{\bar{3}} E \to S_{\bar{3}} S_T, S_4 \to S_4 S_T\}.$$