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SHAPE DESCRIPTION USING WEIGHTED SYMMETRIC AXIS FEATURES*

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Abstract – We present here an application of symmetric axis geometry to shape classification and description. Shapes are segmented into simplified segments, in which a sequential string of features is derived. These are based on both width and axis properties, and two linear combinations of them, as well as their derivatives. These are felt to have intuitively and geometrically simple meanings. A weighting measure is developed which evaluates the importance of these shape descriptors. Consequently, small perturbations are recognizable despite their generation of sizable symmetric axis arcs. A variety of other segment measures for shape is introduced.

Image processing Symmetric axis transform Biomedical shape analysis Segmentation Curvature Feature extraction

INTRODUCTION

The need for a geometry to deal with amorphous objects seems transparently obvious. Topology is so general that all silhouettes without holes are equivalent. Congruence geometries require an exact match, or some distance or area tolerance from it. Some simple examples are affine, euclidean, projective geometries, as well as template schemes. They are too rigid for the forms of biology, which can flex, grow and otherwise modify. Their primitives do not match the archetypes of biology. A variety of *ad hoc* procedures for dealing with shape has been developed in computer science. But by and large, they operate on the boundaries of objects rather than on the objects themselves. These procedures have been successful for a host of applications, particularly those dealing with cultural objects, i.e. alphanumerics, blocks, etc. In the field of natural objects, their success has been very limited. Yet, shape is an essential and unsolved problem in biology. We hope here to provide the base for both the classification and discrimination required by biological problems, as well as to elicit some of the underlying mechanisms at work. It is to that purpose that the symmetric axis transform is here elaborated.

The symmetric axis transform (also called medial axis and skeleton) was introduced by Blum[1,2] and explored by him,[3,4] Calabi and Hartnet[5,6] and others for continuous shapes. This paper continues recent work by Nagel and Blum[8] on continuous properties. Computer implementation and discrete axis theory was developed by Rosenfeld *et al.*,[9,10] and further explored by Montenari[7] and others using discrete space concepts. An extensive, though dated, bibliography exists.[3] Blum and Nagel have recently combined the two approaches, developing a computer program using a true euclidean metric on quantized forms to yield a smoothed description (unpublished)

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2. TRANSFORM DESCRIPTION AND OVERVIEW

The symmetric axis transform (SAT) is a particular description of a shape (boundary plus interior) that is in a precise sense centered on the shape. A boundary description is related to generating the object by cutting it out. The SAT description is related to generating the object by growing it. The conventional description traverses the boundary, while the SAT description goes orthogonal to the boundary and traverses the shape (its outside too, although that is not considered in this paper).

One formalism for obtaining the SAT uses a circular primitive. Objects are described by the collection of maximal discs, ones which fit inside the object but in no other disc inside the object. The object is the logical union of all of its maximal discs. The description is in two parts: the locus of centers, called the symmetric axis (not unlike a central contour or stick figure) and the radius at each point, called the radius function, R. This description is exactly equivalent to the boundary description, but it makes new properties of the shape obvious. Note that the discs intersect and have centers which form a connected set.

An alternative way of describing the axis portion of a symmetric axis transform employs an analogy to a grassfire. Imagine an object whose border is set on fire. The subsequent internal quency points of the fire represent the symmetric axis, the time of quench for unity velocity propagation being the radius function.

The symmetric axis of a 2-dimensional shape is a 1-dimensional planar graph, i.e. it has no area. With proper shape constraints (see next section), this graph has curvature defined everyplace except at a finite number of points where sided curvature exists. Its nodes connect simplified segments whose shape properties are sequentially ordered on their axes and which have distinct left and right sides. By observing the axis and distance function independently, new features of shape appear. These depend on the algebraic sign of the axis and radius function and their derivatives.

The classification scheme gives shape features which are intuitive. Conversely, it gives precise definitions to a number of loosely defined verbal features. A natural and precise language for shape results. The primitive features include worm, wedge, cup, flare, pinch, branch. Simple linear combinations of the axis and width curvature give left and right boundary curvature. These may now be ordered with respect to the earlier object properties.

A notion of flexure is introduced using a complete constraint on the radius function with no constraints on axis curvature. This provides an interesting object deformation which follows the intuitive meaning of flexure, and preserves both perimeter and area. Flexure also defines a canonical shape which is symmetric in the particular sense of the transform.

3. CLASSES OF SHAPES UNDER DISCUSSION

Although the SAT is interesting and important in non-euclidean and higher dimensional spaces, this paper is limited to 2-dimensional euclidean shapes. Shapes, here, are bounded, connected, closed sets of points, with one extension and two constraints. The normal boundary is extended so that it may include designated closed sets of interior points of 0 or 1-dimension, that do not segment the shape. Thus, a disc with a radius or interior point as additional boundary is allowed. Objects which have parts of their boundaries abutting on each other are now allowed. The following constraints exclude degenerate cases which cannot occur in real shapes.[3] The first condition requires the boundary to have tangent and curvature defined everywhere except at a finite number of places. At these places, a sided curvature must exist from each direction along the boundary. When a good tangent exists at such a point, a smooth point of discontinuous curvature results. When the tangent itself has only a sided definition, corners, cusps and such are allowed. This condition rules out a variety of boundary degeneracies, such as unbounded variation and limit sets of discontinuities. The second condition requires that the boundary consists of a finite number of connected pieces. Consequently, an infinite number of holes cannot exist. For tutorial simplicity, objects with holes in them will not be considered, except for occasional comment.

The SATs of objects can exist outside the object. By considering the shapes as holes in a large bounding circle, many of the properties of the external SATs are seen as identical to the properties of internal SATs. Two important properties remain different however. The external SAT of a shape need not exist as is the case in a convex shape; nor if it exists, need it be connected.

4. POINT TYPES

Points on the symmetric axis are classified by the properties of their maximal discs. A normal point is one whose underlying disc touches the object border in exactly two separate and contiguous sets of points. A branch point is one whose underlying disc touches the object border in three or more separate contiguous sets. An end point is one whose underlying disc touches the object border in only one contiguous set. For all but a finite number of discs, a single point contact occurs at each touching of the boundary. For these cases, the contiguous sets are reduced to discrete point contact. Finite contact occurs when the underlying disc and the object boundary coincide over an interval for at least one of its touchings. At such a contact, radius of curvature of the object boundary and the disc are equal.

Figure 1(A) shows a "dog bone"; Fig. 1(B), its symmetric axis; Fig. 1(C), the point types. Note that only normal points exist over an interval. All axis points not branch or end are normal points.

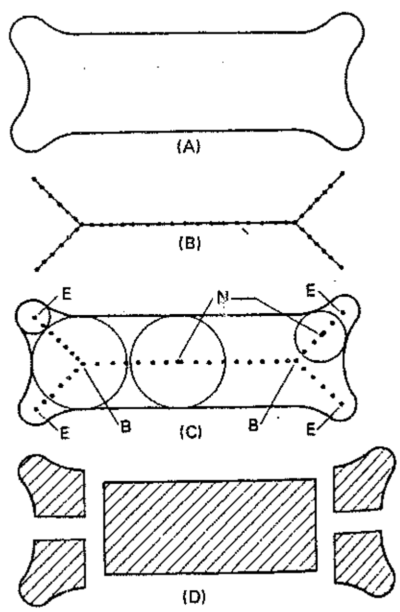

Fig. 1. (A) "bone" shape and (B) its symmetric axis. (C) shows symmetric axis POINT TYPES based on number of non-contiguous boundary touching arcs; "E" for END POINTS where only one such touching exists, "N" for normal points where two exist and "B" for branch points where three or more exist. Normal points are the only kinds of points which can occur over an interval. (D) shows object partitioned into SIMPLIFIED SEGMENTS at branch points. These are later given sequential descriptors.

5. SEGMENTATION

The symmetric axis is broken up for further analysis into parts called simplified segments. This segmentation is based on the point properties of the axis. A simplified segment is a set of contiguous normal points bounded by either a branch point or an end point. Such segmentation has several desirable properties. It is unique, disjoint, and complete, in the sense that the union of all the segments is the total axis. The boundary parts accounted for by the axis segmentation are also unique, disjoint and complete.

Simplified segments basically subdivide an object into two-sided parts. For a large class of biological objects, this has an intuitive appeal. But for some rectilinear figures, the segmentation can sometimes be counter-intuitive. For example, the dog bone of Fig. 1 produces the 5 simplified segments of Fig. 1(D). Four of the segments correspond to lobes in the bone, the fifth forms a connecting segment. The break up into parts seems natural. In contrast, the rectangle shown in Fig. 2(A) has the same symmetric axis. The resulting segmentation shown in Fig. 2(B) seems awkward and unnecessarily complex since straight sides are arbitrarily broken into 2 and 3 parts. The SAT is not the simplest description for rectilinear figures. The reader is reminded that although the axis shape of the dog bone and rectangle are identical, the radius function separates these two figures. Later discussion shows how the SAT deals with these shapes.

Segmentation subdivides the original object and its symmetric axis for an analysis of each part. This permits a complex object to be broken into simpler parts which may be sequentially described. These simplified segments are augmented by the discs formed at their ends. The axes of simplified segments are open sets since branch and end points are used only as delimiters. But that is of no issue. For each of these terminal points, a disc description is also formulated. These describe the segment joinings at branch points, and the object termination at end points. For some purposes, the deliminator properties may be lumped with the segment. Simplified segments are discussed next and the delimiters deferred until later.

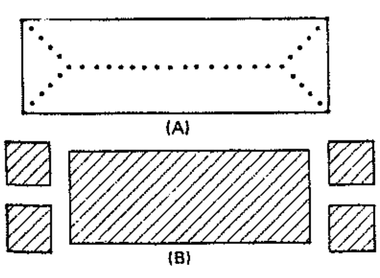

Fig. 2. (A) A rectangle with its symmetric axis and (B) segmentation. Comparison with Fig. 1 shows the importance of the OBJECT WIDTH in object description. The symmetric axes loci of both shapes are identical. Note also that for the rectangle, the symmetric axis segmentation is awkward. Overcoming this handicap is discussed later.

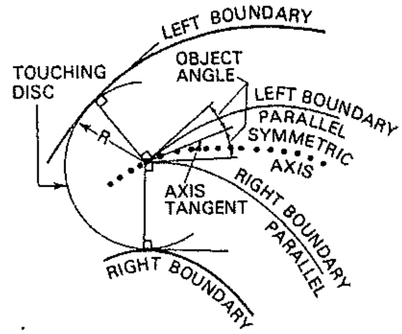

Fig. 3. Underlying geometry of an object at a point where the symmetric axis is continuous. This geometry forms the basis for the definition of: (1) OBJECT ANGLE, (2) OBJECT CURVATURE, the change in object angle along the symmetric axis and (3) AXIS CURVATURE.

6. SEGMENT ANALYSIS AND FEATURES

A simplified segment's axis has no branches and is not a loop since objects with holes are not treated here. The SAT description is assumed to be given in cartesian coordinates. This description is converted to intrinsic Gaussian coordinates using curvature as a function of axis arc length, s, from either end. The curvature of the radius function is also described as a function of axis arc length from the same end. The features extracted are aimed at being related to the object, and not to external coordinate space. External coordinate system independence is thus achieved. This representation creates a dual description for each simplified segment, depending on the direction of traversal. This is important later, when dealing with the total graph.

Our object here is to create a set of well defined object properties based on SAT measures. Figure 3 gives a heuristic view of these before plowing ahead. Object width is the touching circle diameter or $2R$. Object angle is the angle between the boundary parallels. Axis curvature needs no elaboration. Object curvature is the rate of object angle change along the axis.

The analysis of simplified segments is based on a set of mathematical descriptions defined on the axis and radius function of the segment. (The reader who is not interested in the mathematical derivations of these terms should skip to section 6.1.) It is assumed here (it can be proven) that these functions exist everywhere, and are differentiable except at a finite number of points, ones of finite contact, which are discussed later. Figure 4 shows an object piece around an axis point, A. The direction of traversal is important and vectors through A are required for analysis. The two touching points of A lie on the left and right of the axis tangent vector extended. The vectors PL and PR are parallels to the left and right touching point tangents respectively. The vector H is the zero degree line, drawn in the right horizontal direction from A.

The axis angle α is measured from H to AT. Note, α

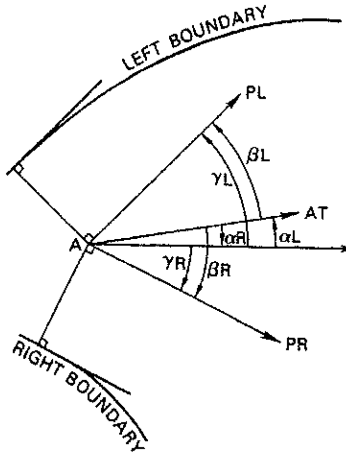

Fig. 4. A careful examination of the angles in the neighborhood of a continuous symmetric axis point. This is necessary for precise statements about the properties of Fig. 3 and their relationships.

$= \alpha_L = -\alpha_R$ in order to keep the angle proper when reversing the direction of traversal.

$$\alpha = \tan^{-1} \frac{dy}{dx}.$$

Note that α varies between $-180°$ and $+180°$, while the arctangent varies between $+90°$ and $-90°$. α is in this interval when dx is positive and outside when dx is negative.

The object angle β is measured from AT to PL. The angle $\beta = \beta_L = \beta_R$ since the object angle is the same on on both sides.

$$\beta = \sin^{-1} \frac{dR}{ds},$$

where ds is the axis arclength differential. When changing direction of traversal, β becomes $-\beta$. The boundary angle γ is measured from H to the parallel to the boundary through A. Thus γ_L is the angle from H to PL, while γ_R is the angle from H to PR.

$$\gamma_L = \beta_L + \alpha_L = \beta + \alpha$$

$$\gamma_R = \beta_R - \alpha_R = \beta - \alpha,$$

or when combined,

$$\gamma_{L,R} = \beta \pm \alpha.$$

Given the angle definitions, two curvatures are defined in terms of the derivatives of α and β, as follows:

axis curvature, $k\alpha = \dfrac{d\alpha}{ds}$

object curvature, $k\beta = -\dfrac{d\beta}{ds}$.

Note the minus sign in the definition of object curvature makes convexity of the boundary correspond to positive object curvature (toward the object).

Proceeding in a similar manner, the sided boundary axis curvature $k_{\gamma L}$, $k_{\gamma R}$ is defined in terms of the total changes in external coordinate angles along the axis. See Fig. 5.

$$k_{\gamma L,R} = \frac{d(\beta \pm \alpha)}{ds} = k_\beta \pm k_\alpha.$$

Since the boundary parallel arc is shorter than the axis arc length, its curvature, $k_{PL,R}$, must be suitably modified.

$$k_{PL,R} = \frac{1}{\rho_{PL,R}} = \frac{k_\beta \pm k_\alpha}{\cos \beta},$$

where ρ is the radius of curvature. The radius of curvature of the boundary, ρ_B, differs from the radius of curvature of the parallel, ρ_P, by the radius function, R.

$$\rho_{BL,R} = \frac{1}{k_{BL,R}} = \frac{\cos \beta}{k_\beta \pm k_\alpha} + R.$$

The values defined to this point are now used to further categorize the simplified segment.

6.1 *Width properties*

Using the object angle, β, and its curvature, k_β, we can categorize the simplified segment into the elementary descriptions of Fig. 6. The assignment of width labels based on the algebraic sign of the two parameters β and k_β is shown in Table 1.

The reader is reminded that the width shapes as categorized are not descriptions of the shape of the boundary; rather they describe the "interior space" of the object. For a straight axis, the boundary takes on the width shapes shown. This can be seen in the rectangle of Fig. 2, where the width descriptors are particularly simple, one per simplified segment. The rectangle has four segments that are wedges, and one

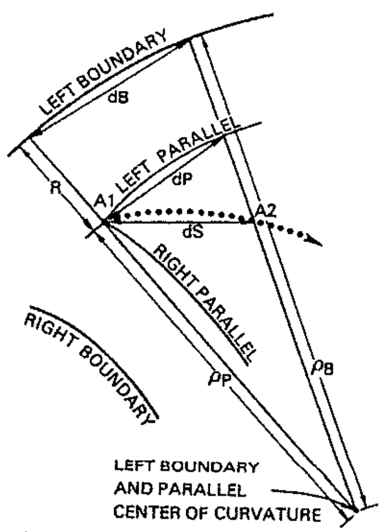

Fig. 5. Basis for the computation of BOUNDARY CURVATURE from object width, object curvature and axis curvature.

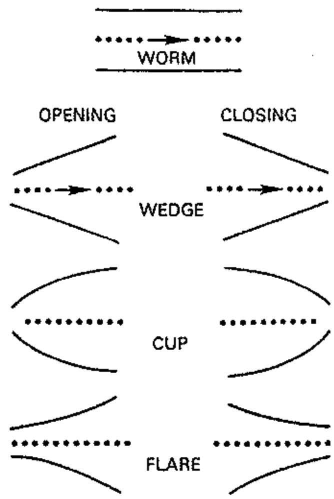

Fig. 6. ELEMENTARY DESCRIPTORS (subsegments) of simplified segments, based on width properties.

Table 1. The assignment of WIDTH LABELS based on the algebraic sign of object angle and object curvature

		Object angle (β) Neg	Zero	Pos
Object curvature (k_β)	Pos	Opening flare		Closing cup
	Zero	Opening wedge	Worm	Closing wedge
	Neg	Opening cup		Closing flare

Table 2. Allowable transition (syntax) of width shapes

from \ to	Worm	Open wedge	Close wedge	Open cup	Close cup	Open flare	Close flare
Worm	.	.	.	.	×	×	.
Opening wedge	.	.	.	×	.	×	.
Closing wedge	.	.	.	.	×	.	×
Opening cup	*	×	.	.	*	×	.
Closing cup	.	.	×	.	.	.	×
Opening flare	.	×	.	×	.	.	.
Closing flare	.	.	×	.	×	*	×

× indicates transition always possible.
* indicates transition at width extreme.

that is a worm. The labels opening and closing depend on the direction of traversal.

A number of the width shape features may be assigned to a simplified segment. Thus a partitioning of the simplified segment is created. The partitioning of the segment into width shapes produces a string description of width labels, with an associated syntax. Thus, each width shape has a restricted set of width shapes which can follow it. These restrictions arise from continuity requirements at transitions. Table 2 shows the allowable transitions.

6.2 *Axis properties*

The shape of the axis is used to further partition the simplified segment. This partition is independent of the partitioning due to width shapes. Using the axis curvature, k_a, and its derivative, k'_a, the simplified segment is partitioned into the curvature descriptors shown in Fig. 7. Note that these descriptors are labeled left or right. The assignment of axis curvature labels based on the sign of the two parameters k_a and k'_a, is the change in axis curvature shown in Table 3. There is no restriction on axis feature syntax.

The application of axis labels produces a second string of features on the simplified segment. This is integrated with the prior width string from width shapes. Either of the two string descriptors is used to form a primary and secondary partition. The decision is based on the analysis goals.

6.3 *Boundary properties*

The preceding two sections discussed width and axis properties of the simplified segment. These are aug-

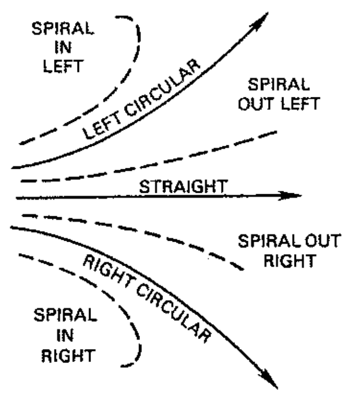

Fig. 7. Elementary descriptors of simplified segments, based on axis properties.

Table 3. The assignment of AXIS LABELS based on the algebraic sign of axis curvature and axis curvature change

		Axis curvature (k_a) Pos	Zero	Neg
Axis curvature change (k'_a)	Pos	Spiral in left		Spiral out right
	Zero	Circular left	Straight	Circular right
	Neg	Spiral out left		Spiral in right

mented now by a characterization of the object border associated with the simplified segment. For a straight axis, the boundary shape is similar in appearance to the width shape. As the axis curvature becomes greater in either direction, the similarity between the two decreases. This has been quantified earlier. Since the two boundaries of a simplified segment can be different, left and right sided boundary descriptions are necessary. Table 4 describes the boundary shape which corresponds to the width shape for various relations of k_α and k_β* As the axis becomes more curved, the convexity properties of boundaries which have the same width shapes change, based on the side of the object to which the axis is curving. Figure 8 illustrates this phenomenon.

Table 4. The local boundary shape as determined by the relationship of axis and width curvature

Curvature relations	Width shapes	Shape of boundary	
		Axis curving side	Opposite side
$\lvert k_\alpha \rvert < \lvert k_\beta \rvert$	Worm	Impossible	Impossible
	Wedge	Impossible	Impossible
	Cup	Convex	Convex
	Flare	Concave	Concave
$\lvert k_\alpha \rvert = \lvert k_\beta \rvert$	Worm	Straight	Straight
	Wedge	Straight	Straight
	Cup	Straight	Convex
	Flare	Concave	Straight
$\lvert k_\alpha \rvert > \lvert k_\beta \rvert$	Worm	Concave	Convex
	Wedge	Concave	Convex
	Cup	Concave	Convex
	Flare	Concave	Convex

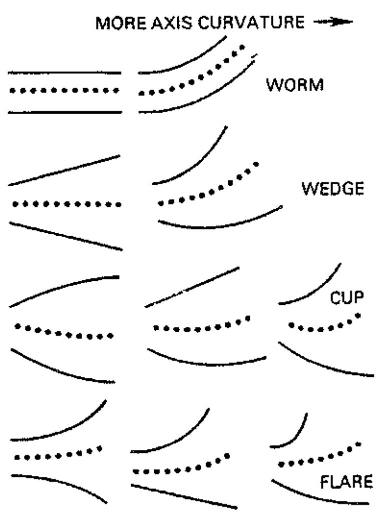

Fig. 8. Effect of increasing the axis curvature of elementary width forms on their boundary curvature. Note that a boundary change between convex and concave goes through a straight transition.

* This can be done on $k_{\gamma L}$ and $k_{\gamma R}$ directly, since they are monotonic to boundary curvature, k_{BL} and k_{BR}, and have the same 0 values.

The boundary shapes are described by a function of both the width and axis properties previously discussed. It is important to point out that when tracing a boundary across simplified segments, a left boundary goes clockwise and a right boundary counterclockwise.

6.4 *Summary and extension*

The simplified segment has been assigned features based on four criteria: width, axis, left boundary and right boundary. The derived features are intervals over which each property is negative, positive or zero. Where these intervals are open, the limit point can be included. Note that these intervals need not be identical for the various criteria.

The axis properties included a curvature change criterion. Maxima and minima of axis curvature become feature terminal points, and axis parts of constant curvature are identified features. This property is extended to width and boundary curvature features. See Fig. 9. Note that these extrema points, though not necessarily their numerical values, will be the same for boundary and width features when the axis is straight.

6.5 *Ligatures and finite contact*

The term semi-ligature is applied to the instance of an outside corner on the object border. Figure 10(A) shows a semi-ligature, and Fig. 10(B), a full ligature. Finite contact, defined earlier, occurs when the touching disc and object border coincide over an interval.

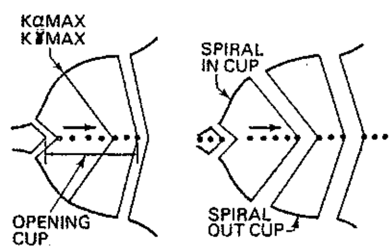

Fig. 9. Extension of elementary width descriptors to include breakup at extrema of boundary curvature.

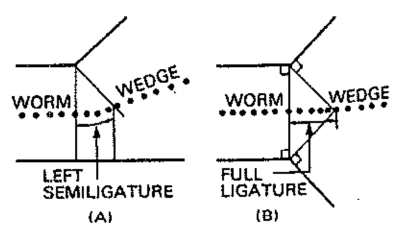

Fig. 10. LIGATURES are formed at location of "outside" object corners. At such points, a single boundary point is represented by an interval of axis points.

The terms ligature and finite contact are applied to the extreme limits of boundary curvature. Recall that positive curvature is defined toward the interior of the object (convex), negative curvature away (concave). Boundary curvature k_B must lie between $+1/R$ (finite contact) and − infinity (ligature). The first limit occurs because the boundary cannot curve toward the object faster than the touching disc. The second limit occurs because any tangent discontinuity on a simplified segment must be an outside corner (away from the object) since an inside corner would be the end point of a different simplified segment.

The center of curvature (c.c.) lies along the normal to the boundary (touching radius extended), at distance ρ_B or $1/k_B$ from the boundary (see Fig. 11). Hence, it lies only on the end points or extension of the touching radius in either direction. It cannot lie on the interior of the touching radius itself.

The outside corner is considered first. See Fig. 10 again. One to one correspondence between boundary and axis is lost. A finite length of axis is created between the normal to each side of the corner by the single corner point. This is an important feature because it is a limit case of the concave boundary and a zero of k_B. Furthermore, a corner is an important shape property. Consequently, an outside corner is named a semi-ligature when it occurs on only one side of an axis interval, and a full ligature when it occurs on both sides. Note again that no new boundary points are described by its axis. Explicit conditions for ligatures are:

for a left ligature, $k_\beta + k_\alpha = \dfrac{\cos\beta}{R}$

for a right ligature, $k_\beta - k_\alpha = \dfrac{\cos\beta}{R}$

for a full ligature, $k_\alpha = 0$ and $k_\beta = \dfrac{\cos\beta}{R}$.

The second limit case of c.c. occurs when boundary curvature, $k_B = +1/R$. It occurs in two different ways, single contact and finite contact. Single point contact of this type is common as an end point property, i.e. it occurs in the axis of a parabola as shown in Fig. 12(A). Its occurrence in a simplified segment is shown in Fig. 12(B), where two parabolas are displaced by twice the minimum radius of curvature. Such a shape property causes no problem as there is still 1:1 correspondence between boundary and axis points. Finite contact is now discussed.

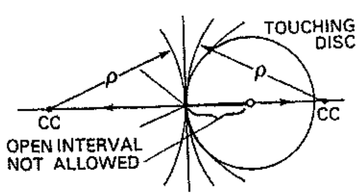

Fig. 11. Allowed center of curvature of object boundary along its normal.

Fig. 12. Limiting positive curvature of boundary is $1/R$, the curvature of its touching circle. (A) Shows this occurring for a parabola at its axis end point. (B) Shows this occurring for two parabolas displaced by $2R$. The touching point now comes from a normal axis point.

Finite contact occurs when the boundary and the touching disc have an interval in common. Figures 13(A) and (B) show finite contact occurring at one or both touching boundaries. The object angle changes discontinuously by the total angle of contact, δ. The axis angle also undergoes a discontinuity, unless both contacts are symmetric about the axis. Thus, k_β does not exist. k_α exists only at a finite contact point that is symmetric. Finite contact is the second failure of 1:1 correspondence between sided boundary and axis. The finite contact boundary arc or arcs correspond to a single point on the axis. Obviously, finite contact points must represent convex boundary parts. Observe also from Figure 13(C) that a finite contact point may oppose or be contiguous with a semi ligature.

6.6 *Boundary axis weight*

A weighing function is now developed to assess the importance of the SA elements in representing the boundary. Length of an axis element is, in general, not equal to length of boundary represented. For example, a ligature which represents a single point of the boundary, gives rise to an arbitrarily large length of SA. In contrast, at a point of finite contact sizable pieces of object border are represented by a single point on the SA. Thus ligatures and finite contact points represent extremes in the boundary axis relationship. The weight derived here quantifies this relationship under general conditions. The boundary/axis weight (B/A) is defined as the limiting ratio of boundary length to SA length, computed at each axis element. Ligatures have a B/A of zero, while the B/A for finite contact is infinite.

Due to the presence of two boundaries at an axis

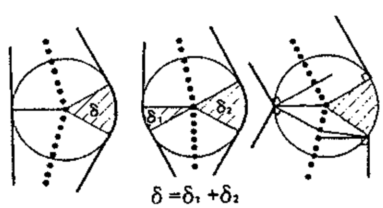

Fig. 13. Examples of FINITE CONTACT boundaries for normal points. (A) Has one touching of point type and one of finite type. (B) Has two touchings of finite type. (C) Has a combination of finite contact and ligatures at one axis point.

element, a left and right sided weight are defined. Given the existence of a B/A weight for an axis neighborhood, it then becomes possible to sum this weight over a portion of the SA to assess the amount of boundary curve produced. An example of this and other uses of the B/A weight can be found in section 11. In what follows the B/A weight is derived and formulated in terms of β, R, k_β and k_x. This derivation parallels that of boundary curvature. Consider Fig. 5 again.

$$k_{PL.R} = \frac{k_\beta \pm k_x}{\cos\beta}$$

where $k_{PL.R}$ is the curvature of the left and right boundaries respectively. Geometrically, the boundary axis weight (B/A) is the product of the projection of the axis length on the boundary parallel and the ratio of the boundary length to boundary parallel length.

$$\frac{dB}{ds} = \left(\frac{dP}{ds}\right)\left(\frac{dB}{dP}\right) = \cos\beta\left(\frac{\rho P + R}{\rho P}\right)$$
$$= \cos\beta\,(1 + R_P).$$

Substituting for k_P gives

$$\frac{dB}{ds} = \cos\beta\left[1 + R\left(\frac{k_\beta \pm k_x}{\cos\beta}\right)\right]$$
$$= \cos\beta + R(k_\beta \pm k_x).$$

Examples of B/A weight are shown in Fig. 14. Part (A) shows the weight becoming small as the object angle approaches 90°, while part (B) shows the case of negative curvature in which the boundary is shorter than the parallel, and finally part (C) shows the case of positive curvature in which the boundary is longer than its parallel.

At a finite contact point, k_β does not exist and the above weight equation is invalid. A finite contact point is given the weight of its total boundary arc length, $R\cos\delta$, where δ is the total contact angle. The weight function over any SA interval up to and including a whole simplified segment is simply,

$$B/A = \int_1^2 W_{\text{point contact}}\, ds + \sum_i W_{\text{finite contact}}.$$

The use of B/A weight over more than one segment

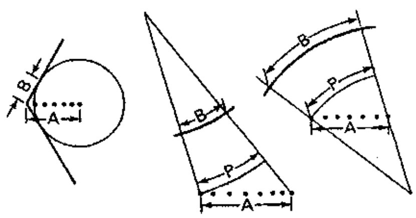

Fig. 14. Geometry of BOUNDARY-AXIS WEIGHT computation. (A) Shows the effect of object angle on this weight. (B) and (C) show the effect of negative and positive boundary curvature, respectively, on this weight.

becomes particularly important later. To compute this, it is necessary to include weights for ends and branches. Since they exist at discrete axis points only, they cannot contribute to the integral normally because they are of point contact type and have 0 weight. However, when they have finite contact ends and branches, they contribute exactly as such points do within the segment. Therefore the above equation may be used between any set of axis points.

7. SEGMENT GLOBAL MEASURES

Global measures of a simplified segment can be extremely varied. That should not be surprising as shape is a richly variable domain. The measures presented here are not intended as the final word, but as an introduction to thinking about segment properties. Particular problems may well require thinking out specialized solutions. For example, different measures are suggested here for inside segments between branches, and end ones, having at least one end disc. (For objects with holes, it is possible to have a segment with no ends. But that problem is not part of this paper.)

Consider the interior simplified segment AB, of Fig. 15. The most obvious length of the segment is its axis length. The segment touching points at A are ligatures. The branch location, hence segment length, depends on the location of the third touching point, a non-contiguous part of the boundary of the segment AB. To avoid this, the length of the axis outside the branch touching point chord CD in Fig. 15 is subtracted. The result is named the symchord length*, arc $A'B'$ of the figure. For reasons that are apparent in due time, this is done without regard to the axis curvature. The excess

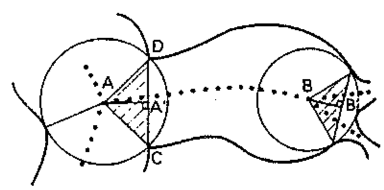

Fig. 15. AXIS LENGTH and ELONGATION of INTERIOR SEGMENT can be defined between touching chords of the segment in order to avoid arbitrarily long distances associated with ligatures at the segment ends. Note that AA' is subtracted from the true SA segment and BB' is added.

* The symmetric chord axis was introduced by Blum.[3] The SA coordinates are replaced by the symmetric chord (SC) coordinates. Each axis point is replaced by the bisector point of the chord between the touching points, and R is replaced by the half chord length. Note that the symchord is perpendicular to its axis and makes equal angles with the boundary. Two changes result. First, the SC is not connected across branches, leaving a residual triangle of area (but not boundary) at the branch. It must be incorporated in branch descriptions. Second, finite contact points lead to disjoint symmetric chord axes. It must be given a suitable definition to include the entire boundary.

subtracted at the branch is $R\cos\beta$. Notice that at branch B, $\beta > 90°$ and the correction must be added. This measure is applied to an end segment in Fig. 16(A). A different measure for them is suggested later, however.

These symchord properties over the whole segment do not concern us here. We simply want a single length determined by the symchord ends. The chord length

$$\ell_{sc} = \int_{\text{branch 1}}^{\text{branch 2}} ds - \sum_{1}^{2} R_i \cos\beta_i.$$

This definition is extended to features as well as segments. The value of ℓ_{sc} has the virtue of being invariant to SA curvature, a property that is important when flexure is considered.

A number of other measures which ignore axis curvature are not considered. Maximum width is the largest symchord found in the segment; minimum width, the shortest. When this width occurs interior to the segment it is the same as the SAT width (2), because the touching radii are 180° apart. An interior max width point can only occur between opening and closing cups. An interior min width point occurs only between closing and opening flares. Average width measures may be based on either symchord or the SAT radius function. Since symchord coordinates are used here it is convenient to define median width as (max + min)/2. In addition, β max, β min, k_β max, k_β min, are interesting measures along a segment that is invariant to axis curvature.

Consider next some relational measures. Expansion, the ratio of far end symchord width to near end symchord width is one measure. It may be measured for both features and segments. Being transitive, it allows the relation of chord widths to be compared over the whole segment and later on, over the whole object. Other end measure ratios may be defined in terms of β and k_β.

Another interesting relational measure is elongation, the ratio of symchord length to mean symchord width (also max and min if so desired). Figure 16(B) and (C) show end segments with elongation > 1 and < 1 respectively. Normally elongation is not ever < 1 because it refers to the ratio of a larger to a smaller quantity. As defined here, length is the direction of the axis, thus it becomes possible for width to exceed length in this measure.

For end segments, the symchord measure of length is not the preferred measure. Rather, segment length is the axis length minus the radius of the branch disc, plus the radius of the end disc. See Fig. 16(B) and 16(C) again. For end segments this seems more natural because the length reflects the change in boundary for the segment. Note however, that other variations are possible.

An even more fundamental aspect of choice must be mentioned. We have used axis and symchord measures. These do not obviate simple Euclidean measures, either alone or in combination with the measures suggested here. A most important example is the Euclidean distance between branch points or between a branch and end point.

A variety of segment curvature measures can also be defined. Some examples follow.

(a) $\dfrac{\text{left } B/A \text{ weight}}{\text{right } B/A \text{ weight}}$

(b) $\displaystyle\int_1^2 |k_\alpha|\, ds$

(c) $\displaystyle\int_1^2 k_\alpha\, ds$

(d) $\dfrac{\int |k_\alpha|\, ds}{\int k_\alpha\, ds}$

(e) $\dfrac{\text{axis length}}{\text{Euclidean distance}}$

The axis length in (e) above may be any of the segment or feature lengths introduced earlier.

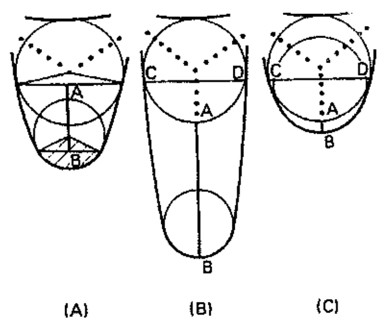

Fig. 16. Axis length and elongation of an END SEGMENT. The length of the end disc is added. Elongation is (axis length)/(branch symchord). (B) and (C) show elongations > 1 and < 1, respectively. Note that elongation can be < 1 since the side of attachment to the branch disc is of concern.

8. SEGMENT ENDS AND JOININGS

The simplified segments were defined as a collection of normal axis points, occurring between either the branch or end point delimiters. The characterization of the branch and end axis points properties are discussed now.

A branch point corresponds to three or more simplified segments coming together. Alternately, one can think of a simplified segment splitting at the branch point into two more segments. For illustration purposes, a three lobed figure with a single branch point, B, is shown in Fig. 17. A disc centered at B touches the object border in three places, $T1$, $T2$, and $T3$. The symmetric axis is dotted and the end points labeled $E1$, $E2$, and $E3$.

Branch points are quantified using seven measures based on order, width, contact, busyness, ligature,

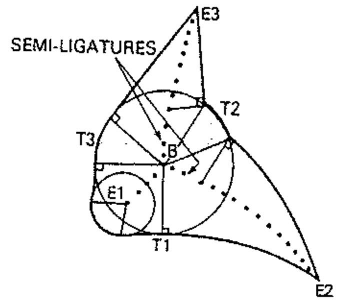

Fig. 17. An object which shows a variety of types of joining at a branch point.

expansion and elongation. While others are possible, these seem intuitive and quantify the joinings of simplified segments nicely.

The branch order is defined as the number of simplified segments connected to the branch disc. Alternatively, it is the number of touchings of the branch disc. The order of point B in Fig. 17 is three.

A branch point is said to be maximal if the angles between adjacent touching points which meet at the branch point are all less than 180°. The branch point is also maximal if the largest angle is equal to 180° and attached to a cup. When a branch point is maximal, the object decreases in width in all directions making it a local width maximum. B in Fig. 17 is such a point. Note that a branch point cannot be minimal as there cannot be three angles of 180° or greater.

The labels, finite or point, are used to categorize the type of contact the branch disc makes with the object border. As defined earlier, a border touching is considered to be finite if the disc which touches the border coincides with the border over an interval. If any of the border contacts of a branch disc are finite, the branch point has finite contact. Otherwise, it is point contact. The branch point, B, of Figs. 1 and 2 have point contact, while B of Fig. 17 has finite contact.

An indication of the complexity of the shape in the vicinity of a branch point is quantified as a busy measure, the number of branch and end points inside the branch disc. A branch point with neither another branch point nor an end point in its branch disc has a busy measure of zero. The branch points in Figs. 1 and 2 have zero busy measure. The branch of Fig. 17 has a busy measure of one due to the inclusion of the end point, $E1$, inside the branch disc.

The ligature number of a branch point is the count of semiligatures on the simplified segments as they are attached to the branch point. It gives an indication of discontinuity in the object border at the connection of simplified segments. In Fig. 17, the branch disc at $T2$ generates two semi ligatures.

The branch expansion is the ratio of each pair of adjacent symchords of the branch. Thus it is multivalued. This allows for the transfer of chord ratios across a branch. The product of all the ratios must equal 1.

Branch elongation is the ratio of the largest touching chord to the smallest touching chord of the branch.

The end points at the end of a simplified segment with nothing attached, represent the closing of the figure. A disc associated with the end point joins in a circular arc of radius R, the right and left pieces of the border which are accounted for by the simplified segment. When $R = 0$, as in $E3$ of Fig. 17, the closing is a corner. Note again that the disc may have a single touching point, as in Fig. 2, or finite contact as in end disc, $E1$, of Fig. 17. The contact angle is also a feature of an end point.

9. GRAPH DESCRIPTION

The SAT description of the total object is combined in a graph structure of its parts. A node is created for each of the branch and end points of the SAT. The simplified segments are represented by a pair of directed arcs. The topology of the graph is not enough to preserve the necessary SAT information. An order relationship on the directed arcs at each node must be imposed. In what follows, a clockwise ordering is assumed. The ordering is based on the order in which simplified segments appear around the branch point when traversed in clockwise order. An end point has exactly one simplified segment attached. A pair of arcs (one to and one from) is created for each simplified segment connected to a node. The outgoing arc of a pair occurs to the left of the incoming arc. In this way the arcs and nodes represent sequential boundary analysis of a clockwise shape traversal.

The dog bone of Fig. 1 provides an example. It has four end points ($E1$, $E2$, $E3$, $E4$) and two branch points ($B1$, $B2$) producing the graph of Fig. 18. Due to the ordering of the arcs in the graph, a traversal beginning at node $E1$ back to $E1$ visits the nodes in the order $E1$, $B1$, $B2$, $E2$, $B2$, $E3$, $B2$, $B1$, $E4$, $B1$, $E1$. Note that end connected segments traverse their arc pairs in succession, while other segments do not. This traversal is unique because the directed graph has an arc order relationship at the nodes. A counterclockwise graph ordering with all arcs reversed, may be considered the dual of the graph in Fig. 18.

A short digression on holes follows. An object with a hole has two unconnected borders: an interior and an exterior one. The graph is no longer a tree, but has a loop in it. If one begins a traversal on the exterior border, the traversal visits only exterior segments. If one begins on an interior segment, one traverses only interior segments. For segments on the loop, both borders are represented. Only half will be traversed. The interior border is traversed in a counterclockwise

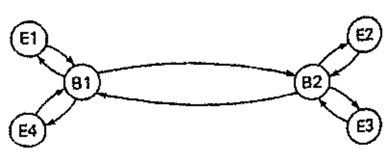

Fig. 18. GRAPH of the "bone" of Fig. 1.

direction. Note that an object with a hole in it may have no end points. A circular disc with an eccentric circular hole is such an object. Its *SA* is an ellipse.

If we let B_i be the order of branch points, then the number of end points, E, expected for a graph with H distinct loops is,

$$E = 2 - 2H + \sum_i (B_i - 2).$$

An in-depth analysis of graphs for *SATs* of objects with holes is beyond the scope of this paper.

In the remainder of this paper, we consider only graphs for objects without holes. Such a graph is a tree, and a path starting and ending at the same node visits all nodes in the graph.

In the discussion of simplified segments, it was pointed out that two equivalent but opposite descriptions are possible, depending on the start point of the intrinsic representation. These two equivalent descriptions correspond to the pair of arcs generated in the graph structure. Thus, they each have an appropriate position in a graph data structure. Similarly, the description that results from the branch and end point analysis is stored at the appropriate nodes in a graph data structure.

An interesting property of the graph structure described here is that if the traversals are always clockwise, all sided properties can be left sided. This is due to the fact that the left side on one arc is equivalent when properly modified to the right side of its dual.

An example of the properties stored with nodes and arcs of the graph appears later.

10. RECTILINEAR OBJECTS AND FLEXURE

The notion of shape features entails a set of equivalence classes of shapes. The features discussed to this point use the algebraic sign and zero of a variety of measures. Using all of these or some subsets we may define an equivalence class of objects. Two of these are discussed here.

Since the 0 condition has no variation, it can only be elongated or widened and stay within the class. Thus, convex rectilinear objects consist of wedges of $R = 0$ ends, and worms of straight axes, with point contact branches. They are objects with $k_\alpha = k_\beta = 0$ simplified segments, $R = 0$ end discs, and point contact branches. For non-convex rectilinear objects, one must add segment features with semi-ligatures and straight opposing boundaries, as well as full ligature segments. These are illustrated by *SL* and *L* of Fig. 19 respectively. As expressed above, this is a property of the boundary condition. It may also be dealt with on the axis. Expressed on the axis these are points at which

$$|k_\alpha| = |k_\beta| = \frac{\cos\beta}{R}.$$

Using these specifications, it is easy to determine that an object consisting of four wedges and a worm is a trapezoid. The object angle of the wedges or the contact angles of the branch point must be used to define it as a rectangle. The internal segment elongation and width specify the complete rectangle. (Using an SAT of higher orders, the rectangle can be defined by just 2 worms.[3])

One very interesting SAT constraint is flexure. The width function is maintained exactly, while no constraint is put on the axis curvature. Flexure is used here to define the permissible deformation of an object, which results in an SAT where width functions are that of the original object. See Fig. 20. Note that the ligature (outside corner) associated with each axis point is the limit of flexure at that point. Points of finite contact have that contact equally divided amongst all touching points under flexure. A canonical form for an object under flexure is called symmetrized, i.e. the axis is straightened, $k_\tau = 0$ everywhere except at branch points. Examples of various forms of flexure of the basic width shapes are found in Fig. 8. Figure 21(A) shows symmetrization for an unbranched object. Figure 21(B) shows that under symmetrization, it is possible to have parts of the object overlap. This must be treated in the same way as a Riemann surface. They form two or more unconnected layers. (Note that Riemann surface objects constitute a greater variety of shapes than allowed here.) One particularly interesting property of flexure in 2 dimensions is its keeping both perimeter and area constant.

11. SHAPE PERTURBATIONS

Small perturbations on an object border can result in creating new branches, end points, and simplified segments of significant length. That sensitivity of an SAT to shape noise on the object border can be large,

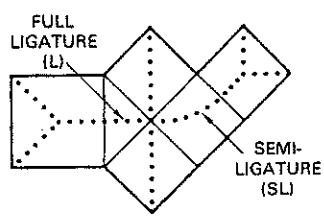

Fig. 19. The symmetric axis of rectilinear object comes from worms, wedges and ligatures, yielding an axis that consists of only straight line segments and parabolic arcs.

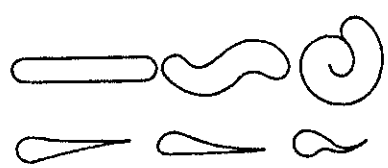

Fig. 20. FLEXURE consists of modifying axis curvature while maintaining object width. Flexure maintains both area and perimeter of the object.

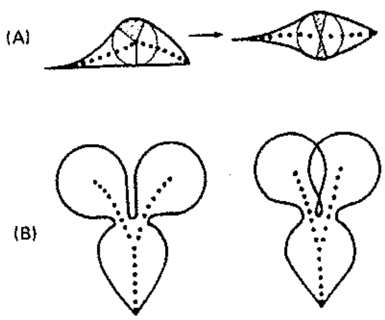

Fig. 21. SYMMETRIZATION is a canonical form of a flexing shape; axis curvature is set =0 everywhere. The axis consequently consists only of straight lines. (A) Shows a case with a normal finite contact point. The contact angle is divided equally among the contacts. (B) Shows a possible difficulty of symmetrization of an object at a branch point, the symmetrized object may overlap itself. Resolution requires the introduction of Riemann type surfaces.

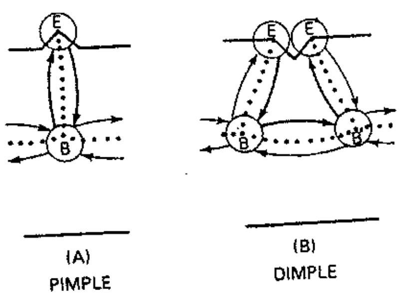

Fig. 22. Change in axes generated by perturbation of a figure boundary. The importance of such perturbations in generating boundary length may be determined by taking the integral of boundary axis weight along EB and BE in the PIMPLE and along the inner loop EB, BB' and $B'E'$ in the DIMPLE.

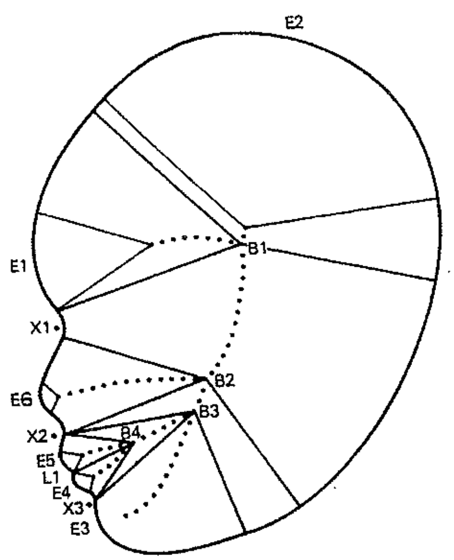

Fig. 23. The symmetric axis of a face in profile, along with the touching radii of its branch and end points. The segmentation that results is labeled in the graph representation. Note that negative (exterior) curvature maxima come from inside descriptors. They are marked $X1$, $X2$ and $X3$.

has been pointed out. A variety of approaches can be used to cope with the problem. Note that in biological forms perturbations are frequently not distortions; rather they are important features. The nose on the profile of Fig. 23 is not noise but a shape "pimple". These effects are detected and evaluated by the B/A weight function here.

Consider the two types of deformation "pimple" and "dimple", illustrated in Fig. 22. Each of these perturbations has a well defined SAT result.

A pimple creates one simplified segment which has a branch point and an end point at the pimple side. A full ligature exists on the axis of the simplified segment starting at the branch point and results in a low weight for the entire segment. Recall, the segment weight is a measure of boundary length to axis length. The exact value of the weight is a function of the branch width, and the pimple size. For pimples due to noise, the weight must be close to zero, i.e. the characterization of noise must assume a small perturbation. The low axis weight can be further described by observing that the weight of individual axis points will be zero, for a portion of the axis (contiguous) and will then have some finite value until the end point.

A dimple results in three additional simplified segments, two of which have end points at the dimple (opposite sides). The third connects the two segments by an interior branch–branch simplified segment. The path from one of the end points to the other passes through the two branch points. The dimple side of the path will have zero weight except at the portion of the axis close to the dimple. The existence of a dimple is characterized by two end points with a small euclidean distance between them and a path with a low sided weight.

For both the pimple and dimple situation, the weight of the path integrated over the axis is the amount of border used in the perturbation. Thus, the decision as to whether or not the perturbance is noise or data can be made with a significant amount of knowledge.

In closing the discussion on perturbation, consider the effect on a graph representation of the unperturbed object. Figure 22 implies the graphs resulting from the pimple and a dimple.

There is no question that these examples are simple idealizations. Real graphs are much more complex. However, the changes in the graph are localized. For our case, one simplified segment is perturbed, changing one pair of arcs into a more complicated substructure.

The authors acknowledge the above to be a simplification and preliminary analysis. Note no algorithm for fixing the boundary is yet claimed. Nevertheless, the claim is made that by the use of weights the SAT can be made insensitive to noise, from an analysis point of view. Furthermore, when small perturbations represent real data, as they often do on biological objects, the SAT has an important asset over supposedly noise insensitive approaches.

12. CONCLUSION: TRYING IT OUT

The time has come to look at the global picture. A profile of a three year old child is used as the object. The profile and its SAT are in Fig. 23. The touching places for all branch and end nodes are marked. Four parts of the profile have outside centers of curvature, labeled $X1$, $X2$, $X3$, and the semi ligature $L1$.

The SAT graph for the profile is in Fig. 24. Each of the end nodes has been labeled with the part of the head which it represents. The details of the nodal and simplified segments are too large for enumeration here. As an example, the simplified segment $B3E3$ which generates a pair of arcs, and has both a branch and end node is discussed in detail.

The node properties are in Table 5. The numeric values represent approximations made on the profile figure (before it was reduced for publication). Contact angle is zero unless a finite contact with the border occurs at the node circle. For the branch node, the top nine values represent the branch label, and the properties discussed for branch nodes. Each branch node has "order" lines of four values to describe the connection to simplified segments. The sequence of these lines is used to represent the clockwise traversal. The four values reflect the node at the other end of the simplified segment: the angle of the opening to the segment, the angle of finite contact on the branch (if any) to the next segment, and the symchord ratio of the current and next segment. "Next" as used here is circular, thus E3 is next to $B2$ as $B2$ is next to $B4$.

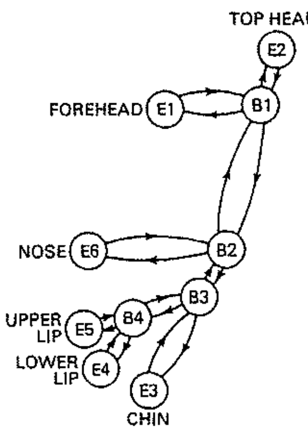

Fig. 24. Graph of the symmetric axis segmentation of face of Fig. 23.

Table 5. An example of point (branch and end) node properties

Branch node			
Label	Radius	Point contact	
B3	35	*PT*	
Order	Business	Width	
3	5	*non max*	
Elongation	Ligature No.	Weight	
3	0	0	
Connect to	Segment angle	Contact angle	Expansion
E3	75	0	2
Connect to	Segment angle	Contact angle	Expansion
B4	230	0	0.333
Connect to	Segment angle	Contact angle	Expansion
B2	255	0	1.5
End node			
Label	Radius	Contact angle	
E3	6	60°	
Ligature	Weight		
0	9		

Table 6. An example of arc node properties

Arc description	*Arc* B3E3	*Arc* E3B3
1. Axis length	40	40
2. Euclidean distance	35	35
3. Symchord near	40	7
4. Symchord far	7	40
5. Min width	7	7
6. Max width	40	40
7. Elongation	11/40	11/40
8. Axis curvature/arc length	0.9	0.9
9. Expansion	0.145	0.145
10. Weight left	44	4
Weight right	4	44
11. No. finite contacts	0	0
12. No. width labels	1	1
13. No. axis labels	2	2
14. No. left boundary labels	1	1
15. No. right boundary labels	1	1

	Arc B3E3		*Arc* E3B3	
Width	Closing cup 40		Opening cup 40	
Axis	Straight 17	Right spiral in 23	Left spiral out 23	Straight 17
Left boundary	Convex 40		Concave 40	
Right boundary	Concave 40		Concave 40	

The two arc descriptions $B3E3$ and $E3B3$ are almost identical as expected. The fifteen segment properties described earlier in the text are presented in Table 6. Also found in Table 6 are the four string descriptions for the simplified segments. Each of the string descriptions has a corresponding length of axis term, to preserve the ability to interrelate the string descriptions.

The SAT produces a wealth of structure data to describe the shape of the profile. The exact form of these data, and their utility is of course dependent on the analysis goals of the user. This paper is meant to open a door rather than completely solve a problem. Having treated shape in an open ended manner, it remains for the reader to choose amongst the described properties, or extend the approach with new measures.

REFERENCES

1. H. Blum, A transformation for extracting new descriptors of shape, in *Models for the Perception of Speech and Visual Form*, (ed. Walthen Dunn), MIT Press. Proc. of Meeting held Nov. 1964, pp. 362–380 (1967).
2. H. Blum, A new model of global brain function, *Perspect. Biol. Med.* **10**, 381–408 (1967).
3. H. Blum, Biological shape and visual science, *J. Theoret. Biol.* **38**, 205–287 (1973).
4. H. Blum, A geometry for biology, in *Mathematical Analysis of Fundamental Biological phenomena*, *Ann. N.Y. Acad. Sci.* **231**, 19–30 (1974).
5. L. Calabi and W. E. Hartnett, A theorem for closed non-convex sets, *Proc. Am. Math. Soc.* **19**, 1495–1498 (1968).
6. L. Calabi and W. E. Hartnett, Shape Recognition, Prairie Fires, Convex Deficiencies and Skeletons, *Am. Math. Mon.* **75**, 335 (1968)
7. U. Montanari, A method for obtaining skeleton using a quasi-euclidean distance, *J. ACM* **15**, 600–624 (1968).
8. R. N. Nagel and H. Blum, A symmetric axis basis for object recognition and description, *Proc. of IEEE Meeting on Decision and Control*, 168–170 (1976).
9. A. Rosenfeld and J. L. Pfaltz, Sequential operations in digital picture processing, *J. Assist. Comput. Mach.* **13**, 471 (1966).
10. A. Rosenfeld and J. L. Pfaltz, Distance functions on digital pictures, *Pattern Recognition* **1**, 33 (1966).

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In 1960, he undertook the present phase of his work: exploring new mechanisms that would expand out understanding of the global integrative process of biology. His work has focused primarily on biological and visual shape, and the perceptual structure of the nervous system. This work started at the Air Force Cambridge Research Laboratories, Bedford, Mass. in 1960. It continued, from 1967 to the present, in the Division of Computer Research and Technology of the National Institutes of Health, where he is currently in the Laboratory for Statistical and Mathematical Methodology.

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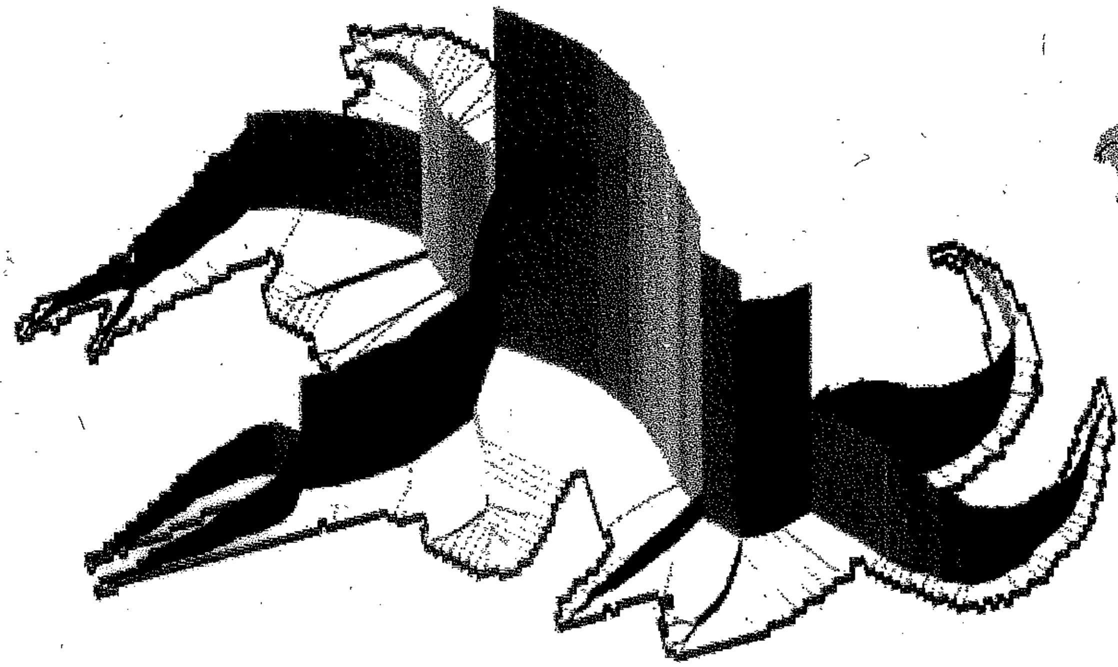
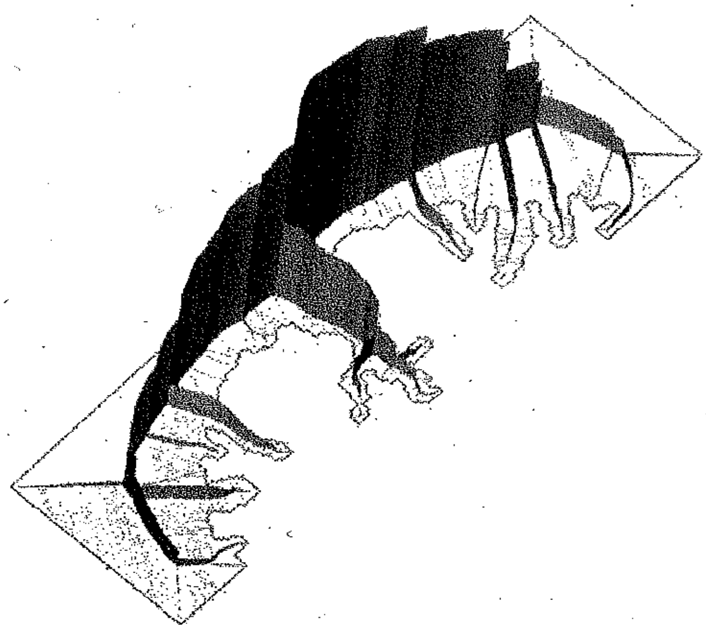
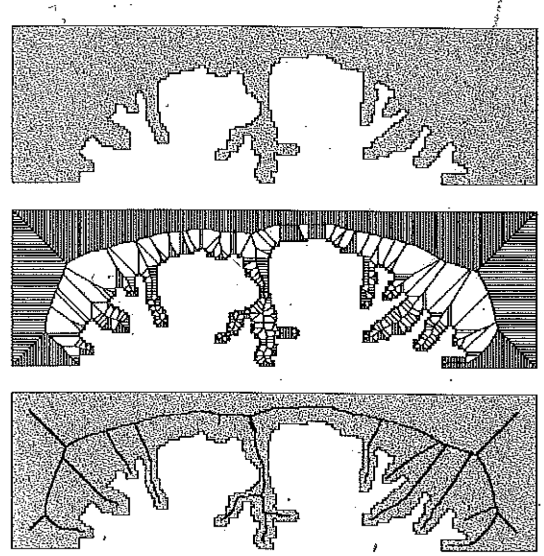
Dr. Nagel has published several scientific papers on computer animation, image processing, shape analysis, and simulation. He is a member of IEEE, ACM, SPIE, MAA, Phi Kappa Phi, and Sigma XI.

Skeleton-Pyramid

3D Plot of Chord Residual







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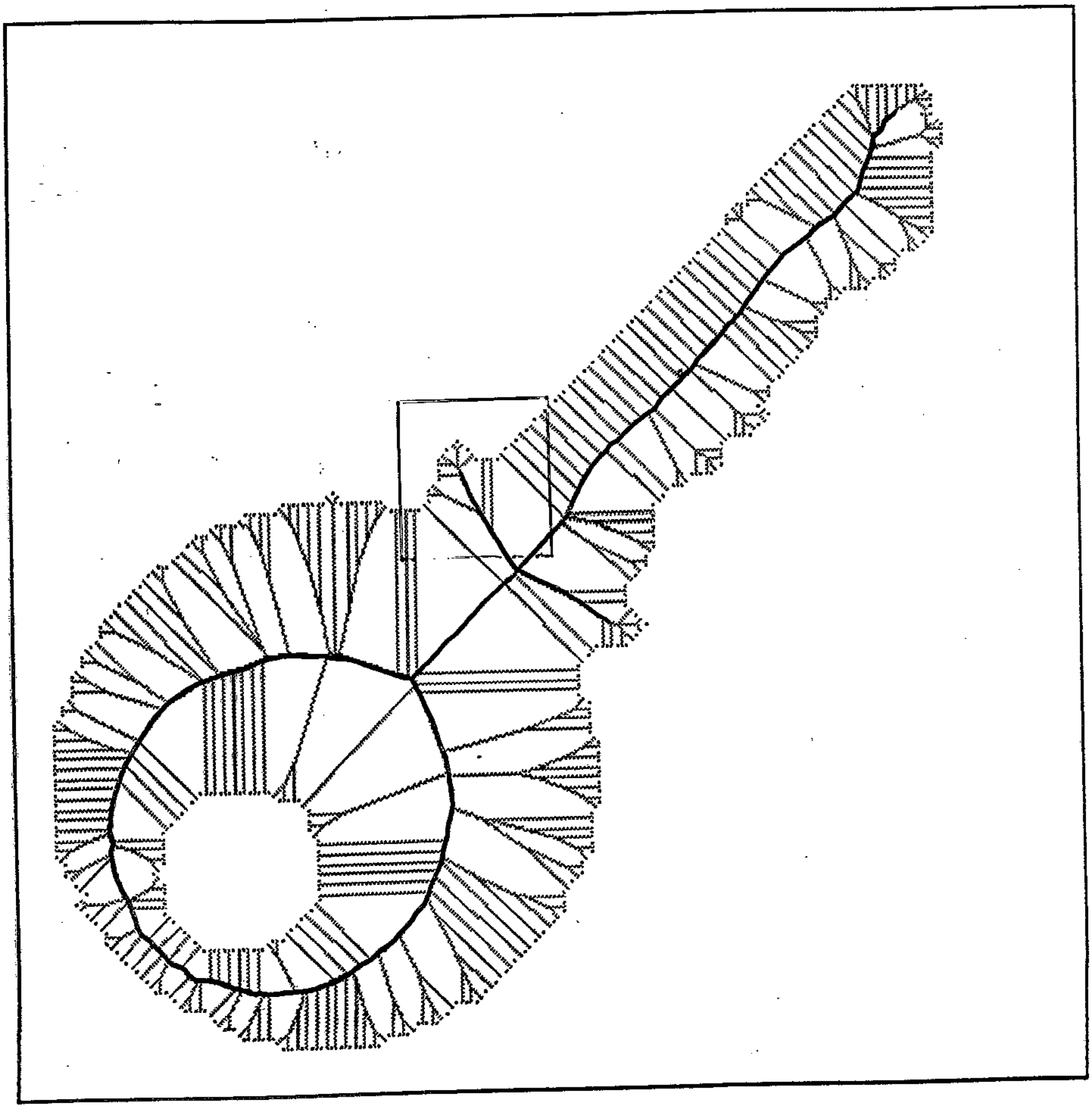
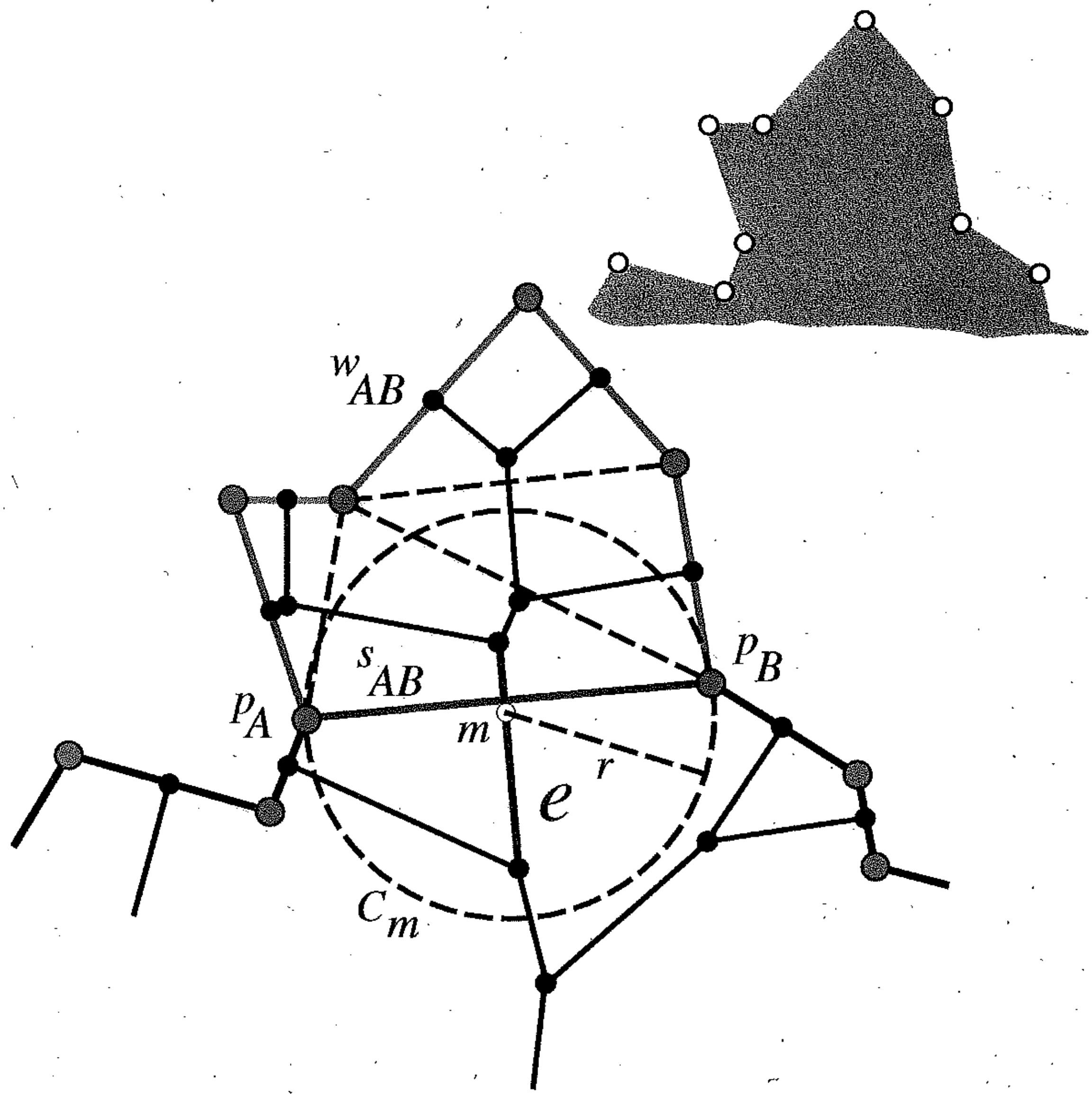
Voronoi-Diagramm des Objektinneren



Residual Functions



$$\Delta R_H(e) \stackrel{\text{def}}{=} w_{AB} - s_{AB}$$

Chord Residual