Lecture 2: FEV 3,4: Gaussian Kernel, Gaussian Derivatives

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Aperture function: Operator

- Unconstrained front-end: Unique solution to aperture function is Gaussian kernel
- Many derivations, all leading to Gaussian kernel

\[
\sigma = 1; \; \text{Plot} \left[ \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{x^2}{2\sigma^2}}, \{x, -4, 4\}, \text{ImageSize} \to \right]
\]

Figure 2.1 The Gaussian kernel with unit standard deviation in 1D.
Gaussian Kernel

\[ G_{1D}(x; \sigma) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{x^2}{2\sigma^2}}, \quad G_{2D}(x, y; \sigma) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}, \quad G_{ND}(\mathbf{x}; \sigma) = \frac{1}{(\sqrt{2\pi} \sigma)^N} e^{-\frac{\|\mathbf{x}\|^2}{2\sigma^2}} \]

Normalization to 1.0

Figure 3.2 The Gaussian function at scales \( \sigma = .3, \sigma = 1 \) and \( \sigma = 2 \). The kernel is normalized, so the total area under the curve is always unity.
Cascade Property, Self Similarity

• Convolution of 2 Gaussian kernel produces another Gaussian kernel
• Squares of σ’s add up

\[ g_{\text{new}}(\vec{x}; \sigma_1^2 + \sigma_2^2) = g_1(\vec{x}; \sigma_1^2) \otimes g_2(\vec{x}; \sigma_2^2) \]

σ := Simplify \[ \int -\infty \quad \text{gauss}[\alpha, \sigma_1] \quad \text{gauss}[\alpha - x, \sigma_2] \quad d\alpha, \quad (\sigma_1 > 0, \sigma_2 > 0) \]

\[ \frac{e^{-\frac{x^2}{2 \sigma_1^2}}}{\sqrt{2 \pi \sigma_1^2}} \frac{e^{-\frac{(x-\alpha)^2}{2 \sigma_2^2}}}{\sqrt{2 \pi \sigma_2^2}} \]
Scale Property

- Gaussian kernel is Green’s function of linear, isotropic diffusion equation

\[ g := \frac{1}{2 \pi \sigma^2} \exp\left[-\frac{x^2 + y^2}{2 \sigma^2}\right] \]

\[ \frac{\partial^2 L}{\partial x^2} + \frac{\partial^2 L}{\partial y^2} = L_{xx} + L_{yy} = \frac{\partial L}{\partial t} \]

\[ t = 2 \sigma^2 \]
Figure 3.4 The error function $\text{Erf}[x]$ is the cumulative Gaussian function.
Gaussian vs. Dirac

- Gaussian function is a blurred version of the Delta Dirac function.
- Cumulative Gaussian function (Erf) is a blurred version of the Heavyside step edge.

\[ \lim_{\sigma \downarrow 0} \left( \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{x^2}{2\sigma^2}} \right) = \delta(x) \]
Separability

\[ g_{2D}(x, y; \sigma_1^2 + \sigma_2^2) = g_{1D}(x; \sigma_1^2) \otimes g_{1D}(y; \sigma_2^2) \]

DisplayTogetherArray[{Plot[gauss[x, \(\sigma = 1\)], {x, -3, 3}],
  Plot3D[gauss[x, \(\sigma = 1\)] gauss[y, \(\sigma = 1\)], {x, -3, 3}, {y, -3, 3}]]},
ImageSize -> 440];

Figure 3.7 A product of Gaussian functions gives a higher dimensional Gaussian function. This is a consequence of the separability.
Relation to binomial coefficients

\[ \text{Expand} \left[ (x + y)^{30} \right] \]

Figure 3.8 Binomial coefficients approximate a Gaussian distribution for increasing order.
Diffusion Equation

- Gaussian is the solution of the linear diffusion equation (with: $t = 2\sigma^2$)
- Intensity is diffused over time (scale) in all directions isotropically

\[
\frac{\partial L}{\partial t} = \frac{\partial^2 L}{\partial x^2} + \frac{\partial^2 L}{\partial y^2} = \Delta L
\]
FEV 4: Gaussian Derivatives
Derivatives

Figure 4.1 Plots of the 1D Gaussian derivative function for order 0 to 7.
Higher Dimensions

x: 1st  y: 0

x: 2nd  y: 0
Other Families: Gabor Filters

\[
gabor[x, \sigma] := \sin[x] \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{x^2}{2\sigma^2}\right]
\]