Appendix

APPENDIX A

Forms and differential forms

In this Appendix, we recall the definitions and the main basic properties of the multi-vectors, the m-forms and the differential m-forms on $\mathbb{R}^d$. We present here the material which is needed for the definition of currents in the framework of this thesis. Some properties are given without any proofs. For more detailed presentation, we refer the reader to any handbook of differential geometry such as [Lang 1962, Sternberg 1964, do Carmo 1994].

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A.1 Wedge product and m-multivectors

A.1.1 Definitions

The wedge product is a generalization of the cross-product, which extends the usual measure of areas and volumes in 3D. Theoretically, the wedge product $u \wedge v$ between u and v, two vectors in $\mathbb{R}^d$, is equal to the tensor product $u \otimes v$ up to any linear combination of the form $x \otimes x$. The set of all wedge product between any pair of vector of $\mathbb{R}^d$ is called the exterior algebra over $\mathbb{R}^d$ and is denoted $\Lambda(\mathbb{R}^d)$. Formally, we have this abstract (and non tractable) definition:

Definition A.1 (exterior algebra over $\mathbb{R}^d$)**.** *The exterior algebra $\Lambda(\mathbb{R}^d)$ is defined as the quotient algebra of the tensor algebra by the two-sided ideal I generated by all elements of the form $x \otimes x$ such that $x \in \mathbb{R}^d$.*

All what we need to know about the wedge product is the two following properties (which is a definition of the wedge product, in some sense): the wedge product is a bilinear operation and vanishes if two vectors are equals:

$$\begin{cases} (\lambda u + v) \wedge w = \lambda(u \wedge v) + u \wedge w \\ u \wedge u = 0 \end{cases} \tag{A.1.1}$$

for all $u, v \in \mathbb{R}^p$ and $\lambda \in \mathbb{R}$.

As a direct consequence of these properties, we have that: $u \wedge v = -v \wedge u$. Indeed, we have $(u+v) \wedge (u+v) = 0 = u \wedge v + v \wedge u$.

Then, we extend the wedge product between two vectors to the wedge product between any family of m-vectors via the associativity law: $u \wedge v \wedge w = (u \wedge v) \wedge w$. This leads to the definition of the mth exterior power of $\mathbb{R}^d$:

Definition A.2 (mth exterior power of $\mathbb{R}^d$)**.** *We call the* mth *exterior power of* $\mathbb{R}^p$ *the vector space spanned by the vectors of the kind* $u_1 \wedge \ldots \wedge u_m$ *for all* $u_i \in \mathbb{R}^d$*. We denote this space* $\Lambda^m \mathbb{R}^p$*. The vectors in* $\Lambda^m \mathbb{R}^p$ *are called* m*-multivectors.*

As a consequence of this definition, the m-multivector $u_1 \wedge \ldots \wedge u_m$ is totally antisymmetric. This means that it vanishes as soon as two u_i are equals. More generally, we have for any permutation of $\{1, \ldots, m\}$ σ:

$$u_{\sigma(1)} \wedge \ldots \wedge u_{\sigma(m)} = \operatorname{sign}(\sigma) u_1 \wedge \ldots \wedge u_m, \tag{A.1.2}$$

where $\operatorname{sign}(\sigma)$ denotes the signature of the permutation σ.

Moreover, we have the following property:

Proposition A.3. *Let* $(u_i)_{i=1\ldots m}$ *be* m *vectors in* $\mathbb{R}^d$ *and* A *a* m*-by-*m *matrix. Let* $v_i = \sum\limits_{j=1}^{m} A_{ij} u_j$*, then*

$$v_1 \wedge \ldots \wedge v_m = |A| \, u_1 \wedge \ldots \wedge u_m, \tag{A.1.3}$$

where $|A|$ *denotes the (signed) determinant of the matrix* A*.*

Proof. By linearity, we have:

$$\begin{aligned} v_1 \wedge \ldots \wedge v_m &= \left(\sum_{j=1}^{m} A_{1j} u_j \wedge \ldots \wedge \sum_{j=1}^{m} A_{mj} u_j \right) = \\ &= \sum_{p \in \mathcal{P}_m} A_{1p(1)} \ldots A_{mp(m)} \left(u_{p(1)} \wedge \ldots \wedge u_{p(m)} \right) \\ &= \left(\sum_{p \in \mathcal{P}_m} \operatorname{sign}(\sigma) A_{1p(1)} \ldots A_{mp(m)} \right) u_1 \wedge \ldots \wedge u_m \\ &= |A| \, u_1 \wedge \ldots \wedge u_m, \end{aligned} \tag{A.1.4}$$

by definition of the determinant ($\mathcal{P}_m$ denotes the set of $m!$ permutations of $\{1, \ldots, m\}$). ■

A.1.2 Euclidean basis for multivectors

Let $(\epsilon_i)_{i=1\ldots d}$ be the canonical basis of $\mathbb{R}^d$, so that each vector u_i is decomposed into $\sum_{k=1}^{d} u_i^k \epsilon_k$ Thanks to the linearity and the alternating properties of the wedge product we have:

$$
\begin{aligned}
u_1 \wedge \ldots \wedge u_m &= \left(\sum_{k_1}^{d} u_1^{k_1} \epsilon_{k_1} \right) \wedge \ldots \wedge \left(\sum_{k_m=1}^{d} u_m^{k_m} \epsilon_{k_m} \right) \\
&= \sum_{p \in C_m^d} \sum_{\sigma \mathcal{P}_m} u_1^{\sigma(p(1))} \ldots u_m^{\sigma(p(m))} \epsilon_{\sigma(p(1))} \wedge \ldots \wedge \epsilon_{\sigma(p(m))} \\
&= \sum_{p \in C_m^d} \left(\sum_{\sigma \in \mathcal{P}_m} \operatorname{sign}(\sigma) u_1^{\sigma(p(1))} \ldots u_m^{\sigma(p(m))} \right) \epsilon_{p(1)} \wedge \ldots \wedge \epsilon_{p(m)},
\end{aligned}
\tag{A.1.5}
$$

where C_m^d denotes the set of all subsets of m elements in $\{1, \ldots, d\}$ and $\mathcal{P}_m$ the set of all permutations of $\{1, \ldots, m\}$. This shows that the vectors $\varepsilon_{i_1} \wedge \ldots \wedge \varepsilon_{i_m}$ for $1 \leq i_1 < \ldots < i_m \leq d$ spanned the vector space $\Lambda^m \mathbb{R}^d$. One can easily show that these vectors are linearly independent. Therefore, the space $\Lambda^m \mathbb{R}^d$ is of dimension $\binom{d}{m}$. Then we write any m-multivectors on $\mathbb{R}^d$ as:

$$
u = \sum_{1 \leq i_1 < \ldots < i_m \leq d} u_{i_1 \ldots i_m} \epsilon_{i_1} \wedge \ldots \wedge \epsilon_{i_m} \tag{A.1.6}
$$

We provide $\Lambda^m \mathbb{R}^p$ with the standard Euclidean inner-product and norm:

$$
|u|^2 = \sum_{1 \leq i_1 < \ldots < i_m \leq d} (u_{i_1 \ldots i_m})^2 \tag{A.1.7}
$$

Of course, this definition does not depend on the choice of the basis.

A.1.3 Particular cases

We study now some particular cases of interest:

- if $m > d$, $\Lambda^m \mathbb{R}^d = \{0\}$,
- if $m = 0$, $\Lambda^0 \mathbb{R}^d$ is of dimension 1: this is the space of scalars $\mathbb{R}$ itself,
- if $m = 1$, $\Lambda^1 \mathbb{R}^d$ is of dimension d: this is the vector space $\mathbb{R}^d$ itself,
- if $m = d - 1$, $\Lambda^{d-1} \mathbb{R}^d$ is of dimension d. The decomposition of a $d-1$-multivector $u_1 \wedge \ldots \wedge u_{d-1}$ on the basis $(\tilde{\varepsilon}_i^d = \varepsilon_1 \wedge \ldots \varepsilon_{i-1} \wedge \varepsilon_{i+1} \ldots \wedge \varepsilon_d)_{i=1,\ldots,d}$ (which denotes the set of d-multivectors $\varepsilon_1 \wedge \ldots \wedge \varepsilon_d$ in which the vector ε_i is missing) leads to:

$$
u_1 \wedge \ldots \wedge u_{d-1} = \sum_{i=1}^{d} \eta_i \tilde{\varepsilon}_i^d, \tag{A.1.8}
$$

where

$$
\eta_i = \sum_{\sigma \in \mathcal{P}_m; \sigma(d) = i} \operatorname{sign}(\sigma) u_1^{\sigma(1)} \ldots u_{d-1}^{\sigma(d-1)} \tag{A.1.9}
$$

This shows that the vector η is such that for any vector $\alpha \in \mathbb{R}^d$:

$$\begin{aligned}\eta^t\alpha = \sum_{i=1}^{d}\eta_i\alpha_i &= \sum_{i=1}^{d}\sum_{\sigma\in\mathcal{P}_m;\sigma(d)=i}\operatorname{sign}(\sigma)u_1^{\sigma(1)}\dots u_{d-1}^{\sigma(d-1)}\alpha_i \\ &= \sum_{i=1}^{d}\sum_{\sigma\in\mathcal{P}_m;\sigma(d)=i}\operatorname{sign}(\sigma)u_1^{\sigma(1)}\dots u_{d-1}^{\sigma(d-1)}\alpha_{\sigma(d)} \\ &= \sum_{\sigma\in\mathcal{P}_m}\operatorname{sign}(\sigma)u_1^{\sigma(1)}\dots u_{d-1}^{\sigma(d-1)}\alpha_{\sigma(d)} = \det(u_1,\dots,u_{d-1},\alpha)\end{aligned} \tag{A.1.10}$$

Therefore, any $d-1$-multivector $u_1\wedge\dots\wedge u_{d-1}$ is associated to a vector η such that $\eta^t\alpha = \det(u_1,\dots,u_{d-1},\alpha)$ for every vector α (see below the instance in 3D)

- If $m=d$, $\Lambda^d\mathbb{R}^d$ is of dimension 1: it is spanned by the vector $\epsilon_1\wedge\dots\wedge\epsilon_d$. Thanks to Eq. (A.1.5), we have:

$$\begin{aligned}u_1\wedge\dots\wedge u_d &= \sum_{\sigma\in\mathcal{P}_d}\operatorname{sign}(\sigma)u_1^{\sigma(1)}\dots u_d^{\sigma(d)}\epsilon_1\wedge\dots\wedge\epsilon_d \\ &= \det(u_1,\dots,u_d)\epsilon_1\wedge\dots\wedge\epsilon_d\end{aligned} \tag{A.1.11}$$

  All d-multivector are proportional to the basis vector $\epsilon\wedge\dots\wedge\epsilon_d$. There is a one-to-one map between d-mutlivectors in $\mathbb{R}^d$ and the determinant of the vectors.

Let u and v be two vectors in dimension 3. Then the 2-multivector $u\wedge v$ is given in coordinates as:

$$u\wedge v = (u^2v^3-u^3v^2)\epsilon_1\wedge\epsilon_2 + (u^3v^1-u^1v^3)\epsilon_3\wedge\epsilon_1 + (u^1v^2-u^2v^1)\epsilon_1\wedge\epsilon_2. \tag{A.1.12}$$

We notice that the coordinates of $u\wedge v$ in the canonical basis of $\Lambda^2\mathbb{R}^3$ are precisely the coordinates of the cross product between u and v: $u\times v$. Any 2-multivector $u\wedge v$ in dimension 3 can be mapped isometrically to $u\times v\in\mathbb{R}^3$. Moreover, we all know that $(u\times v)^t w = \det(u,v,w)$.

A.2 m-forms as antisymmetric tensors

We define now the forms on the space of m-multivectors: the m-forms.

Definition A.4 (m-forms). *A m-form ω on $\mathbb{R}^d$ is an linear map from $\Lambda^m\mathbb{R}^d$ to $\mathbb{R}$: $\omega : (u_1\wedge\dots\wedge u_m) \longrightarrow \omega(u_1\wedge\dots\wedge u_m)\in\mathbb{R}$, where every u_i is a vector in $\mathbb{R}^d$. We denote by $(\Lambda^m\mathbb{R}^d)^*$ the space of m-forms on $\mathbb{R}^d$.*

If we write $\omega(u_1,\dots,u_m) = \omega(u_1\wedge\dots\wedge u_m)$, we see that ω can be written as a m-covariant tensor. Due to the symmetries of the wedge product, this m-covariant tensor is totally antisymmetric (i.e. alternated forms). For example, in 3D, $(u,v,w)\to\det(u,v,w)$ is a 3-form and $u,v\longrightarrow(u\times v)^t z$ for a fixed vector z is a 2-form.

As the dual space of $\Lambda^m\mathbb{R}^d$, the space of m-forms in $\mathbb{R}^d$ is of dimension $\binom{d}{m}$. As an alternated tensor, ω is decomposed into:

$$\omega = \sum_{1\le i_1<\dots<i_m\le d}\omega_{i_1\dots i_m}dx_{i_1}\wedge\dots\wedge dx_{i_m}, \tag{A.2.1}$$

where dx_i denotes the dual basis of $\mathbb{R}^d$ (i.e. $dx_i(\epsilon_j) = \delta_{i,j}$) and $dx_1 \wedge \ldots \wedge dx_m$ the antisymmetric part of the tensor $dx_1 \otimes \ldots \otimes dx_m$. In particular, $dx \wedge dy = dx \otimes dy - dy \otimes dx$.

The space of m-forms inherits from the same properties as the space of m-multivectors:

- If $m = 0$, ω is simply a constant mapping on $\mathbb{R}$.
- If $m = 1$, ω is a linear form on $\mathbb{R}^d$: for all $u \in \mathbb{R}^d$, $\omega(u) \in \mathbb{R}$. Thanks to the Riesz representation theorem, this linear form can be represented by the inner-product with a fixed vector $\overline{\omega}$:
$$\omega(u) = \overline{\omega}^t u. \tag{A.2.2}$$
- If $m = d-1$, the space of $d-1$-forms is also of dimension d. With the notations of Section A.1.3, we have for any $d-1$-multivectors $u_1 \wedge \ldots \wedge u_{d-1} = \sum_{i=1}^{d} \eta_i \tilde{\varepsilon}_i^d$. Therefore, by linearity a $d-1$-form satisfies:
$$\omega(u_1 \wedge \ldots \wedge u_{d-1}) = \sum_{i=1}^{d} \eta_i \omega(\tilde{\varepsilon}_i^d) = \eta^t \overline{\omega}, \tag{A.2.3}$$
where $\overline{\omega}$ denotes the vector whose coordinates equal $\omega(\varepsilon_1 \wedge \ldots \varepsilon_{i-1} \wedge \varepsilon_{i+1} \ldots \wedge \varepsilon_d)$ for $i = 1$ to d. Therefore, a $d-1$-form also can be represented by an inner-product such that:
$$\omega(u_1 \wedge \ldots \wedge u_{d-1}) = \eta^t \overline{\omega} = \det\left(u_1, \ldots, u_{d-1}, \overline{\omega}\right), \tag{A.2.4}$$
according to Section A.1.3.
- If $m = d$, all d-forms are proportional to the determinant (the space of d-form in dimension d is of dimension 1). Indeed, every d-multivector $u_1 \wedge \ldots \wedge u_d$ is equal to $\det(u_1, \ldots, u_d)(\varepsilon_1 \wedge \ldots \wedge \varepsilon_d)$. Therefore every d-forms in dimension d is written as:
$$\omega(u_1 \wedge \ldots \wedge u_d) = \overline{\omega} \det(u_1, \ldots, u_d), \tag{A.2.5}$$
for a given scalar $\overline{\omega} = \omega(\varepsilon_1 \wedge \ldots \wedge \varepsilon_d)$.

We define the Euclidean norm of a m-form ω as the spectral norm (which corresponds to the Euclidean norm on $(\Lambda^m \mathbb{R}^d)^*$):

Definition A.5. *Let ω a m-form in $\mathbb{R}^d$. The norm of ω is defined as:*
$$|\omega|_{(\Lambda^m \mathbb{R}^d)^*} = \sup_{|u_1 \wedge \ldots \wedge u_m| = 1} |\omega(u_1 \wedge \ldots \wedge u_m)| = \left(\sum_{1 \le i_1 < \ldots < i_m \le m} (\omega_{i_1 \ldots i_m})^2 \right)^{1/2}, \tag{A.2.6}$$
where $\omega_{i_1 \ldots i_m}$ are the coordinates of the m-forms in the basis $dx_{i_1} \wedge \ldots \wedge dx_{i_m}$.

A.3 Differential forms as multi-covariant tensor fields

A.3.1 Definition

Like we extend the concept of vectors to vector fields on a smooth sub-manifold, we extend the concept of m-forms to differential m-forms. Each point x of a manifold is associated to a m-form $\omega(x)$ whose input vectors are chosen in the tangent-space of the manifold at point x. This leads to the following definition:

Definition A.6 (differential m-forms)**.** *A differential m-form on $\mathbb{R}^d$ (or on an open subspace of $\mathbb{R}^d$) maps every $x \in \mathbb{R}^d$ to $\omega(x)$ a m-form in $(\Lambda^m\mathbb{R}^d)^*$. We denote $\mathcal{C}^0(\mathbb{R}^d, (\Lambda^m\mathbb{R}^d)^*)$ the space of the differential m-forms which are continuous and tend to zero at infinity. It is provided with the norm:*

$$\|\omega\|_\infty = \sup_{x\in\mathbb{R}^d} \sup_{|u_1\wedge\ldots\wedge u_m|\leq 1} |\omega(x)(u_1 \wedge \ldots \wedge u_m)| . \tag{A.3.1}$$

If $m = 0$, a differential 0-form is simply a scalar function on $\mathbb{R}^d$.

If $m = 1$, a differential 1-form is a vector field on $\mathbb{R}^d$.

If $m = d-1$, the $d-1$ differential form can be associated to a vector field on $\mathbb{R}^d$ thanks to the isometric mapping between the $d-1$-form on $\Lambda^{d-1}\mathbb{R}^d$ and the vectors on $\mathbb{R}^d$.

If $m = d$, the d differential forms are all of the form $\omega = \overline{\omega}(x)\det$ where $\overline{\omega}(x)$ is a scalar function on $\mathbb{R}^d$ and det denotes the determinant form on $\mathbb{R}^d$.

A.3.2 Integration of differential forms on a colored sub-manifold

In order to model sub-manifolds of $\mathbb{R}^d$ as currents, we need to define the integration of differential m-forms on this manifold.

Definition A.7. *Let $\mathcal{M}$ be an oriented sub-manifold of dimension m in $\mathbb{R}^d$ and I a integrable function on $\mathcal{M}$ with respect to the Lebesgue measure on $\mathcal{M}$. Let $\omega \in \mathcal{C}^0(\mathbb{R}^d, (\Lambda^m\mathbb{R}^d)^*)$ be a m-differentiable form (Note that the degree of ω equals the dimension of the sub-manifold).*

For all $x \in \mathcal{M}$, we denote by $u_1(x), \ldots, u_m(x)$ a positively oriented basis of the tangent-space of $\mathcal{M}$ at point x (defined almost everywhere). Then, we define the integral of ω on $(\mathcal{M}, I)$ as:

$$\int_{\mathcal{M}} I\omega = \int_{\mathcal{M}} I(x)\omega(x)\left(\frac{u_1(x)\wedge\ldots\wedge u_m(x)}{|u_1(x)\wedge\ldots\wedge u_m(x)|}\right)d\lambda(x), \tag{A.3.2}$$

where the integral on the right hand denotes the usual Lebesgue integral of a scalar function on $\mathcal{M}$ $d\lambda$ the usual Lebesgue measure on $\mathcal{M}$.

Proposition A.8. *The definition of the integral in Eq. (A.3.2) does not depend on the choice of the positively oriented basis of the tangent-space of $\mathcal{M}$ at point x.*

Proof. Let A be a m-by-m matrix which change the basis $u_1(x), \ldots, u_m(x)$ to the basis $v_1(x) = \sum_{k=1}^m A_{1k}u_k, \ldots, v_m(x) = \sum_{k=1}^m A_{mk}u_k$. Since the change of basis is supposed not to change the orientation, the determinant of A is positive. Thanks to Proposition A.3, we have that:

$$\begin{aligned}\frac{u_1(x)\wedge\ldots\wedge u_m(x)}{|u_1(x)\wedge\ldots\wedge u_m(x)|} &= \frac{|A|\,(v_1(x)\wedge\ldots\wedge v_m(x))}{\big||A|\big|\,|v_1(x)\wedge\ldots\wedge v_m(x)|} \\ &= \frac{v_1(x)\wedge\ldots\wedge v_m(x)}{|v_1(x)\wedge\ldots\wedge v_m(x)|},\end{aligned} \tag{A.3.3}$$

since $|A| = \big||A|\big|$ (the absolute value of the determinant of A). ■

Remark A.9. This definition still holds if $\mathcal{M}$ is of dimension 0. In this case, $\mathcal{M}$ is a discrete set of points. The Lebesgue measure on $\mathcal{M}$ must be replaced by the measure $\sum_{x\in\mathcal{M}} \delta_x$ which counts the number of elements in $\mathcal{M}$. An integrable function on $\mathcal{M}$ is therefore a function which satisfies: $\sum_{x\in\mathcal{M}} I(x) < \infty$. The integral of a 0-form on $\mathcal{M}$ is simply the integral of a scalar function on $\mathcal{M}$. □

To compute the integral in Eq. (A.3.2) in practice, we need to write it with local charts. Let $\{U_i, \pi_i\}$ be an atlas of $\mathcal{M}$ and χ_i a partition of unity of the open cover $\{U_i\}$. This means that $\mathcal{M}$ is parametrized locally (on $U_i \subset \mathbb{R}^m$) by a piecewise differentiable chart $\pi_i : U_i \to \mathcal{M}$. We suppose moreover that every chart are positively oriented.

Let $x = \pi_i(\mathbf{p})$ be a point on $\mathcal{M}$ for $\mathbf{p} \in U_i$. We can choose $u_k(x) = \frac{\partial \pi_i(\mathbf{p})}{\partial p_k}$ as the positively oriented basis of the tangent plane of $\mathcal{M}$ at point $\pi_i(\mathbf{p})$. These vectors are considered in $\mathbb{R}^d$. However, they all belong to the tangent-space of dimension m. Let Π_x denote the orthogonal projection on this tangent-space. Therefore,

$$\begin{aligned}\left|\frac{\partial \pi_i(\mathbf{p})}{\partial p_1} \wedge \ldots \wedge \frac{\partial \pi_i(\mathbf{p})}{\partial p_m}\right| &= \left|\Pi_x\left(\frac{\partial \pi_i(\mathbf{p})}{\partial p_1}\right) \wedge \ldots \wedge \Pi_x\left(\frac{\partial \pi_i(\mathbf{p})}{\partial p_m}\right)\right| \\ &= \det\left(\Pi_x\left(\frac{\partial \pi_i(\mathbf{p})}{\partial p_1}\right), \ldots, \Pi_x\left(\frac{\partial \pi_i(\mathbf{p})}{\partial p_m}\right)\right) \\ &= \det\left(\frac{\partial \pi_i(\mathbf{p})}{\partial p_1}, \ldots, \frac{\partial \pi_i(\mathbf{p})}{\partial p_m}\right) \\ &= |d_{\mathbf{p}}\pi_i|\end{aligned} \tag{A.3.4}$$

since the magnitude of a d-multivector in dimension d is equal to the determinant of these vectors (See Section A.1.3). Since the charts are positively oriented, this determinant is positive.

Moreover, the Lebesgue measure written in the charts π_i is equal to: $d\lambda(x) = |d_x\pi_i|\, d\mathbf{p}$ for $x = \mathbf{p}$. Therefore, in the charts π_i, the norm of the multivector and the normalizing factor of the Lebesgue measure cancel ($|d_x\pi_i|$ in the numerator and denominator). The integral in Eq. (A.3.2) finally is written as:

$$\int_{\mathcal{M}} I\omega = \sum_i \int_{U_i} \chi_i(\mathbf{p}) I(\pi_i(\mathbf{p}))\omega(\pi_i(\mathbf{p}))\left(\frac{\partial \pi_i}{\partial p_1}(\mathbf{p}) \wedge \ldots \wedge \frac{\partial \pi_i}{\partial p_m}(\mathbf{p})\right) d\mathbf{p}, \tag{A.3.5}$$

where the integrals of the right-hand side denotes the usual Lebesgue integral on open subset of $\mathbb{R}^d$. Proposition A.8 shows that this expression is independent of the choice of the basis.

We remark that the argument of ω within the integrals written in local charts is *not* normalized in Equation (A.3.5). If $\mathcal{M}$ is a surface parametrized by $S(u, v)$, then the argument of ω is the non-normalized normal $\frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v}$. By contrast, in the intrinsic formulation in Eq. (A.3.2), the argument of ω is the *unit* normal of S (the Lebesgue measure on $\mathcal{M}$ taking care of the right scaling of the normal).

A.3.3 Change of variable formula

Let $\mathcal{M}$ be a sub-manifold of $\mathbb{R}^d$ and I an integrable function on $\mathcal{M}$. This function plays the role of an image (i.e. a map of colors) drawn on the manifold. The purpose of this section is to define the geometrical transport of such a colored manifold and to compute the integration of a differential form on the transported manifold, namely by the definition of a proper change of variable formula.

For $\mathcal{M}$ a sub-manifold of $\mathbb{R}^d$ and ϕ a diffeomorphism of $\mathbb{R}^d$, then we define $\phi(\mathcal{M})$ the geometrical transport of $\mathcal{M}$, namely the set of points $\phi(x)$ for all $x \in \mathcal{M}$. Since ϕ is a diffeomorphism, the regularity of $\phi(\mathcal{M})$ is the same as the regularity of the original sub-manifold $\mathcal{M}$.

If I is an image drawn on $\mathcal{M}$, then we define the transport of I by the diffeomorphism ϕ as $I\circ\phi^{-1}$. This means that the intensities on the manifold are carried along the deformation without any change. This action $((\phi, I) \to I \circ \phi^{-1})$ is the usual transport of intensities for image registration.

This leads to the following definition:

Definition A.10 (geometric transport of colored sub-manifolds)**.** *Let $\mathcal{M}$ be a rectifiable sub-manifold of $\mathbb{R}^d$ and I a scalar function on $\mathcal{M}$. Let ϕ be a diffeomorphism of $\mathbb{R}^d$. We define the geometrical transport of the couple $(\mathcal{M}, I)$ as:*

$$\phi\left(\mathcal{M}, I\right) = (\phi(\mathcal{M}), I \circ \phi^{-1}). \tag{A.3.6}$$

Our purpose now is to compute the integration of a m-differential form ω over the couple $\phi\left(\mathcal{M}, I\right)$: $\int_{\phi(\mathcal{M})} I \circ \phi^{-1}\omega$. If $u_1(x), \ldots, u_m(x)$ is a positively oriented basis of the tangent-space of $\mathcal{M}$, then $d_x\phi(u_1(x)), \ldots d_x\phi(u_m(x))$ is a basis of the tangent-space of $\phi(\mathcal{M})$, where $d_x\phi$ is a d-by-d Jacobian matrix of ϕ at point x. Therefore, the integral $\int_{\phi(\mathcal{M})} I \circ \phi^{-1}\omega$ is written as (using the linearity of the form $\omega(x)$):

$$\begin{aligned}\int_{\phi(\mathcal{M})} I \circ \phi^{-1}\omega &= \int_{\mathcal{M}} I \circ \phi^{-1}(\phi(x))\omega(\phi(x)) \left(\frac{d_x\phi(u_1(x)) \wedge \ldots \wedge d_x\phi(u_m(x))}{|d_x\phi(u_1(x)) \wedge \ldots \wedge d_x\phi(u_m(x))|} \right) d\lambda^{\phi}(\phi(x)) \\ &= \int_{\mathcal{M}} I(x)\omega(\phi(x)) \left(d_x\phi(u_1(x)) \wedge \ldots \wedge d_x\phi(u_m(x))\right) \frac{d\lambda^{\phi}(\phi(x))}{|d_x\phi(u_1(x)) \wedge \ldots \wedge d_x\phi(u_m(x))|},\end{aligned} \tag{A.3.7}$$

where $d\lambda^{\phi}$ denotes the Lebesgue measure on $\phi(\mathcal{M})$.

We can restrict the tangential map $d_x\phi$ to map the tangent-space of $\mathcal{M}$ at x to the tangent-space of $\mathcal{M}$ at $\phi(x)$. We denote $d_x\tilde{\phi}$ this m-by-m matrix. Therefore, $|d_x\phi(u_1(x)) \wedge \ldots \wedge d_x\phi(u_m(x))| = \left|d_x\tilde{\phi}(u_1(x)) \wedge \ldots \wedge d_x\tilde{\phi}(u_m(x))\right| = \left|d_x\tilde{\phi}\right| |u_1(x) \wedge \ldots \wedge u_m(x)|$. Moreover, the Lebesgue measure on $\phi(\mathcal{M}))$ is given by $d\lambda(\phi(x)) = \left|d_x\tilde{\phi}\right| d\lambda(x)$, so that the factor $d_x\tilde{\phi}$ in the numerator and denominator cancels:

$$\begin{aligned}\int_{\phi(\mathcal{M})} I \circ \phi^{-1}\omega &= \int_{\mathcal{M}} I(x)\omega(\phi(x)) \left(d_x\phi(u_1(x)) \wedge \ldots \wedge d_x\phi(u_m(x))\right) \frac{d\lambda(x)}{|u_1(x) \wedge \ldots \wedge u_m(x)|} \\ &= \int_{\mathcal{M}} I(x)\phi^*\omega(x) \left(\frac{u_1(x) \wedge \ldots \wedge u_m(x)}{|u_1(x) \wedge \ldots \wedge u_m(x)|} \right) d\lambda(x),\end{aligned} \tag{A.3.8}$$

where we denote $\phi^*\omega(x)(u_1 \wedge \ldots \wedge u_m) = \omega(\phi(x))(d_x\phi(u_1) \wedge \ldots \wedge d_x\phi(u_m))$.

This justifies the introduction of the pullback action of a diffeomorphism on a differential m-form:

Definition A.11 (pullback action on differential forms)**.** *Let ω be a m-differential form on $\mathbb{R}^d$ and ϕ a diffeomorphism of $\mathbb{R}^d$ such that $\sup_{x\in\mathbb{R}^d} |d_x\phi| < \infty$. We define $\phi^*\omega$ a m-differential form (of the same regularity as ω) as:*

$$\phi^*\omega(x)(u_1 \wedge \ldots \wedge u_m) = \omega(\phi(x))\left(d_x\phi(u_1) \wedge \ldots \wedge d_x\phi(u_m)\right), \tag{A.3.9}$$

for all points $x \in \mathbb{R}^d$ and every vectors $u_i \in \mathbb{R}^d$. The differential form $\phi^*\omega$ is called the pullback action of the diffeomorphism ϕ on the differential form ω.

We can verify easily that the vector field $\phi^*\omega$ still belong to our space of differential m-forms $\mathcal{C}^0(\mathbb{R}^d, \Lambda^m\mathbb{R}^d)$ (since we suppose that $\sup_{x\in\mathbb{R}^d} |d_x\phi| < \infty$). Moreover, the pullback action is really an action of the group of diffeomorphism on the space of differential form, namely that $(\phi \circ \psi)^*\omega = \phi^*(\psi^*\omega)$ for all diffeomorphism ϕ and ψ.

Proposition A.12. *Let $\mathcal{M}$ be a sub-manifold of dimension m in $\mathbb{R}^d$ and I an integrable function on $\mathcal{M}$. Let ϕ be a diffeomorphism of $\mathbb{R}^d$. Then:*

$$\int_{\phi(\mathcal{M})} I \circ \phi^{-1}\omega = \int_{\mathcal{M}} I\phi^*\omega. \tag{A.3.10}$$

Proof. This is exactly what we proved in Equation (A.3.8). ■

We can write the pullback action on a differential m-forms on $\mathbb{R}^d$, ω, in some particular cases of interest according to the dimension m:

- If $m = 0$, then $\omega(x)$ is a scalar field and $\phi^*\omega = \omega \circ \phi$.
- If $m = 1$, then $\omega(x)$ is represented by a vector field $\overline{\omega}(x)$: $\omega(x)(u) = \overline{\omega}(x)^t u$. Therefore,

$$\phi^*\omega(x)(u) = \omega(\phi(x))(d_x\phi(u)) = \overline{\omega}(\phi(x))^t d_x\phi(u) = \left(d_x\phi^t\overline{\omega}(\phi(x))\right)^t u. \tag{A.3.11}$$

  The vector field associated to $\phi^*\omega$ is $\phi^*\overline{\omega}(x) = d_x\phi^t\overline{\omega}(\phi(x))$.

- If $m = d-1$, the $\omega(x)$ is represented by a vector field $\overline{\omega}(x)$ such that $\omega(x)(u_1 \wedge \ldots \wedge u_{d-1}) = \det(\overline{\omega}(x), u_1, \ldots, u_{d-1})$. Therefore,

$$\begin{aligned}
\phi^*\omega(x)(u_1 \wedge \ldots \wedge u_{d-1}) &= \omega(\phi(x))(d_x\phi(u_1) \wedge \ldots \wedge d_x\phi(u_{d-1})) \\
&= \det\left(\overline{\omega}(\phi(x)), d_x\phi(u_1), \ldots, d_x\phi(u_{d-1})\right) \\
&= |d_x\phi| \det\left(d_x\phi^{-1}\overline{\omega}(\phi(x)), u_1, \ldots, u_{d-1}\right) \\
&= \det\left(|d_x\phi|\, d_x\phi^{-1}\overline{\omega}(\phi(x)), u_1, \ldots, u_{d-1}\right),
\end{aligned} \tag{A.3.12}$$

  so that the vector field associated to $\phi^*\omega$ is $|d_x\phi|\, d_x\phi^{-1}\overline{\omega}(\phi(x))$.

- If $m = d$, the $\omega(x)$ is represented by a scalar field $\overline{\omega}(x)$ such that $\omega(x)(u_1 \wedge \ldots \wedge u_d) = \overline{\omega}(x) \det(u_1, \wedge \ldots \wedge, u_d)$. Therefore,

$$\begin{aligned}
\phi^*\omega(x)(u_1 \wedge \ldots \wedge u_d) &= \overline{\omega}(\phi(x)) \det\left(d_x\phi(u_1) \wedge \ldots \wedge d_x\phi(u_d)\right) \\
&= |d_x\phi|\, \overline{\omega}(\phi(x)) \det(u_1, \wedge \ldots \wedge, u_d),
\end{aligned} \tag{A.3.13}$$

  so that the scalar field associated to $\phi^*\omega$ is $|d_x\phi|\, \overline{\omega}(x)$.

Appendix B

Construction of RKHS and their dual spaces

Contents

A Hilbert space is a vector space provided with an inner-product which is topologically complete (i.e. in which every Cauchy sequence converges). These spaces play a tremendous role in almost all area of science, since they are the more natural extension of the usual Euclidean spaces $\mathbb{R}^n$. The structure of vector space and the inner-product allows us to perform standard computation in a way similar to linear algebra. Roughly speaking, Hilbert spaces give a rigorous framework to work with infinite-dimensional vectors and matrices. The completeness enables to consider such infinite-dimensional vectors as the limit of finite-dimensional vectors.

Among the functional Hilbert spaces (i.e. Hilbert spaces of functions), the reproducing kernel Hilbert spaces (RKHS) are of great practical interest. They make the framework of Hilbert spaces even more similar to the finite-dimensional Euclidean spaces. In $\mathbb{R}^n$, any symmetric definite-positive matrix K defines a metric. The inner product between two vectors X and Y is given simply by: $\langle X, Y \rangle_K = X^t K Y$. This can be seen as the Euclidean inner-product between $K^{1/2}X$ and $K^{1/2}Y$ (as if the matrix $K^{1/2}$ maps $\mathbb{R}^n$ with the metric K to the Euclidean space $\mathbb{R}^n$). Similarly, a RKHS is entirely determined by the choice of a function K, called a kernel. The squared root of the kernel (in a sense to be defined) maps the space of L^2 functions to the RKHS. Computations in the RKHS involves only standard operations with the kernel, as we shall see in this appendix. Besides the computational benefit, the framework of RKHS offers a way to adapt the metric to any particular applications, since defining a metric is equivalent to choosing a single function K.

The purpose of this appendix is to recall the basic properties of RKHS. We emphasize two important aspects. First, RKHS are build as a completion of the linear span of some basis vectors. This gives a way for the definition of finite-dimensional approximation spaces like in Chapter 2. Second, there is a canonical isometric mapping between a RKHS and its dual space. This isometric mapping, which is directly linked to the kernel, plays a central role in the computation of standard operations on currents and vector fields. This allows us to define a whole computational framework for dealing with currents, as introduced in Chapter 2 and 3.

The material presented in this appendix results from standard mathematical constructions. We introduce here only what is needed in the framework of this thesis and we refer the reader to [Aronszajn 1950, Schwartz 1964, Saitoh 1988] for more details on the theory.

B.1 Where does it come from?

There are two different (but equivalent) ways to construct RKHS. The first way comes from the theory of differential equations: under some hypotheses, it is possible to express the solution of a differential equation as a convolution with a kernel. In this case, the space of solution is naturally a RKHS. The second construction starts choosing a kernel K and then builds a RKHS so that its kernel is given by K. In this appendix, we will present the second construction in details, since it is better suited for computational purposes. However, in this section, we will give a sketch of the construction from the point of view of the differential operators, since it gives better insights into the emergence of such spaces and their "reproducing property".

The rigorous framework of this construction is the one of the Friedrichs' extensions in functional analysis, as introduced in [Zeidler 1991]. Here, we recall simply the main steps of the construction.

Let L be a linear, self-adjoint operator which maps a space E, which is dense in the space of L^2 functions, to L^2. We suppose moreover that L is such that $\|u\|_{L^2}^2 \leq C \langle Lu, u \rangle_{L^2}$ for every $u \in L^2$. For instance, the Laplacian operator ($Lu = -\Delta u$) defined on the space E of twice differentiable functions with compact support satisfies these requirements. The differential operator L defines an inner-product on E by: $\langle u, v \rangle_E = \langle Lu, v \rangle_{L^2}$ for all functions u, v in E.

The space E provided with this inner-product is not yet a Hilbert space, since it is not topologically complete (the limit of Cauchy sequence in E may not be in E). Nevertheless, we can build the completion of E to give the Hilbert space W (one adds to E every limit of the Cauchy sequence of E), still included in L^2. In W, we have still $\langle u, v \rangle_W = \langle Lu, v \rangle_{L^2}$.

Under some assumptions (in particular that the evaluation functionals $\delta_x(u) = u(x)$ are continuous on W), one can prove that the differential operator is invertible, that L^{-1} maps the space L^2 to W and that L^{-1} admits a Green function K. This Green function satisfies, for every function $h \in L^2$:

$$L^{-1}h(x) = \int K(x, y)h(y)dy = \langle K(x, .), h \rangle_{L^2}. \tag{B.1.1}$$

Combining this equation with the definition of the inner-product in W leads to:

$$L^{-1}h(x) = \left\langle K(x, .), L^{-1}h \right\rangle_W. \tag{B.1.2}$$

This last equation shows that the function $h'(x) = L^{-1}h(x)$ in W satisfies the "reproducing property": $h'(x) = \langle K(x,.), h'\rangle_W$.

Applying this equation to the Green function K itself, called kernel is this context, leads to: $K(x,y) = \langle K(x,.), K(.,y)\rangle_W$. That's why K is called "auto-reproducing kernel".

Eventually, this construction shows that we can build a Hilbert space W of solutions of the differential equation $h = Lh'$. Such solutions satisfy the reproducing property, meaning that their evaluation on a point x is given by a convolution with a kernel K. Such Hilbert space W are then called reproducing kernel Hilbert spaces (RKHS).

This approach is usually followed in the field of fluid mechanics, for which the differential operator L is given by the laws of mechanics. Then, the Green function is defined implicitly. In our case, however, we prefer to control the kernel K which determines the metric and leaves the differential operator implicit. From a numerical point of view, it is better to write the operations on the RKHS with the kernel K which is a regularizing convolution instead of L which is a numerically unstable differential operator.

B.2 Construction of RKHS

In this section, we show how to construct a RKHS whose kernel is a given function K. First, we give rigorous definition of kernels and RKHS for scalar and vectorial functions.

B.2.1 Kernels and RKHS

Definition B.1 (auto-reproducing kernel Hilbert space (RKHS) (scalar case))**.** *Let W be a Hilbert space of scalar field on $\mathbb{R}^d$ (i.e. mapping from $\mathbb{R}^d$ to $\mathbb{R}$). W is a RKHS if the evaluation functions (linear forms on W) $\delta_x : W \to \mathbb{R}$ defined by:*

$$\delta_x(\omega) = \omega(x) \tag{B.2.1}$$

are continuous.

If W is a RKHS, then the Riesz representation theorem guarantees that for all $x \in \mathbb{R}^d$ there is a function $K_x \in W$ such that:

$$\omega(x) = \delta_x(\omega) = \langle K_x, \omega\rangle_W \tag{B.2.2}$$

We denote $K(x,y)$ the scalar mapping from $\mathbb{R}^d \times \mathbb{R}^d$ to $\mathbb{R}$: $K(x,y) = K_x(y)$. From the previous equation, we get $K_x(y) = \langle K_x, K_y\rangle_W = \langle K_y, K_x\rangle_W = K_y(x)$. This shows that K is symmetric: $K(x,y) = K(y,x)$.

Since we deal with vector fields, we give the slightly more general definition:

Definition B.2 (auto-reproducing kernel Hilbert space (RKHS) (vectorial case))**.** *Let W be a Hilbert space of mapping from $\mathbb{R}^d$ to $\mathbb{R}^p$. W is a RKHS is the evaluation functions (linear forms on W) $\delta_x^\alpha : W \to \mathbb{R}$ defined by:*

$$\delta_x^\alpha(\omega) = \omega(x)^t\alpha \tag{B.2.3}$$

are continuous for all point $x \in \mathbb{R}^d$ and all vectors $\alpha \in \mathbb{R}^p$ (i.e. each coordinates are continuous).

If W is a RKHS, then the Riesz representation theorem guarantees that for all points $x \in \mathbb{R}^d$ and all vectors $\alpha \in \mathbb{R}^p$ there is a function $K_x(\alpha) \in W$ such that:

$$\omega(x)^t \alpha = \delta_x^\alpha(\omega) = \langle K_x(\alpha), \omega \rangle_W \tag{B.2.4}$$

Applying this equation with $\alpha + \lambda\beta$ (for α, β two vectors an λ a real) shows that the mapping $\alpha \to K_x(\alpha)$ is linear. We denote therefore $K(x,y)$ the p-by-p matrix such that $K(x,y)\alpha = K_x(\alpha)(y)$ for all vectors α. Eventually, we have: $\alpha^t K(x,y)\beta = \langle K_x(\alpha), K_y(\beta)\rangle_W = \langle K_y(\beta), K_x(\alpha)\rangle_W = \beta^t K(y,x)\alpha$. This shows that $K(x,y) = K(y,x)^t$.

This shows that any RKHS contains a function K (i.e. a kernel as this will be shown in Theorem B.6) which satisfies the reproducing property in Eq. (B.2.2) and Eq. (B.2.3). The following proposition shows that this is actually a characterization of the RKHS.

Proposition B.3. *Let W be a Hilbert space which contains vector fields of the form $K(x,.)\alpha$ where K is a function from $\mathbb{R}^d \times \mathbb{R}^d$ to the space of p-by-p matrices. If every $\omega \in W$ satisfy the "reproducing property":*

$$\omega(x)^t \alpha = \langle \omega, K(.,x)\alpha \rangle_W \tag{B.2.5}$$

for all $x \in \mathbb{R}^d$ and all $\alpha \in \mathbb{R}^p$, then W is a RKHS.

Proof. Thanks to the Cauchy-Schwarz inequality, the evaluation functional verify:

$$|\delta_x^\alpha(\omega)| = \left|\omega(x)^t\alpha\right| \leq \|\omega\|_W \, \|K(.,x)\alpha\|_W \tag{B.2.6}$$

and therefore are continuous. ■

The following proposition gives an important example of RKHS, which is used to give a generic definition of the space of currents in Chapter 1.

Proposition B.4. *If W is a Hilbert space continuously embedded in the space of continuous mapping from $\mathbb{R}^d$ to $\mathbb{R}^p$ which tend to zero at infinity (i.e. such that for every $\omega \in W$, $\|\omega\|_\infty \leq C_W \|\omega\|_W$) for a fixed constant C_W, then W is a RKHS.*

Proof. If the condition is satisfied, then for every point x and vector α, $|\omega(x)^t\alpha| \leq \|\omega\|_\infty |\alpha| \leq C\,|\alpha|\,\|\omega\|_W$: the evaluation functions in Eq. (B.2.3) are continuous.

The condition means in particular that small errors measured in W are numerically small. ■

B.2.2 To each kernel its RKHS

Neither the definition of RKHS nor the propositions in the previous section give a practical way to construct RKHS. In this section, we show that, given a positive kernel K, there is a generic way to construct a RKHS whose kernel is K. The RKHS is therefore entirely determined by its kernel and we every operations in the RKHS can be written with the kernel.

First, we give the definition of positive kernel:

Definition B.5 (positive kernels). *A positive definite scalar kernel K on $\mathbb{R}^d$ is a scalar function on $\mathbb{R}^d \times \mathbb{R}^d$ such that*

- $K(x, y) = K(y, x)$ *for all* $x, y \in \mathbb{R}^d$
- $\sum_{i,j} a_i K(x_i, x_j) a_j \geq 0$ *for all finite set of reals* (a_i) *and points* (x_i) *in* $\mathbb{R}^d$
- *If* $\sum_{i,j} a_i K(x_i, x_j) a_j = 0$ *when the* (x_i) *are all distinct, then all* $a_i = 0$*.*

If only the first two properties are satisfied, K *is a positive semi-definite kernel.*

A positive definite vectorial kernel K *on* $\mathbb{R}^d$ *is a mapping* $\mathbb{R}^d \times \mathbb{R}^d$ *to the space of p-by-p matrix, such that*

- $K(x, y) = K(y, x)^t$ *for all* $x, y \in \mathbb{R}^d$
- $\sum_{i,j} a_i^t K(x_i, x_j) a_j \geq 0$ *for all finite set of vectors* (a_i) *in* $\mathbb{R}^d$ *and points* (x_i) *in* $\mathbb{R}^d$
- *If* $\sum_{i,j} a_i^t K(x_i, x_j) a_j = 0$ *when the* (x_i) *are all distinct, then all* $a_i = 0$*.*

If only the first two properties are satisfied, K *is a positive semi-definite kernel.*

The following theorem shows that a unique RKHS corresponds to any positive kernel K. The idea is to build the vector space spanned by the vector fields of the form $K(x, .)\alpha$ and to make this space complete by adding to it the limit of every Cauchy sequence. This construction allows us in Chapter 1 to process in the same setting discrete meshes (finite linear combination of $K(x, .)\alpha$) and the continuous surfaces (limit of such finite combination).

Theorem B.6. *We have the two properties:*

- *The kernel of a RKHS is a positive semi-definite kernel,*
- *If* K *is a positive semi-definite kernel, then it exists a unique RKHS* W *such that* K *is its kernel.*

Proof. We prove the previous theorem in the vectorial case. It can be easily simplified to apply in the scalar case. If W is a RKHS and K its kernel, then

$$\left\| \sum_i K(., x_i) a_i \right\|_W^2 = \sum_{i,j} a_j K(x_j, x_i) a_i \geq 0 \tag{B.2.7}$$

for all finite set of (x_i) and (α_i). K is positive semi-definite kernel.

Conversely, let K be a positive semi-definite kernel and E the vector space spanned by the function of the type $K(x, .)\alpha$ for all points x and vector α. Note that these vectors do not build a basis of E since the kernel is supposed to be only positive semi-definite. We provide E with the bilinear form defined on the $K(x, .)\alpha$ elements by: $\langle K(x, .)\alpha, K(y, .)\beta \rangle_E = \alpha^t K(x, y)\beta$. This bilinear form does not depend on the decomposition of the vectors $\omega \in E$. If a vector $\omega \in E$ has two different decompositions ω and $\tilde{\omega}$, one wants to prove that $\langle \omega, \omega' \rangle_E = \langle \tilde{\omega}, \omega' \rangle_E$. Assume that $\omega = \sum_i K(x_i, .)\alpha_i = 0$, then for any y and β, $\langle \omega, K(y, .)\beta \rangle_E = \beta^t \sum_i K(x_i, y)\alpha_i = \beta^t \omega(y) = 0$. By linearity, we get that $\langle \omega, \omega' \rangle_E = 0$ for every $\omega' \in E$.

We prove now that this bilinear form is an inner-product on E. Due to the definition of a positive kernel, this bilinear form is symmetric and positive. Let $\omega \in E$ such that

$\langle\omega,\omega\rangle_E = 0$. By linearity, the reproducing property which is satisfied for every $K(.,x)\alpha$ extends to every $\omega \in E$: $\omega(x)^t\alpha = \langle K(x,.)\alpha,\omega\rangle_E$. This implies thanks to the Cauchy-Schwarz inequality that: $|\omega(x)| = \sup_{|\alpha|=1} |\omega(x)^t\alpha| \leq \sup_{|\alpha|=1} \alpha^t K(x,x)\alpha \langle\omega,\omega\rangle_E = 0$. And $\omega = 0$.

E is therefore provided with an inner-product and satisfies the reproducing property. However, E is not Hilbert, since it is not complete. We build from E the space W which contains E and the limits of any Cauchy sequences of E.

Let ω_n be a Cauchy sequence in E. From the Cauchy-Schwarz inequality, we get:

$$|\omega_p(x) - \omega_q(x)| \leq \|\omega_p - \omega_q\|_E \sqrt{\sup_{|\alpha|=1} \alpha^t K(x,x)\alpha} \tag{B.2.8}$$

Therefore, $\omega_p(x)$ is a Cauchy sequence in $\mathbb{R}^d$ and hence converges. Let $\omega(x)$ be its limit. We define now W as the set of functions ω which are limits from Cauchy sequence in E: $W = \left\{\omega;\quad \exists(\omega_n) \in E(\text{Cauchy}), \forall x \in \mathbb{R}^d, \omega(x) = \lim_{n\to\infty}\omega_n(x)\right\}$. For any Cauchy sequence ω_n in E, $\|\omega_n\|_E$ is a Cauchy sequence in $\mathbb{R}$ and therefore converges. This allows us to provide W with the norm (and inner-product): $\|\omega\|_W = \lim_{n\to\infty}\|\omega_n\|_E$. Nevertheless, we have to check that this definition does not depend on the Cauchy sequence used to approximate ω. For this purpose, assume that ω_n is a Cauchy sequence in E, such that $\omega_n(x)$ converges to 0 for all x. We will prove that $\|\omega_n\|_E$ will converge to 0. Indeed, $\omega_n(x)^t\alpha = \langle\omega_n, K(x,.)\alpha\rangle_E \to 0$. By linearity, for all $\omega' \in E$, $\langle\omega_n,\omega'\rangle_E \to 0$. Then, since ω_n is a Cauchy sequence, there is an integer n such that for all $n \geq p$,

$$\|\omega_n\|_E - 2\langle\omega_p,\omega_n\rangle_E \leq \|\omega_p - \omega_n\|_E^2 \leq \varepsilon \tag{B.2.9}$$

for all $\varepsilon > 0$. Since $\langle\omega_p,\omega_n\rangle_E \to_{n\to\infty} 0$, for n large enough, $\|\omega_n\|_E \leq 2\varepsilon$ and therefore $\|\omega_n\|_E$ tends to 0.

Now, we prove that the construction of the Hilbert space W leads to a RKHS of kernel K. By definition of $\omega(x)$, we have $\omega(x)^t\alpha = \lim_{n\to\infty}\omega_n(x)^t\alpha$. We have $\omega_n(x)^t\alpha = \langle\omega_n, K(x,.)\alpha\rangle_W$ which converges to $\langle\omega, K(x,.)\alpha\rangle_W$ by definition of the norm in W. Therefore K is the kernel of the RKHS W.

We still need to prove that W is the unique RKHS whose kernel is K. If $\tilde{W}$ is a RKHS of kernel K, the every function of the type $K(x,.)\alpha$ are in $\tilde{W}$, and by linearity E is included in $\tilde{W}$. Let $\omega \in W$ as a limit of the Cauchy sequence ω_n in E. Due to the reproducing property, the inner-product $\langle.,.\rangle_W$, $\langle.,.\rangle_{\tilde{W}}$ and $\langle.,.\rangle_E$ all coincide on E. Therefore, ω_n is also a Cauchy sequence in $\tilde{W}$. This sequence converge pointwise to ω, the same limit as in W since this pointwise convergence does not depend on the Hilbert inner-product. Therefore $\omega \in \tilde{W}$ and W is a closed subset $\tilde{W}$. To prove the equality of the two spaces, we show that the orthogonal subspace of W in $\tilde{W}$ is equal to $\{0\}$. Let $\tilde{\omega} \in \tilde{W}$, such that for all $\omega \in W$, $\langle\omega,\tilde{\omega}\rangle_{\tilde{W}} = 0$. Then, $\tilde{\omega}(x)^t\alpha = \langle\tilde{\omega}, K(x,.)\alpha\rangle_{\tilde{W}} = 0$ and therefore $\tilde{\omega} = 0$. ■

A direct consequence of this proof is the following corollary:

Corollary B.7 (dense vector space in the RKHS)**.** *The span of vector fields of the form $K(x,.)\alpha$ for every $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}^p$ is dense in W.*

This corollary offers a way to define a approximation spaces of the space W by limiting the point x to belong to a particular discrete subset (see Chapter 2).

B.2.3 Choice of the kernel

The previous theorem shows that the choice of the kernel determines the RKHS and especially its metric. The choice of this metric is therefore crucial and must be adapted to every particular problems. Here, we give some examples of parametric kernels. They are translation-invariant isotropic scalar kernels, which means of the form $K(x, y) = k(|x-y|)\mathrm{I}_p$. The following functions k lead to positive kernels, as shown in [Glaunès 2005]:

- Gaussian kernel: $k(x) = \exp\left(\frac{-x^2}{\lambda_W^2}\right)$
- Cauchy kernel: $k(x) = \left(1 + \frac{x^2}{\lambda_W^2}\right)^{-1}$
- Sobolev kernel: k is the inverse Fourier transform of $(1+x^2)^{-s}$ for $s > d + 1/2$

See [Glaunès 2005] and the theorem of Bochner for more details on translation-invariant kernels. In particular, it is shown that the Sobolev spaces $\mathbb{H}^s(\mathbb{R}^d, \mathbb{R}^m)$ are RKHS if $s > d + 1/2$.

However, how to choose the "best" kernel according a particular application is still an open question. Through the applications of chapter 6, 7 and 8, we will give some clue to adjust kernel's parameters in different context.

From now on, we consider only symmetric kernel so that we do make differences between $K(x, y)$ and $K(y, x)$.

B.3 A RKHS is isometric to its dual space

B.3.1 W^*: dual space of RKHS W

Let W be a RKHS of kernel K. We denote W^* the dual space of W (i.e. the space of continuous linear forms on W). This means that $T : W \to \mathbb{R}$ is in W^* if there is a constant C_T such that for all ω, $|T(\omega)| \leq C_T \|\omega\|_W)$.

By definition of a RKHS in B.2, the evaluation functional δ_x^α are continuous linear forms on W. They belong therefore to W^*. They will play the role of Dirac delta currents in Chapter 1.

As a vector space of linear maps, W^* is provided with the operator norm:

$$\|T\|_{W^*} = \sup_{\|\omega\|_W \leq 1} |T(\omega)| \qquad \text{(B.3.1)}$$

B.3.2 Isometric mapping $\mathcal{L}_W$

One of the key property of the RKHS is that there is a canonical isometric map between a RKHS W and its dual space W^*. This isometric map is used intensively throughout the thesis.

Definition B.8. *Let $\mathcal{L}_W$ be the mapping:*

$$\begin{array}{rccc} \mathcal{L}_W: & W & \longrightarrow & W^* \\ & \omega & & \mathcal{L}_W(\omega) \end{array} \tag{B.3.2}$$

where $\forall \omega' \in W, \mathcal{L}_W(\omega)(\omega') = \langle \omega, \omega' \rangle_W$. $\mathcal{L}_W(\omega)$ is continuous thanks to the Cauchy-Schwarz inequality and therefore belongs to W^*.

Proposition B.9. *$\mathcal{L}_W$ is an isometric mapping between W and W^*.*

Proof. The following equalities apply for all $\omega \in W$:

$$\begin{aligned} \|\mathcal{L}_W(\omega)\|_{W^*} &= \sup_{\|\omega'\|_W=1} |\mathcal{L}_W(\omega)(\omega')| \\ &= \sup_{\|\omega'\|_W=1} |\langle \omega, \omega' \rangle_W| = \|\omega\|_W \end{aligned} \tag{B.3.3}$$

■

This proposition shows that the operator norm on the dual space W^* (see Eq. (B.3.1)) derives from an inner-product. Indeed, the norm on W comes from the inner-product. Since, $\langle \omega, \omega' \rangle_W = (\|\omega + \omega'\|_W^2 - \|\omega - \omega'\|_W^2)/4$ and $\|\mathcal{L}_W(\omega)\|_{W^*} = \|\omega\|_W$, we have:

$$\langle T, T' \rangle_{W^*} = \left\langle \mathcal{L}_W^{-1}(T), \mathcal{L}_W^{-1}(T') \right\rangle_W \tag{B.3.4}$$

The isometric map $\mathcal{L}_W$ carries the Hilbert structure in W to W^*. This makes W^* a Hilbert space.

Moreover, let $T \in W^*$, then by definition of $\mathcal{L}_W$, the vector field $\mathcal{L}_W^{-1}(T)$ satisfies $T(\omega) = \mathcal{L}_W(\mathcal{L}_W^{-1}(T))(\omega) = \left\langle \mathcal{L}_W^{-1}(T), \omega \right\rangle_W$. Using the isometric map, we obtain these two equalities:

$$T(\omega) = \langle T, \mathcal{L}_W(\omega) \rangle_{W^*} = \left\langle \mathcal{L}_W^{-1}(T), \omega \right\rangle_W \tag{B.3.5}$$

In particular, this allows us to show that the vector field which achieves the supremum in the definition of the norm in W^* in Eq. (B.3.1) is given by $\mathcal{L}_W^{-1}(T)/\left\|\mathcal{L}_W^{-1}(T)\right\|_W$. Indeed we have:

$$\begin{aligned} \|T\|_{W^*} &= \sup_{\|\omega\|_W=1} |T(\omega)| \\ &= \sup_{\|\omega\|_W=1} \left| \left\langle \mathcal{L}_W^{-1}(T), \omega \right\rangle_W \right| \end{aligned} \tag{B.3.6}$$

whose supremum is achieved for $\omega = \pm \mathcal{L}_W^{-1}(T) \left\|\mathcal{L}_W^{-1}(T)\right\|_W$.

In Eq. (B.3.5), we write $T(\omega)$ via the map $\mathcal{L}_W$. Actually, any operations on W and W^* can be expressed using this map. In particular, the inner-product in these two spaces are given as:

$$\begin{aligned} \langle \omega, \omega' \rangle_W &= \mathcal{L}_W(\omega)(\omega') \\ \langle T, T' \rangle_{W^*} &= T\left(\mathcal{L}_W^{-1}(T')\right) \end{aligned} \tag{B.3.7}$$

The first equality is a direct consequence of Definition B.8. The second one results from the application of Eq. (B.3.5) with $\omega = \mathcal{L}_W^{-1}(T')$.

B.3.3 Link between $\mathcal{L}_W$ and the kernel

We have just shown that the metric on W and W^* can be expressed via the isometric map $\mathcal{L}_W$. Actually, this isometric map is closely related to the kernel K of the RKHS W. This allows us to express the metric in terms of operations with the kernel. Therefore, once the kernel is chosen, any operations in the RKHS will have a closed form.

First, we apply the previous equations in the particular case when $T \in W^*$ is an evaluation functional δ_x^α, as defined in Eq. (B.2.2). For every $\omega \in W$, we have by application of Eq. (B.3.5): $\delta_x^\alpha(\omega) = \left\langle \mathcal{L}_W^{-1}(\delta_x^\alpha), \omega \right\rangle_W$. Moreover, thanks to the reproducing property satisfied in the RKHS W, we have: $\delta_x^\alpha(\omega) = \omega(x)^t\alpha = \langle \omega, K(x,.)\alpha \rangle_W$. This proves that:

$$\mathcal{L}_W^{-1}(\delta_x^\alpha) = K(x,.)\alpha \tag{B.3.8}$$

This equation shows that the kernel K may be seen as the Green function of the mapping $\mathcal{L}_W$ (which is implicitly a differential operator).

The application of Eq. (B.3.4) and the reproducing property leads to the explicit computation of the inner-product between evaluation functionals:

$$\left\langle \delta_x^\alpha, \delta_y^\beta \right\rangle_{W^*} = \langle K(.,x)\alpha, K(.,y)\beta \rangle_W = \alpha^t K(x,y)\beta \tag{B.3.9}$$

By linearity, the inner-product between $T = \sum_{i=1}^n \delta_{x_i}^{\alpha_i}$ and $U = \sum_{j=1}^m \delta_{y_j}^{\beta_j}$ is given by:

$$\langle T, U \rangle_{W^*} = \sum_{i=1}^n \sum_{j=1}^m \alpha_i^t K(x_i, y_j)\beta_j \tag{B.3.10}$$

This equation may be written in a matrix form:

$$\langle T, U \rangle_{W^*} = \boldsymbol{\alpha}^t \mathbf{K} \boldsymbol{\beta}, \tag{B.3.11}$$

where $\boldsymbol{\alpha}$ (resp. $\boldsymbol{\beta}$) denotes the nd (resp. md) dimensional vector obtained by the concatenation of every vectors α_i (resp. β_j). $\mathbf{K}$ denotes the nd-by-md block matrix whose block (i,j) is given by the d-by-d matrix $K(x_i, y_j)$ (for $i = 1, \dots, n$ and $j = 1, \dots, m$). This shows that the map $\mathcal{L}_W^{-1}$ is computed via the matrix $\mathbf{K}$ when applied to finite linear combination of evaluation functionals.

This way to compute the metric on W^* in a matrix form involving only the kernel K is the core of the numerical framework for computing with currents, as introduced in Chapter 2 and 3. Indeed, by construction of the RKHS the span of the functions $K(x,.)\alpha$ is dense in W (see Corollary B.7). By isometry, the span of the evaluation functionals δ_x^α is a dense vector space in W^*. This means that we can always approximate a current in W^* as a finite linear combination of evaluation functionals and use this matrix form to compute the metric in W^*. The true map $\mathcal{L}_W^{-1}$ can be considered then as a multiplication with the matrix $\mathbf{K}$ whose dimensions tend to infinity.

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