

SNAKES

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 CS 6640 (4)

Euler Lagrange DE: Variational calculus

$$J_{snake} = \int \alpha \cdot \left| \frac{d\underline{v}}{ds} \right|^2 + \beta \left| \frac{d^2 \underline{v}}{ds^2} \right|^2 + P(\underline{v}(s)) ds = \text{min}$$

Variational calculus: concept

$\underline{u}_0(s)$ curve in minimal energy state

$\underline{u}(s) = \underline{u}_0(s) + \delta \underline{u}$ curve with minimal perturbation variational

[Euler: for $\delta J = 0$: $\underline{u}(s) \rightarrow \underline{u}_0(s)$...]

$$\begin{aligned} \textcircled{1} \quad \frac{d\underline{v}}{ds} \cdot \frac{d\underline{v}}{ds} &= \frac{d}{ds} (\underline{u}_0(s) + \delta \underline{u}) \cdot \frac{d}{ds} (\underline{u}_0(s) + \delta \underline{u}) \\ &= \left(\frac{d\underline{u}_0}{ds} \right)^2 + 2 \frac{d\underline{u}_0}{ds} \cdot \frac{d}{ds} \delta \underline{u} + \frac{d}{ds} \delta \underline{u} \cdot \frac{d}{ds} \delta \underline{u} \end{aligned}$$

$O(\delta u^2)$
 very small variation

$$\textcircled{3} \quad P(\underline{u}(s)) \cdot P(\underline{u}_0(s) + \delta \underline{u}) \stackrel{\text{Taylor series exp.}}{=} P(\underline{u}_0) + \nabla P \cdot \delta \underline{u} + \dots$$

External condition:

$$\begin{aligned} &\int_0^1 \left(\alpha \cdot \frac{d\underline{u}_0}{ds} \cdot \frac{d}{ds} \delta \underline{u} + \beta \dots + \nabla P \cdot \delta \underline{u} \right) ds = 0 \\ &= \int_0^1 \left(\alpha \cdot \frac{d\underline{u}_0}{ds} \cdot \frac{d}{ds} \delta \underline{u} \right) ds + \int_0^1 \beta \dots ds + \int_0^1 \nabla P \cdot \delta \underline{u} ds = 0 \\ &\quad \alpha \cdot \frac{d\underline{u}_0}{ds} \cdot \delta \underline{u} \Big|_0^1 - \int_0^1 \delta \underline{u} \frac{d}{ds} \left(\alpha \cdot \frac{d\underline{u}_0}{ds} \right) ds \end{aligned}$$

Euler
 Lagrange DE
 4th order

$$\int \left[- \frac{d}{ds} \left(\alpha \frac{d\underline{u}_0(s)}{ds} \right) + \frac{d^2}{ds^2} \left(\beta \cdot \frac{d^2 \underline{u}_0(s)}{ds^2} \right) + \nabla P \right] \delta \underline{u} ds$$

Euler Lagrange DE 4th order: initial values $\underline{u}(0)$ $\underline{u}''(0)$
 $\underline{u}'(0)$ $\underline{u}'''(0)$
 boundary values: $\underline{u}(1)$ $\underline{u}'(1)$
 $\underline{u}''(1)$ $\underline{u}'''(1)$

⇒ write in coordinates:

(1)

$$\Delta x_{SS} + \beta x_{SSSS} + \frac{\partial E_{ext}}{\partial x} = 0$$

$$\Delta y_{SS} + \beta y_{SSSS} + \frac{\partial E_{ext}}{\partial y} = 0$$

$$\left. \begin{aligned} \Delta x_{SS} + \beta x_{SSSS} + \frac{\partial E_{ext}}{\partial x} &= 0 \\ \Delta y_{SS} + \beta y_{SSSS} + \frac{\partial E_{ext}}{\partial y} &= 0 \end{aligned} \right\} \boxed{-\Delta v_{SS} + \beta v_{SSSS} + \nabla P = 0}$$

solve analytically, or with perturbation methods

abstract Na:

$$\boxed{-\Delta u_{SS} + \beta u_{SSSS} + \nabla P = 0}$$

finite differences:

$$u_S = \frac{1}{\Delta} \left\{ u(s+\frac{\Delta}{2}) - u(s-\frac{\Delta}{2}) \right\}$$

$$u_{SS} = \frac{1}{\Delta^2} \left\{ u(s+\Delta) - 2u(s) + u(s-\Delta) \right\}$$

$$u_{SSSS} = \frac{1}{\Delta^4} \left\{ u(s+2\Delta) - 4u(s+\Delta) + 6u(s) - 4u(s-\Delta) + u(s-2\Delta) \right\}$$

		1			
		1	-1		
		1	-2	1	
		1	-3	3	
		1	-4	6	-4

DE: $\Delta u_{SS} = \frac{1}{\Delta^2} (1 \ -2 \ 1) \begin{pmatrix} u(s-\Delta) \\ u(s) \\ u(s+\Delta) \end{pmatrix}$

$$\Rightarrow \frac{1}{\Delta^2} \begin{pmatrix} 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & & \ddots & \\ & & & & & 1 & -2 & 1 \\ & & & & & & & & \ddots & \\ & & & & & & & & & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} u(-1) \\ u(0) \\ u(1) \\ \vdots \\ u(n-1) \end{pmatrix} = \begin{pmatrix} \nabla P(u(-1)) \\ \vdots \\ \nabla P(u(n-1)) \end{pmatrix}$$

$$\Rightarrow \frac{1}{\Delta^4} \begin{pmatrix} 1 & -4 & 6 & -4 & 1 & & & \\ & 1 & -4 & 6 & -4 & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 & -4 & 6 & -4 & 1 \\ & & & & & & & & & & & \ddots & \\ & & & & & & & & & & & & 1 & -4 & 6 & -4 & 1 \end{pmatrix} \begin{pmatrix} u(-1) \\ u(0) \\ \vdots \\ u(n-1) \end{pmatrix}$$

shortcut: $u(i) = \begin{pmatrix} x(i) \\ y(i) \end{pmatrix}$



Structure

$$A \cdot x + P_x(x, y)$$

$$A \cdot y + P_y(x, y)$$

A: periodicity matrix

$$\begin{pmatrix} x(i) \\ y(i) \end{pmatrix} = \underline{v}(i)$$

$$\Rightarrow K \cdot \underline{v}^{[t]} = -P \underline{v} \mid \underline{v}^{[t-1]}$$

$$K \cdot x^{[t]} = -P_x \mid x^{[t-1]}$$

$$K \cdot y^{[t]} = -P_y \mid y^{[t-1]}$$