2.0 Generalization of Hough Transform (HT)

straight line : $x\cos\theta + y\sin\theta - \rho = 0$

$\bar{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\qquad \bar{a} = (\rho, \theta)$ = parametervector

$$f(\bar{x}, \bar{a}) = \phi$$

$\bar{x}$: image points
$\bar{a}$: parameter vector

Transformation:

$$\forall \bar{x}_i \in \text{contours} : \left(\bar{x}_i \longrightarrow \{\bar{a} \mid f(\bar{x}_i, \bar{a}) = \phi\}_i \right)$$

Incrementation:

$$\forall \bar{a}_j \in \{\bar{a}\}_i : \quad acc(\underline{a}_j) = acc(\underline{a}_j) + inc$$

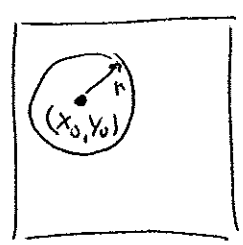
Example: Detection of circles

$$(x - \mathbf{x_0})^2 + (y - \mathbf{y_0})^2 - \mathbf{r}^2 = \phi$$

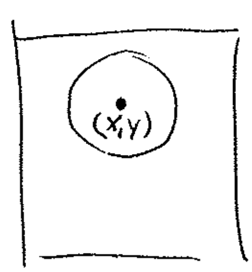
image points: $\bar{x}_i = \begin{pmatrix} x \\ y \end{pmatrix}$
parameter Vecto: $\bar{a} = \begin{pmatrix} x_0 \\ y_0 \\ r \end{pmatrix}$

Symmetry of points/parameters:

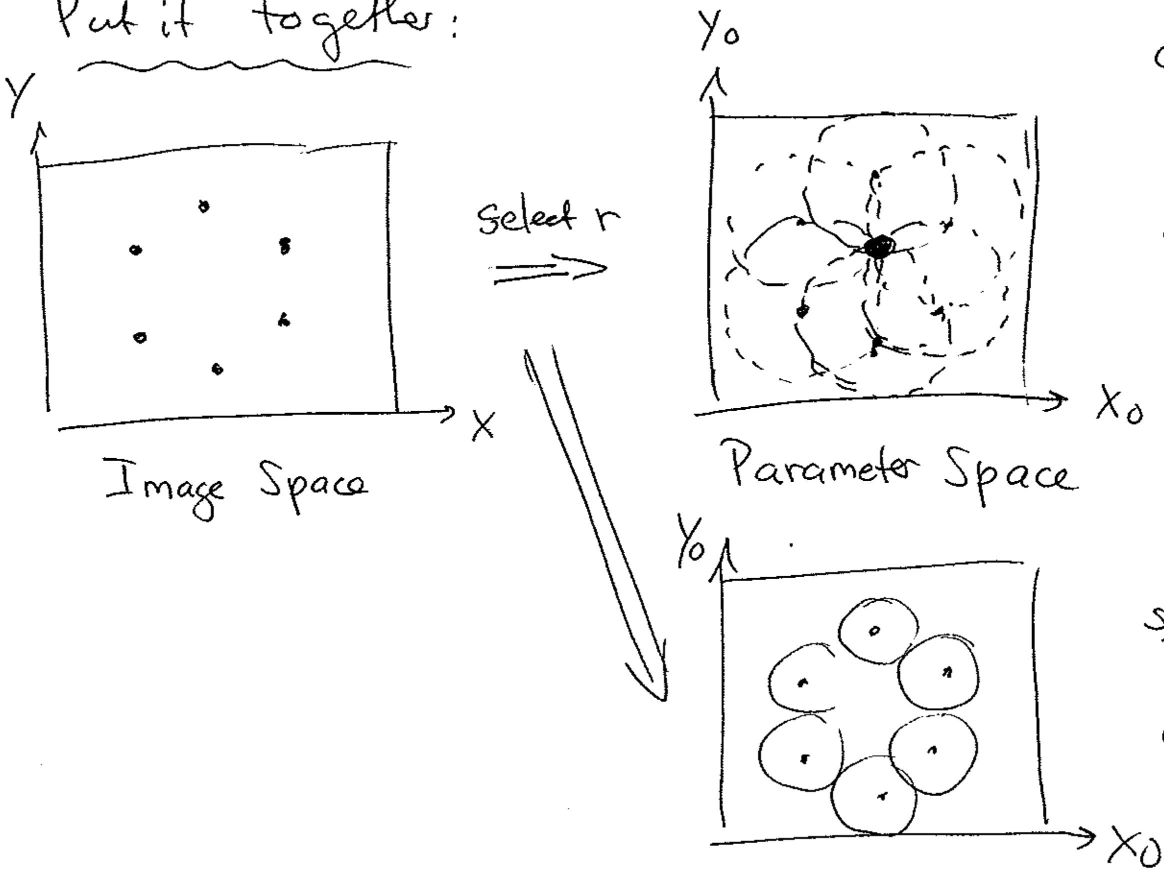
- fix center (x_0, y_0) and radius r: image points (x, y) describe circle centered at (x_0, y_0).



- fix an image point (x, y) and choose a radius r: possible centes (x_0, y_0) describe circle centered at (x, y).



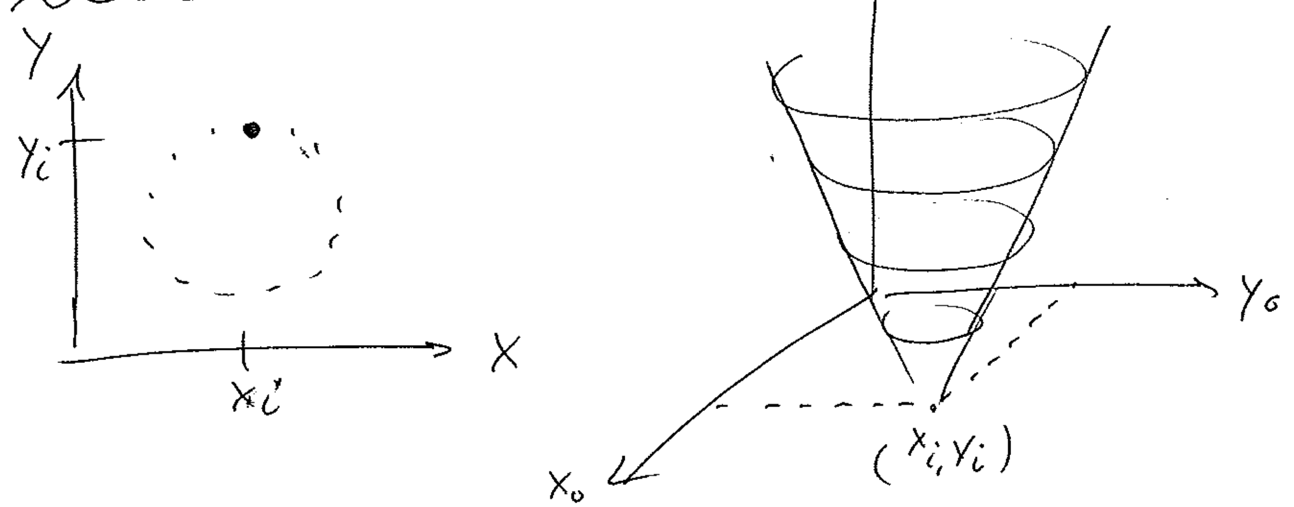
Put it together:



correct r:
high density of votes indicates center

smaller r:
no accumulation of center

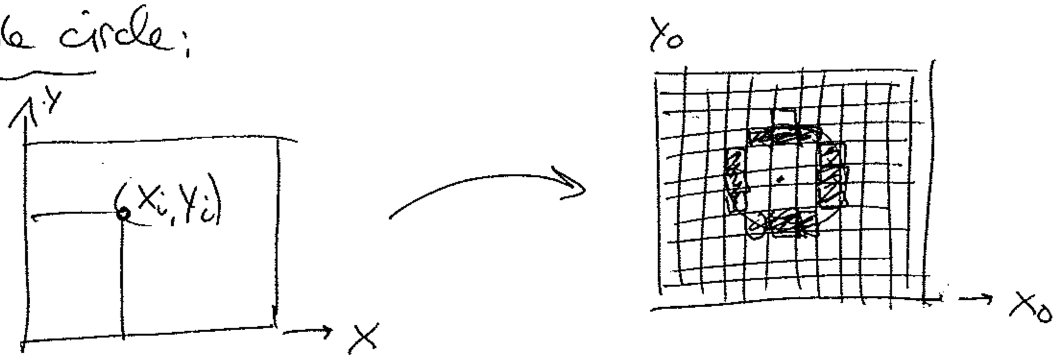
if r is not known:

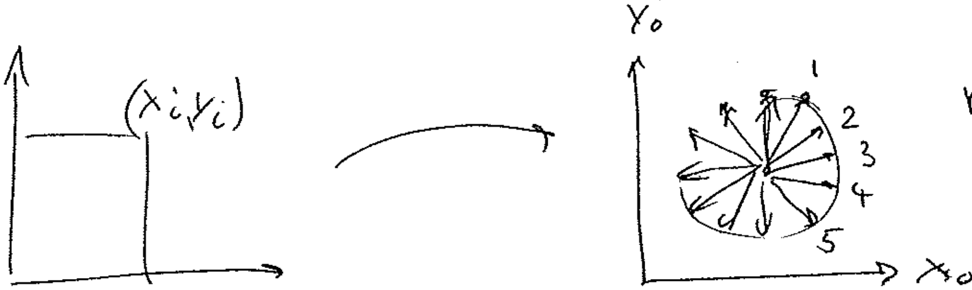


- Image point $\bar{x}_i$ transforms into right cone in a 3D parameter space (x_0, y_0, r).
- Each point $\bar{x}_i$: accumulation of votes of cells intersected by right cones.

(2.1) Extension to arbitrary 2D shapes

Example circle:



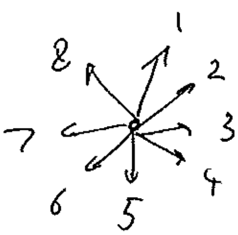


model: $m = \{\bar{m}_k\}$

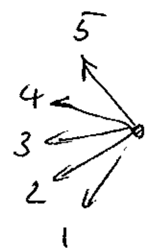
model given by set of discrete vectors

Algorithm:

- construct model $m = \{\bar{m}_k, k = 1 \dots n\}$ of discrete model points



- incrementation: for each image contour point $\bar{x}_i$: in parameter space, map vectors backwards and increment parameter cells:



$\forall_k \in \text{model}: \{\bar{x}_i - \bar{m}_k\} \Rightarrow acc(\bar{x}_i - \bar{m}_k)++$

Formal:

$$\boxed{\underbrace{\forall \bar{x}_i}_{\text{image contour points}} \left[\underbrace{\forall \bar{m}_k}_{\text{model curve}} \left(acc(\bar{x}_i - \bar{m}_k)++ \right) \right]}$$

HT Algorithm for arbitrary 2D curves

(1) Define discrete model curve, chose center, represent model as set of vectors from center to curve points $\Rightarrow \{\bar{m}_k ; k=0 \ldots m-1\}$

(2) Incrementation of accumulator : $\underbrace{\forall \bar{x}_i}\left[\forall \bar{m}_k \left(acc(\bar{x}_i - \bar{m}_k)++\right)\right]$

$\bar{x}_i$: image points that are part of contours/edges (after Canny edge detection and nonmaximum suppression)

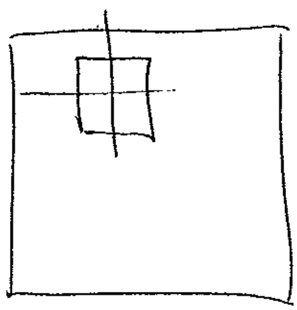
(3) For rotation and scaling: transform discrete model and start new incrementation in new accumulator buffer

(4) Find accumulator cells with high number of votes $\Rightarrow$ centers of likely structures

Comparison of HT and Template Matching

HT is efficient implementation of a general template matching strategy.

a) Matched Filtering: $\forall \bar{x}_i \quad F(x,y) = \sum_j \sum_k T(j,k)\, I(x-j, y-k)$



T: template

F: sum of all products over template size

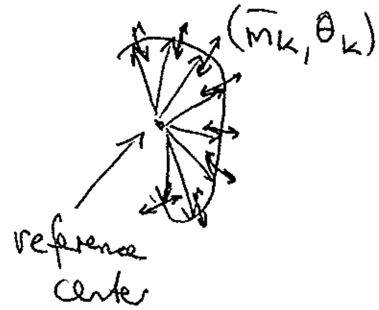
Computational expense:

$$(x \cdot y) \cdot (j \cdot k)$$

b) Hough Transform: $\underbrace{\forall \bar{x}_i \in \text{edges}}_{\#\text{ edge pixels}} \cdot \underbrace{\forall \bar{m}_k}_{\#\text{ template vectors}}$

Models with local edge/gradient direction

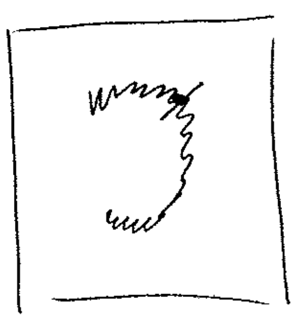
Model: $\{(\bar{m}_k, \theta_k) ; k = 0 \ldots m-1\}$



each model vector has edge gradient orientation at tip location as additional attribute

$\Rightarrow$ edge gradient and model orientation have to match!

Incrementation:



- each image edge point with gradient orientation $(\bar{x}_i, \theta_i)$:
- accumulate not whole model but only model vectors with same gradient orientation:

$$(\bar{x}_i, \theta_i) \longleftrightarrow (\bar{m}_k, \theta_k),\ \theta_k = \theta_i$$

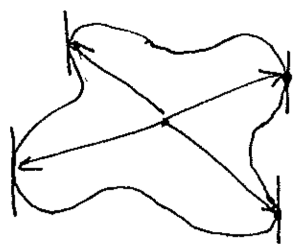
$$\forall (\bar{x}_i, \theta_i) \left[\forall (\bar{m}_k, \theta_k \mid \underline{\underline{\theta_k = \theta_i}}) \ acc(\bar{x}_i - \bar{m}_k)++ \right]$$

$\Rightarrow$ leads to concept presented by Ballard 1981: Generalized HT (GHT

Generalized Hough Transform GHT

(see original paper Ballard 1981)

Basic idea: sort model vectors $(\bar{m}_k, \theta_k)$ as a function of the associated contour normal θ_k



R-table:

θ_1	$\bar{m}_{11}, \bar{m}_{12}, \bar{m}_{13} \dots$
θ_2	$\bar{m}_{21} \dots$
θ_3	$\bar{m}_{31}, \bar{m}_{32} \dots$
$\vdots$	
θ_m	$\bar{m}_{m1}, \bar{m}_{m2} \dots$

Operation with GHT:

a) construct discrete model curve, chose reference center, store model vectors and associated contour normal (gradient direction)

b) generate R-table by sorting the $(\bar{m}_k, \theta_k)$ vectors by angle and putting them into a list with discrete bins of angles

c) $\forall (\bar{x}, \bar{\theta})_i \in$ image contours:
- index R-table at θ_i $\Rightarrow$ get model vectors $\{\bar{m}_{ik}\}$
- increment accumulator at positions $(\bar{x}_i - \bar{m}_{ik})$, $k = 1 \dots c$

Properties of R-table

- scaling: $\bar{m}_{\theta_j}^{(s)} = \bar{m}_{\theta_j} \cdot s$
- rotation: $\theta_j^{(r)} = \theta_j + \alpha$
  $\bar{m}_{\theta_j} = \bar{m}_{\theta_j} \cdot [R(\alpha)]$