Fourier Transform

Part II

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Repetition:

$$\mathcal{F}[f(t)] = F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi st}\, dt$$

inverse:

$$\mathcal{F}^{-1}(F(s)) = f(t) = \int_{-\infty}^{\infty} F(s)\, e^{j2\pi st}\, ds$$

$$\boxed{f(t) = \mathcal{F}^{-1}(\mathcal{F}(f(t)))}$$ Fourier Integral theorem

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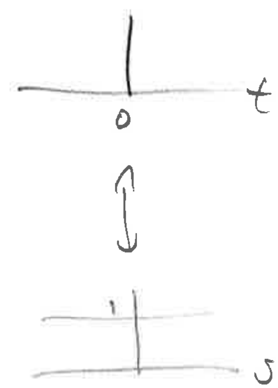
Example: a) impulse at origin:

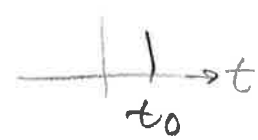
$$F(s) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi st}\, dt$$



$$\underset{\text{sifting}}{=} e^{-j2\pi s \emptyset} = e^{\emptyset} = 1$$

b) impulse at $t = t_0$

$$F(s) = \int e^{-j2\pi st}\, \delta(t - t_0)\, dt$$



$$= e^{-j2\pi s t_0} = \cos(2\pi s t_0) - j \sin(2\pi s t_0)$$

last time:

a) $\mathcal{F}(\cos(2\pi s_0 t)) \overset{?}{=} \dots = \frac{1}{2}\left[\delta(t-t_0) + \delta(t+t_0)\right]$

solution not so trivial! in continuous domain

see before:

$$\delta(t-t_0) \xrightarrow{\mathcal{F}} \overbrace{\cos(2\pi s t_0) - j\sin(2\pi s t_0)}^{e^{-j2\pi s t_0}}$$

$$\delta(t+t_0) \longrightarrow \dots$$

$\mathcal{F}^{-1}$

use integral theorem

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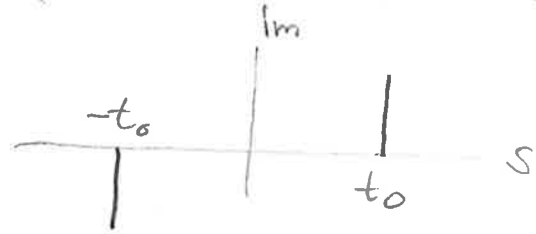
b) $\mathcal{F}(\sin(2\pi s_0 t)) = \dots = \frac{1}{2j}\left(-\delta(t-t_0) + \delta(t+t_0)\right)$



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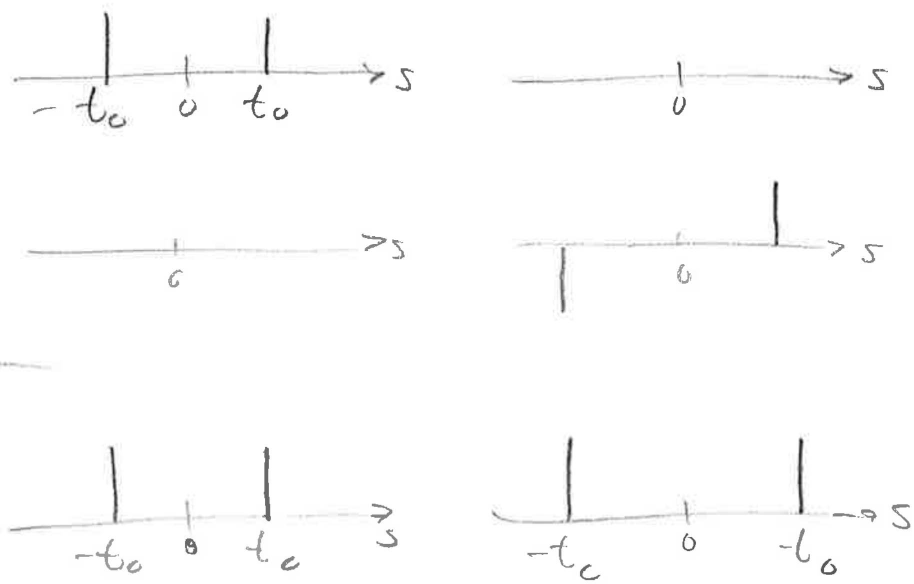
a word about Re/Im
and amplitude/phase :

cosine | sine

real

imaginary

amplitude



phase $\qquad \emptyset \qquad\qquad \pi/2$

Negative frequencies ?

$$\cos(-f) = \cos(f)$$
$$\sin(-f) = -\sin(f)$$

} negative part is redundant*

=> often: we only plot the positive frequency components

* provide no additional information

Example

Gaussian

$$\boxed{f(t) = e^{-\pi t^2}}$$

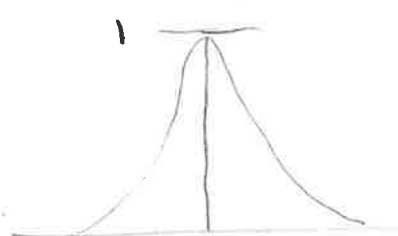
$$\left(\pi = \frac{1}{2\sigma^2} \Rightarrow \sigma^2 = \frac{1}{2\pi} , \ \sigma = \frac{1}{\sqrt{2\pi}} \right.$$

$$F(s) = \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-j 2\pi s t} \, dt = \int_{-\infty}^{\infty} e^{-\pi (t^2 + j2st)} \, dt$$

$$= \int_{-\infty}^{\infty} e^{-\pi \underbrace{(t+js)^2}_{u}} \cdot e^{-\pi s^2} \, dt$$

$$(t+js)^2 = t^2 + 2tjs + \overset{-1}{j^2} s^2 = t^2 + 2jst - s^2$$

$$= e^{-\pi s^2} \underbrace{\int_{-\infty}^{\infty} e^{-\pi u^2} \, du}_{1} = \boxed{e^{-\pi s^2}}$$



$$\boxed{\text{general: } \mathcal{F}\left(e^{-ax^2}\right) = \sqrt{\frac{\pi}{a}} \, e^{-\pi^2 s^2 / a}}$$

Large σ → small a → wide in $F(s)$

normalized Gaussian: $g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ | $\mu = 0$, $\sigma = 1$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \overset{?}{=} e^{-\pi x^2} = \quad \cdot e^{-\frac{1}{2}x^2}$$

general Gaussian:

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$$\mathcal{F}\left[e^{-\pi(at)^2}\right] = \frac{1}{a} e^{-\pi\left(\frac{s}{a}\right)^2}$$

↑

similarity theorem!
change of scale of abscissa

$$F(s) = \mathcal{F}[f(at)] = \int_{-\infty}^{\infty} f(at)\, e^{-i2\pi st}\, dt$$

$$= \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

$$f(t) = e^{-\pi(at)^2}$$

$$\downarrow$$

$$F(s) = e^{-\pi\left(\frac{s}{a}\right)^2}$$

large a → →|←
narrows Gaussian

⇒

Frequency:

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FT of convolution

(follows "Proof of ..." 5)

Part II: shift invariance:

$$f(x) \longrightarrow \boxed{\mathcal{F}} \xrightarrow{F(s)} \quad \text{then} \quad \xrightarrow{f(x-a)} \boxed{\mathcal{F}}\; e^{-j2\pi sa} F(s)$$

shifting $(x-a)$ does not change spectrum $F(u)$ but adds linear phase (mult by $e^{-j2\pi ua}$)*.
Multiplying $F(u)$ by $e^{j2\pi ua}$ for different a translates/shifts $f(x)$ by a.

$x' = x - a \qquad dx = dx'$

$$F[f(x-a)] = F[f(x')] = \int_{-\infty}^{\infty} f(x') e^{-j2\pi s(x'+a)} dx'$$

$$\left(e^{-j2\pi s(x'+a)} = \underbrace{e^{-j2\pi sa}}_{\text{constant}} e^{-j2\pi s x'} \right)$$

$$\Rightarrow F[f(x-a)] = e^{-j2\pi sa} F(s)$$

$\Rightarrow$ FT is shift invariant

show demo

(* Adds linear phase $\theta = sa$ to original phase.
Or: adding linear phase filter produces transl. of signal.)

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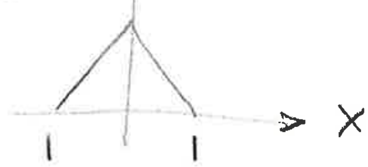
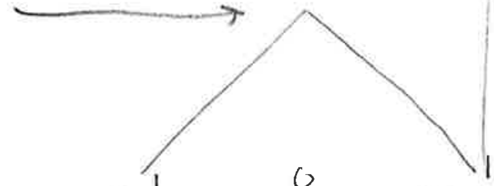
Quiz

See slide

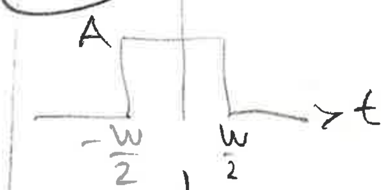
$f(x)$



$\mathcal{F}$?



notes

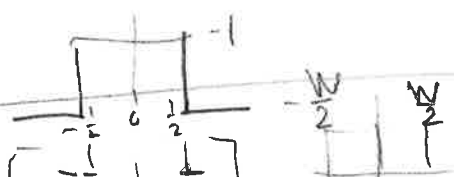


$$AW \frac{\sin \pi s W}{\pi s W}$$

$$\frac{1}{2}\ \underset{\text{box}}{\overset{-\frac{1}{2}\ 0\ \frac{1}{2}}{|\,1\,|\,1\,|}} \otimes \underset{\text{box}}{\overset{-\frac{1}{2}\ 0\ \frac{1}{2}}{|\,1\,|\,1\,|}} = \overset{-1 \quad\quad 0 \quad\quad 1}{|\ 0\ |\ 0{,}5\ |\ 1\ |\ 0{,}5\ |\ 0\ |}$$

$\downarrow \mathcal{F}$ $\qquad$ $\downarrow \mathcal{F}$ $\qquad$ $\downarrow \mathcal{F}$

sinc $\cdot$ sinc $=$ sinc^2



rect : $[-\frac{1}{2} \mid \frac{1}{2}]$

domain

$\to \text{sinc}(\pi s W)$

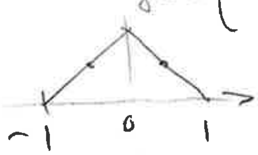
$\text{rect}(x) \begin{cases} 1 & \frac{x}{2} < x < \frac{x}{2} \\ 0 & \text{otherwise} \end{cases}$

See slide

$-\frac{W}{2}$ $+\frac{W}{2}$

$\Rightarrow$ scaled to $[-2, 2]$ from $[-\frac{1}{2}, \frac{1}{2}]$ $\Rightarrow \frac{x}{4}$ $(\frac{2}{4} = \frac{1}{2})$

triangle: $[-1 \mid 1]$ $\quad$ scaled to $[-2, 2]$ from $[-1, 1]$ $\Rightarrow \frac{x}{2}$



$\Rightarrow$

$$f(x) = \sqcap\left(\frac{x}{4}\right) - \wedge\left(\frac{x}{2}\right) + 0{,}5 \wedge$$

$$\Rightarrow \boxed{F(s) = 4\ \text{sinc}(4s) - 2\ \text{sinc}^2(2s) + 0{,}5\ \text{sin}^2(s)}$$

FT Properties

Linear (shift invariant) filters : LSI (see ess522)

$$x(t) \rightarrow \boxed{T} \rightarrow y(t) \quad \Big| \quad y(t) = T[x(t)]$$

1. shift invariant: $T[x(t+a)] = y(t+a)$
2. scale invariant: $T[a\,x(t)] = a\,y(t)$
3. superposition invariant: $T[x_1(t) + x_2(t)] = T[x_1(t)] + T[x_2(t)]$

Convolution

$$c(t) = a(t) \otimes b(t) = \int_{-\infty}^{\infty} a(\tau)\, b(t-\tau)\, d\tau$$

- commutative: $a(t) \otimes b(t) = b(t) \otimes a(t)$
- associative: $(a(t) \otimes b(t)) \otimes c(t) = a(t) \otimes (b(t) \otimes c(t))$
- distributive: $a(t) \otimes [b(t) + c(t)] = [a(t) \otimes b(t)] + [a(t) \otimes c(t)]$

sum of 2 convolved signals = convolution of sum

- shift invariant:

$$x(t+a) \xrightarrow{T} y(t+a)$$

produces shifted output when given shifted input

(following "Proof of ..")

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Part II: Convolution Theorem

$$f(t) \otimes g(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau \qquad (7)$$

$\mathcal{F}$ - transform:

$$\mathcal{F}[f \otimes g] = \int_{-\infty}^{\infty} \Big[\underbrace{\int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau}_{\text{function of } t \text{ after evaluation of } \int \text{ over } \tau} \Big] e^{-j2\pi s t}\, dt$$

reverse order of integration

$$\int_{-\infty}^{\infty} f(\tau) \Big[\int_{-\infty}^{\infty} g(t-\tau)\, e^{-j2\pi s t}\, dt \Big] d\tau$$

(- $f(\tau)$ can be treated as constant when integrating over dt
- move bracket)

use Part I: $\int g(t-\tau) e^{-j2\pi s t} dt = \mathcal{F}[g(t-\tau)] = e^{-j2\pi s \tau} G(s)$

$$\int_{-\infty}^{\infty} f(\tau)\, e^{-j2\pi s \tau} G(s)\, d\tau$$

$$= \underbrace{\Big[\int_{-\infty}^{\infty} f(\tau)\, e^{-j2\pi s \tau}\, d\tau \Big]}_{} G(s)$$

$$= F(s)\,G(s) \quad \Rightarrow \quad \boxed{\mathcal{F}[f \otimes g] = F(s)\,G(s)}$$

Convolution in space/time domain is equivalent to multiplication in the frequency domain.

$f(t)\,g(t) \longleftrightarrow F(s) \otimes G(s)$

mult in time/space domain — convolution in frequency space

Symmetry

Inverse filtering, Reconstruction

$f(t)$: filter

$g(t)$: signal

$h(t)$: filtered signal

$$H(s) = \overbrace{\underset{\uparrow \text{known}}{F(s)}\,G(s)}^{\text{measured}}$$

$$\downarrow$$

$$G(s) = \frac{H(s)}{F(s)} \quad \Big| \; F(s) \neq 0$$

$$g(t) = \mathcal{F}^{-1}\left[\frac{H(s)}{F(s)}\right]$$

<u>example</u>: image is corrupted by blurring (out of focus)

$\Rightarrow$ sharpening of image ?

✓ filter is known (e.g. Gaussian)

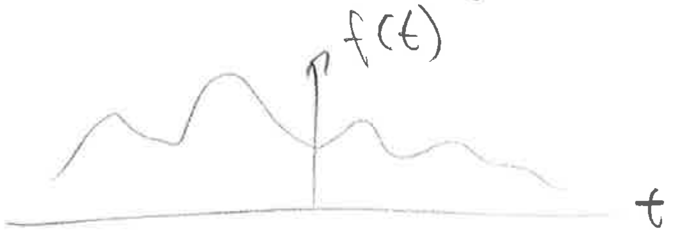
✓ image $f(t) \otimes g(t)$ is acquired

$$\downarrow$$

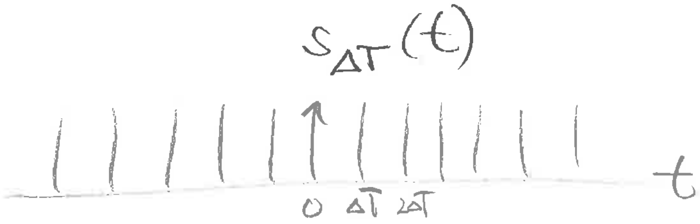
estimate $g(t)$

not so simple, but idea is right!

4.3.1 Sampling (DIP 4.3.1, page 211/212)



(e.g. camera sensor, scanner: signal measured at discrete locations)



train of impulses

$$s_{\Delta T}(t) = \sum_{k=-\infty}^{\infty} \delta(t - n\Delta T)$$

sampled values

→ Sifting property (4.2.3)

sampling: $$\tilde{f}(t) = \underbrace{f(t)\, s_{\Delta T}(t)}_{\text{multiply signal } f(t) \text{ by sampling function}} = \sum_{n=-\infty}^{\infty} f(t)\, \delta(t - n\Delta T)$$

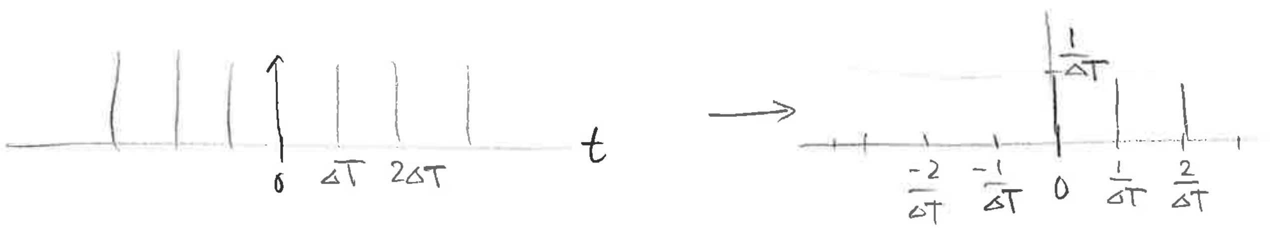
$$\Rightarrow f_k = \int_{-\infty}^{\infty} f(t)\, \delta(t - k\Delta T)\, dt = f(k\Delta T)$$

(sifting property: value of $f(t)$ at $t = k\Delta T$)

4.3.2 Fourier Transform of Sampled Function

$$\tilde{F}(s) = \mathfrak{F}\big(\overset{\text{sampled}}{\tilde{f}}(t)\big) = \mathfrak{F}\big(f(t)\, s_{\Delta T}(t)\big) \overset{\text{convolution theorem}}{=} F(s) \otimes S(s)$$

- $F(s)$: $\mathfrak{F}(f(t))$
- $S(s)$: $\frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta\left(s - \frac{n}{\Delta T}\right)$ (see p. 208/209)



(book 213)

$$\Rightarrow \boxed{\tilde{F} = F(s) \otimes S(s) = \int_{-\infty}^{\infty} F(\tau)\, S(s-\tau)\, d\tau}$$

$$= \frac{1}{\Delta T} \int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{\infty} \delta\left(s - \tau - \frac{n}{\Delta T}\right) d\tau$$

$$= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} F(\tau)\, \delta\left(s - \tau - \frac{n}{\Delta T}\right) d\tau}_{\text{sifting property}}$$

$$\boxed{= \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(s - \frac{n}{\Delta T}\right)}$$

What does that mean?

$\tilde{F}(s)$: infinite, periodic sequence of copies of $F(s)$, with $F(s)$ transform of original, continuous function!

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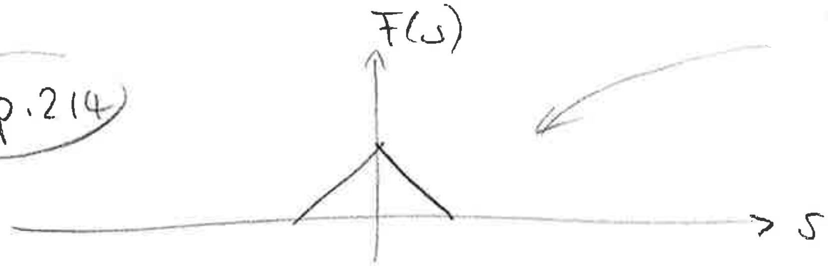
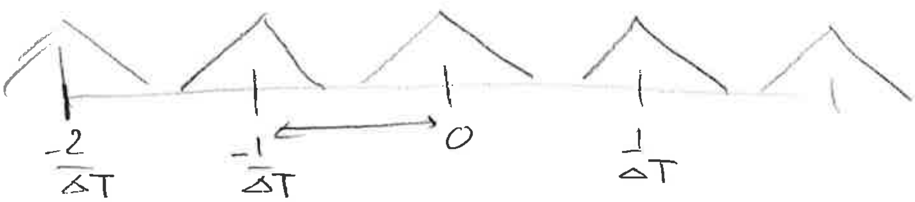
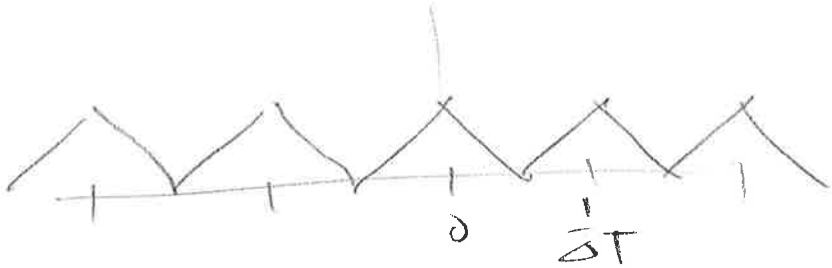
$\mathcal{F}(f(t))$



$\frac{1}{\Delta T}$: sampling rate (cycles/mm, e.g.)

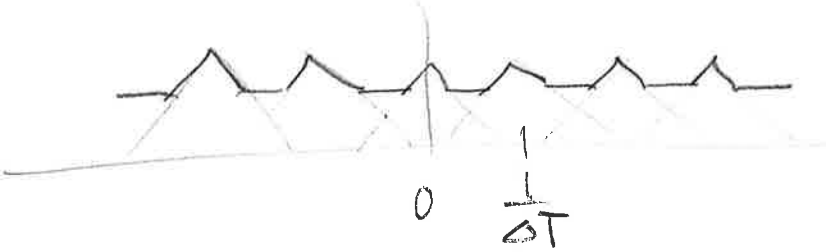


sampling rate high enough to provide sufficient separation between periods: oversampled signal



just enough

critically-sampled



low value of $\frac{1}{\Delta T}$ causes periods to merge

coarse!

sampling rate below the minimum required to maintain distinguished copies of $F(s)$ → fails to preserve original transform!

under-sampled signal

Recover continuous $f(t)$ from $\tilde{F}(s)$ (FT of sampled fct $\tilde{f}(t)$)

need one complete period to characterize entire transform!

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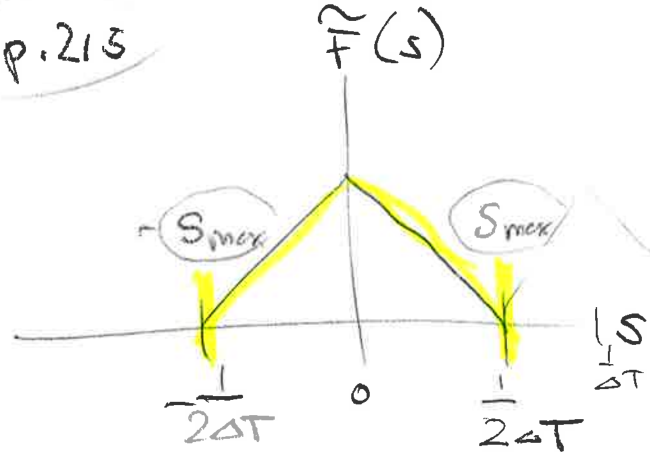


$$\frac{1}{2\Delta T} > s_{max}$$

$$\frac{1}{\Delta T} > 2\, s_{max}$$

Sampling Theorem

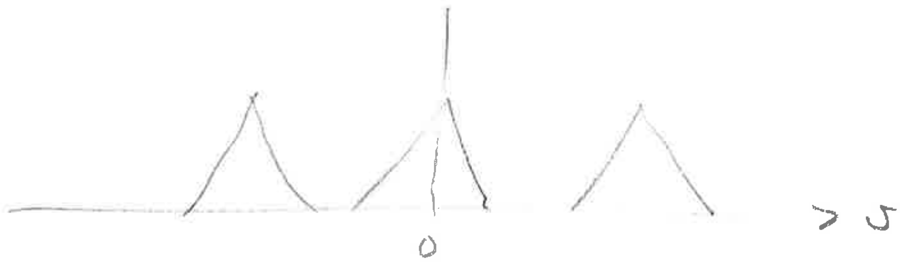
- Continuous, band-limited, function can be recovered completely from a set of its samples if samples acquired at a rate exceeding twice the highest frequency content of the function.
- Exactly twice the highest frequency: $\boxed{\text{Nyquist rate}}$
- Conversely:

  Maximum frequency that can be "captured" by sampling a signal at a rate $\frac{1}{\Delta T}$ is $s_{max} = \frac{1}{2\Delta T}$.

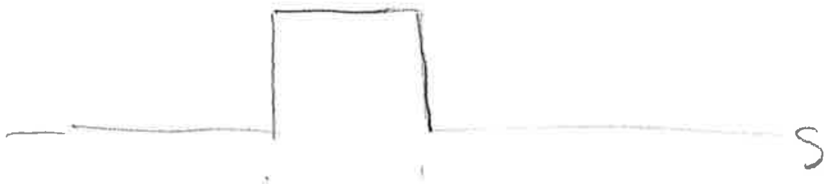
Procedure

sampled

$\tilde{\mathcal{F}}(s)$



$\tilde{F}(s)$

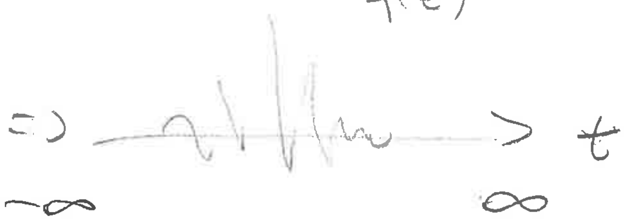


$H(s)$: low-pass filter

$F(s) = H(s)\tilde{F}(s)$

$\updownarrow \mathcal{F}^{-1}$

$f(t)$



$f(t) = \mathcal{F}^{-1}(F(s))$

real world: $f(t)$ has limited range $\int_{-t_{min}}^{t_{max}}$

$\Rightarrow$ ideal recovery impossible

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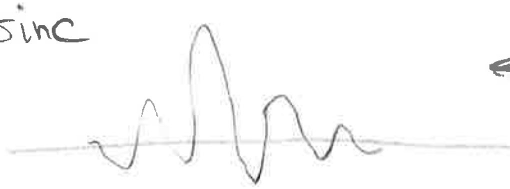
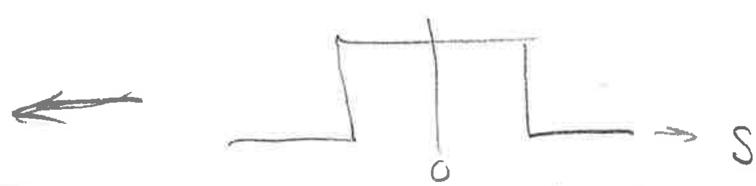
rect function in frequency domain?

sinc





convolution with sinc in space/time domain

cutting higher frequencies in frequency domain

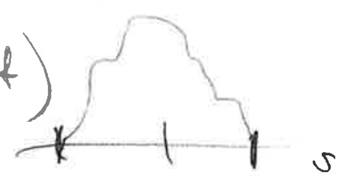
$$\Rightarrow \overset{\text{reconstr}}{f(t)} = \overset{\text{sampled}}{\tilde{f}(t)} \otimes \mathrm{sinc}\left(\frac{t}{\Delta T}\right)$$

$$= \sum_k \overset{\text{samples}}{f_k} \, \mathrm{sinc}\left(\frac{t - k\Delta T}{\Delta T}\right)$$

See slide

Problem: most fcts not band limited!

(power spectrum has finite support)



$\Rightarrow$ forcing fcts to be band limited

$\longrightarrow$ ringing artifacts

See slides

Aliasing 4.3.4

(Example book p.219)

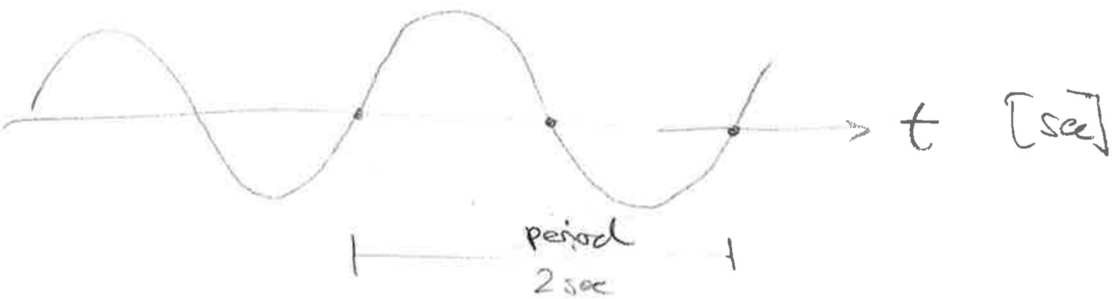
sine wave → single frequency → band limited?

$\sin(\pi t)$ period is 2s, frequency $\frac{1}{2}$ cycles/s

(sampling theorem: $\frac{1}{\Delta T} >$ twice highest frequency of signal

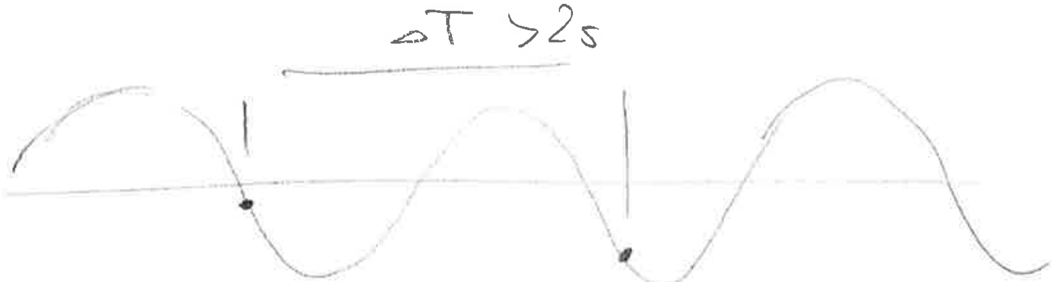
$\frac{1}{\Delta T} > 1$ sample/sec

$\Delta T < 1s$



if exactly twice: $\Delta T = 1s \Rightarrow \sin(-\pi), \sin(0), \sin(\pi)$ all $\emptyset$?

$\Rightarrow \Delta T < 1s$



what if samples taken with separation $\Delta T > 2s$?

$\Rightarrow$ sampling rate $\frac{1}{\Delta T} < \frac{1}{2}$ samples/sec

$\Rightarrow$ looks like sine wave, but frequency $\approx \frac{1}{10}$ th ?

$\Rightarrow$ Aliasing