CS/BIOEN 6640 Guido Gerig | DIP | Chapter 4 (1)

Fourier Transforms

10/6/2014

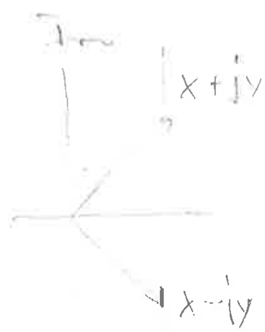
Jean Baptiste Joseph Fourier (1768 ..)

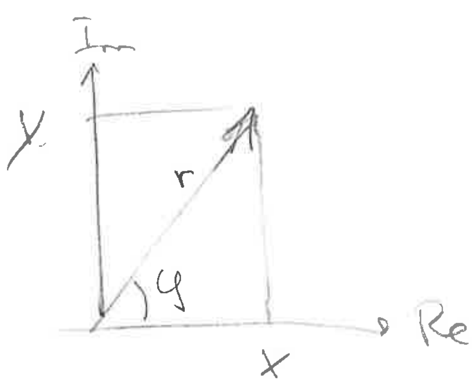
Basic contribution: 1807

- any periodic function expressed by a weighted sum of sines/cosines of different frequencies → Fourier Series

- non-periodic functions: Expressed as the integral of sines/cosines with weights → Fourier Transform

Review complex numbers:





$$z = x + jy$$

$$r = |z| = \sqrt{x^2 + y^2}$$

polar form: $z = r(\cos\varphi + j\sin\varphi)$

$= r\, e^{j\varphi}$ (Euler)

amplitude   phase

$= r \angle \varphi$ (Electronics, phasor)

trigonometric identity:

$$e^{j\varphi} = \cos\varphi + j\sin\varphi$$

$$\cos\varphi = \frac{1}{2}\left(e^{j\varphi} + e^{-j\varphi}\right)$$

$$\sin\varphi = \frac{1}{2j}\left(e^{j\varphi} - e^{-j\varphi}\right)$$

$$e^{j\varphi} + e^{-j\varphi} = \cos\varphi + \cancel{j\sin\varphi} + \cos\varphi \cancel{- j\sin\varphi} = 2\cos\varphi$$

$$e^{j\varphi} - e^{-j\varphi} = \cancel{\cos\varphi} + j\sin\varphi \cancel{- \cos\varphi} + j\sin\varphi = 2j\sin\varphi$$

frequency domain   time/spatial domain

FT: $\boxed{F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi st}\, dt} \boxed{= \mathcal{F}(f(t))}$

$$= \int_{-\infty}^{\infty} f(t)\cdot[\underbrace{\cos(2\pi st) - j\sin(2\pi st)}_{\text{cosine / sin functions}}]\, dt$$

t: time, space (x), continuous variable

s: ? frequency, also continuous
t: unit s: $\frac{\#\text{cycles}}{\text{unit}}$

unit [t vs. s; if t = mm, s = $\frac{\text{cycles}}{\text{mm}}$ *]

inverse FT:

$$\boxed{\mathcal{F}^{-1}(F(s)) =} \boxed{f(t) = \int_{-\infty}^{\infty} F(s)\, e^{+j2\pi st}\, ds}$$

(function can be recovered from it's Fourier pair)

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* (other conventions (e.g. Mathematica): $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{j\omega t} dt$
=> ω in radians

Fourier Integral Theorem

$$\boxed{f(t) = \mathcal{F}^{-1}(\mathcal{F}(f(t)))}$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) e^{-j2\pi st} dt \right] e^{j2\pi st} ds$$

Fourier Series:

weighted sum of basis function

general: $f(x) = \sum_i w_i B_i(x)$

slide

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-j\frac{2\pi n}{T}t} dt$$

$$\Rightarrow f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t}$$

c_n: complex

or $f(t) = a_0 + \sum_{n=1}^{\infty} a_n \sin(j2\pi \frac{n}{T} t) + \sum_{n=1}^{\infty} b_n \cos(j2\pi \frac{n}{T} t)$

slide with animations

Demonstrate

- Square wave

- magnitude / phase

  $\cos(x + \pi) = \cos(x)$

  $\cos(x + \frac{\pi}{2}) = \sin(x)$

- triangle magnitude, phase

- low-pass filter: cut higher frequencies ?

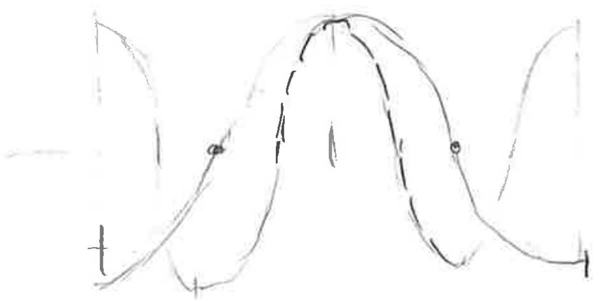
demo - low-pass filter

Fourier Basis

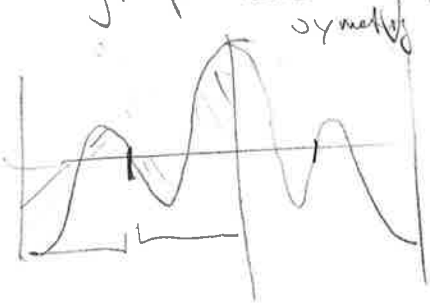
- orthonormal in $[-\pi, \pi]$

can be shown!

$$\int_{-\pi}^{\pi} \cos(x) \cdot \cos(2x)\, dx = 0$$



graphically:



$$\cos(2x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$
$$= \cos^2(x) - \sin^2(x)$$

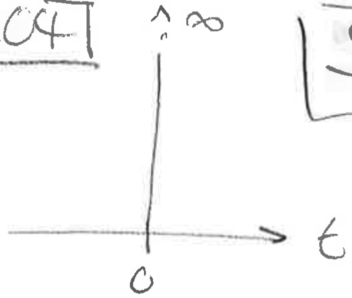
$$\left[\cos(x)\left(\cos^2(x) - \sin^2(x)\right) = \cos^3(x) - \cos(x)\sin^2(x)\right.$$

$$\cos(x)\left(2\cos^2(x) - 1\right) = 2\cos^3(x) - \cos(x)$$

DIP 203 (204)
continuous

Sifting Property

$\delta(t)$



delta funct' / Dirac $\begin{cases} \infty & t=0 \\ 0 & t\neq 0 \end{cases}$ $\displaystyle\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

spike in amplitude, zero duration

sifting property: $\boxed{\displaystyle\int_{-\infty}^{\infty} f(t)\,\delta t\,dt = f(0)}$

(convolution with spike at 0 → function at 0!)

sifting: value of $f(t)$ at location of impulse

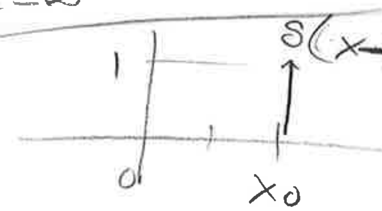
⇒ general: $\boxed{\displaystyle\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)}$

(example: $f(t) = \cos(t)$, impulse at $\delta(t-\pi)$

$\Rightarrow f(\pi) = \cos(\pi) = -1$)

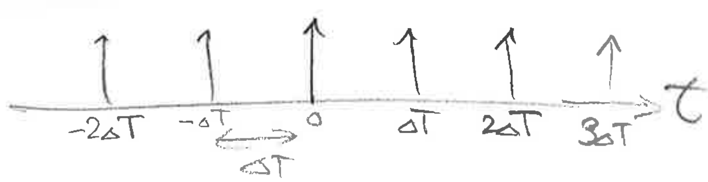
discrete:

$\delta(x) = \begin{cases} 1 & x=0 \\ 0 & \text{otherwise} \end{cases}$ $\boxed{\displaystyle\sum_{x=-\infty}^{\infty} \delta(x) = 1}$ $\boxed{\displaystyle\sum_{x=-\infty}^{\infty} f(x)\,\delta(x-x_0) = f(x_0)}$



Impulse train

$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T) \qquad s_{\Delta T}(t)$$



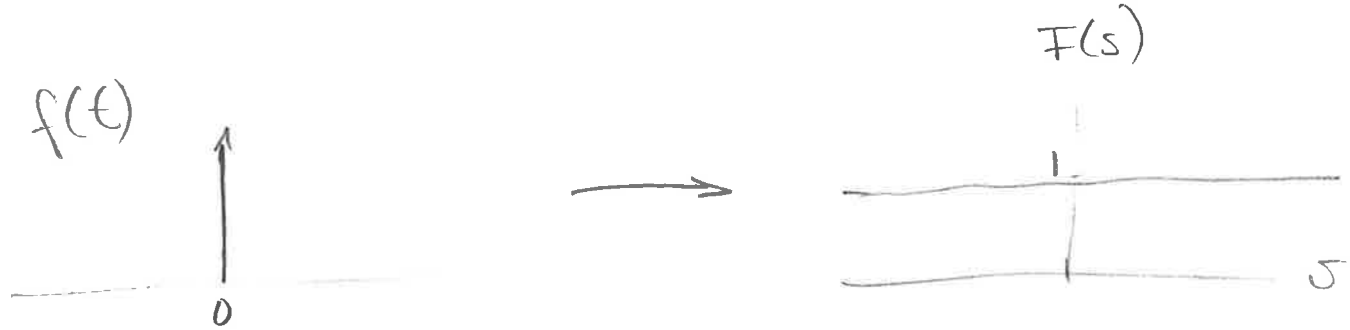
Example

Impulse at origin:

$$F(s) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi st}\, dt$$

sifting prop.

$$= \int_{-\infty}^{\infty} e^{-j2\pi st} \cdot \delta(t)\, dt = e^{-j2\pi s \phi} = e^{\phi} = 1$$

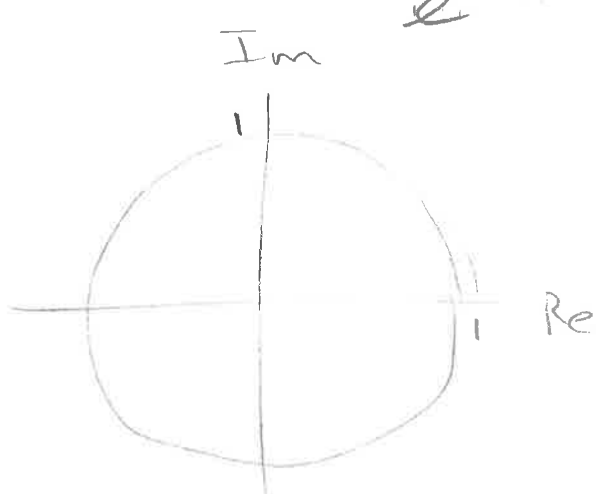


at $t = t_0$:

$$F(s) = \int_{-\infty}^{\infty} e^{-j2\pi st}\, \delta(t-t_0)\, dt$$

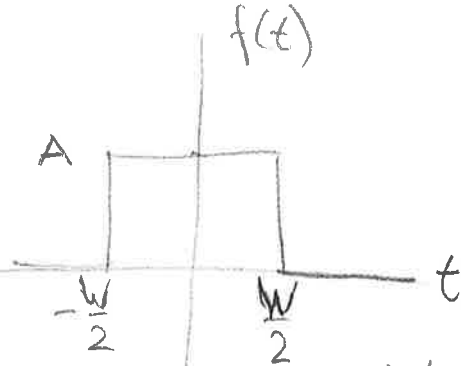
# of repetitions

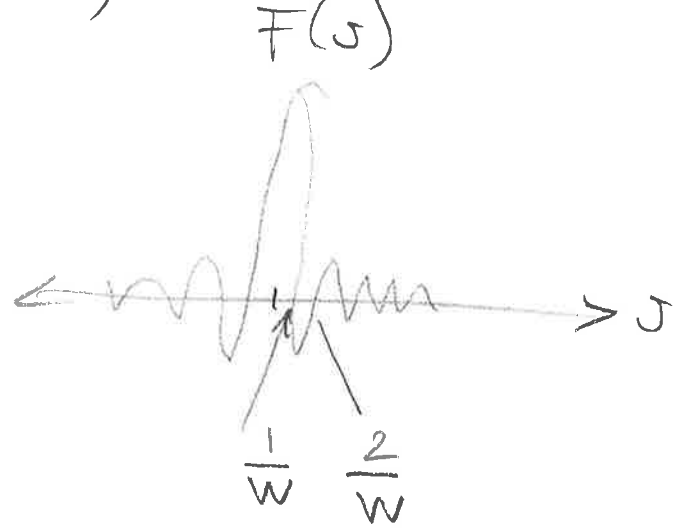
$$= e^{-j2\pi s t_0} = \cos(2\pi s t_0) - j\sin(2\pi s t_0)$$



?

Example: $\mathcal{F}(\text{Rect}[t])$





$$\text{Rect}[t] \begin{cases} 1 & |t| \le \frac{W}{2} \\ 0 & \text{otherwise} \end{cases}$$

(HeavisidePi Mathematica)

(Book DIP p 206, ~~example~~ 4.1)

$$\Rightarrow\Rightarrow \quad F(s) = AW \underbrace{\frac{\sin \pi s W}{(\pi s W)}}_{\text{known as sinc}} = AW \sin(\pi s W)$$

important:

- location of Ø's inversely prop to W
- height decreases as fct of distance from origin
- extend to infinity

$$\text{sinc}(m) = \frac{\sin \pi (m)}{(\pi m)}$$

$\text{sinc}(0) = 1$

$\text{sinc}(m = \text{integer}) = 0$

in general: FT contains complex terms

$\Rightarrow$ for display purposes, magnitude of the transform

$$\Rightarrow |F(s)|$$

Examples

$\cos(2\pi k_0 t)$

$$\mathcal{F}\left(\cos(2\pi k_0 t)\right) = \int_{-\infty}^{\infty} e^{-j2\pi kt}\,\frac{1}{2}\left(e^{j2\pi k_0 t} + e^{-j2\pi k_0 t}\right)dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\left(e^{-j2\pi(k-k_0)t} + e^{-j2\pi(k+k_0)x}\right)dt$$

$$= \frac{1}{2}\left[\delta(k-k_0) + \delta(k+k_0)\right]$$

$\sin(2\pi k_0 \cdot)$

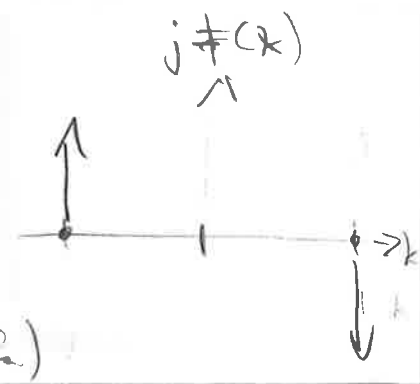
$$\mathcal{F}\left(\sin(2\pi k_0 t)\right) = \int_{-\infty}^{\infty} e^{-j2\pi kt}\cdot\frac{1}{2j}\left(e^{j2\pi k_0 t} - e^{-j2\pi k_0 t}\right)dt$$

$j^2 = -1$

$$= \frac{1}{2j}\int_{-\infty}^{\infty}\left(e^{-j2\pi(k-k_0)t} - e^{-j2\pi(k+k_0)t}\right)dt$$

$$= -\frac{1}{2}j\left(\delta(k-k_0) - \delta(k+k_0)\right)$$

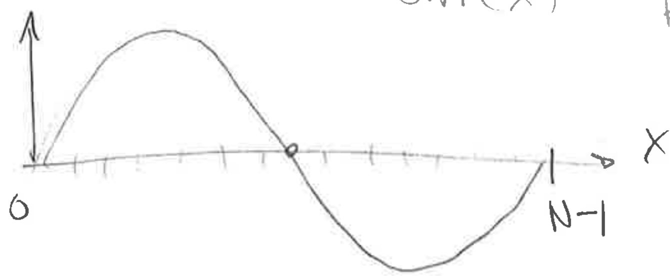
$$= \frac{1}{2}j\left(\delta(k+k_0) - \delta(k-k_0)\right)$$

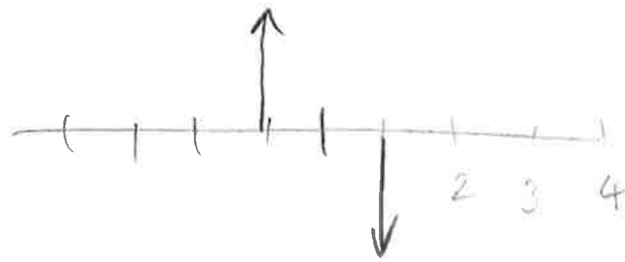


(no real part, imaginary Fourier transf.)

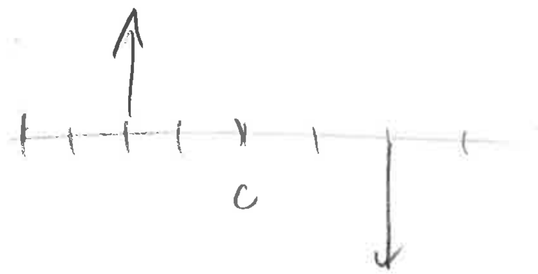
Note:

$\sin(x)$ $k_0=1$

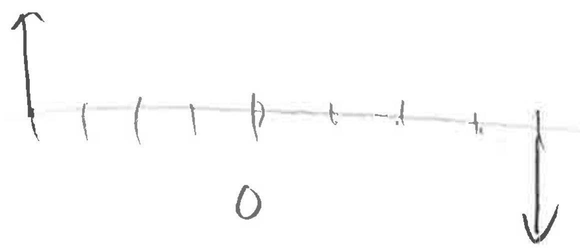




$\sin(2x)$ $k_0=2$



$\sin(4x)$ $k_0=4$



low frequencies

high frequencies

Example
Gaussian

$$\boxed{f(t) = e^{-\pi t^2}}$$

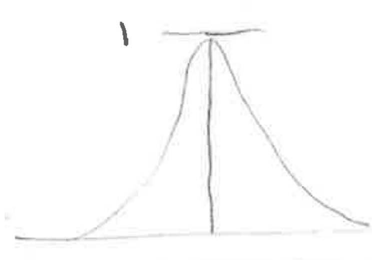
$$\left(\pi = \frac{1}{2\sigma^2} \Rightarrow \sigma^2 = \frac{1}{2\pi},\ \sigma = \frac{1}{\sqrt{2\pi}} \right.$$

$$F(s) = \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-j 2\pi s t} dt = \int_{-\infty}^{\infty} e^{-\pi(t^2 + j2st)} dt$$

$$= \int_{-\infty}^{\infty} e^{-\pi \underbrace{(t+js)^2}_{u}} \cdot e^{-\pi s^2} dt$$

$$(t+js)^2 = t^2 + 2tjs + \overset{-1}{j^2} s^2 = t^2 + 2jst - s^2$$

$$= e^{-\pi s^2} \underbrace{\int_{-\infty}^{\infty} e^{-\pi u^2} du}_{1} = \boxed{e^{-\pi s^2}}$$



$$\boxed{\text{general: } \mathcal{F}\left(e^{-ax^2}\right) = \sqrt{\frac{\pi}{a}}\, e^{-\pi^2 s^2/a}}$$

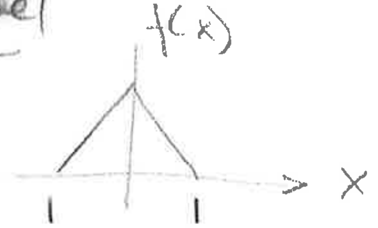
Large $\sigma \rightarrow$ small a
$\rightarrow$ wide in $F(s)$

normalized Gaussian: $g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ | $\mu = 0$, $\sigma = 1$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \overset{?}{=} e^{-\pi x^2} = \cdot e^{-\frac{1}{2}x^2}$$

Quiz

$\boxed{\text{see slide}}$



$\xrightarrow{\mathcal{F}\,?}$

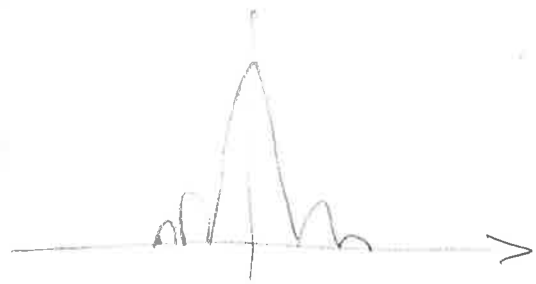
$\frac{1}{2}$ | 1 | 1 | $\otimes$ | 1 | 1 | = | 0 | 0,5 | 1 | 0,5 | 0 |

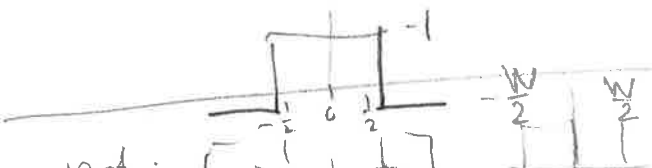
box ↓ $\mathcal{F}$ ↓ sinc

box ↓ $\mathcal{F}$ ↓ sinc

↓ $\mathcal{F}$ ↓ sinc^2

sinc · sinc = sinc^2





rect: $\left[-\frac{1}{2} \,\middle|\, \frac{1}{2}\right]$ domain

$-\frac{W}{2}$ $\frac{W}{2}$ $\rightarrow \text{sinc}(\pi s W)$

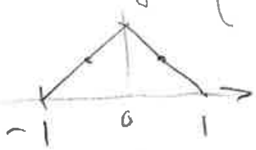
$\text{rect}(x) \begin{cases} 1 & -\frac{X}{2} < x < \frac{X}{2} \\ 0 & \text{otherwise} \end{cases}$

$-\frac{W}{2}$ $+\frac{W}{2}$

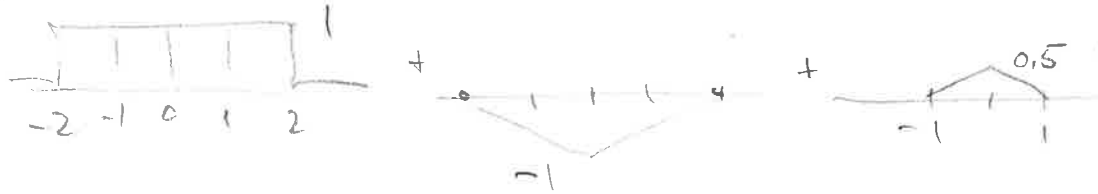
$\boxed{\text{see slides}}$

$\Rightarrow$ scaled to $[-2, 2]$ from $[-\frac{1}{2}, \frac{1}{2}]$ $\Longrightarrow \frac{x}{4}$ $\left(\frac{2}{4} = \frac{1}{2}\right)$

triangle: $[-1 \,|\, 1]$ scaled to $[-2, 2]$ from $[-1, 1]$ $\Rightarrow \frac{x}{2}$



$\Rightarrow$



$$f(x) = \sqcap\left(\frac{x}{4}\right) - \wedge\left(\frac{x}{2}\right) + 0{,}5 \wedge$$

$$\Rightarrow \boxed{F(s) = 4\,\text{sinc}(4s) - 2\,\text{sinc}^2(2s) + 0.5\,\text{sinc}^2(s)}$$

FT Properties

Linear (shift invariant) filters : LSI (see ess522)

$$x(t) \rightarrow \boxed{T} \rightarrow y(t) \qquad \Big| \qquad y(t) = T[x(t)]$$

1. shift invariant: $T[x(t+a)] = y(t+a)$
2. scale invariant: $T[a\,x(t)] = a\,y(t)$
3. superposition invariant: $T[x_1(t) + x_2(t)] = T[x_1(t)] + T[x_2(t)]$

Convolution

$$c(t) = a(t) \circledast b(t) = \int_{-\infty}^{\infty} a(\tau)\, b(t-\tau)\, d\tau$$

- commutative: $a(t) \otimes b(t) = b(t) \otimes a(t)$
- associative: $(a(t) \otimes b(t)) \otimes c(t) = a(t) \otimes (b(t) \otimes c(t))$
- distributive: $a(t) \otimes [b(t) + c(t)] = [a(t) \otimes b(t)] + [a(t) \otimes c(t)]$

sum of 2 convolved signals = convolution of sum

- shift invariant:

$$x(t+a) \xrightarrow{T} y(t+a)$$

produces shifted output when given shifted input

FT of convolution

(following "Proof of ...")

Part I: shift invariance:

$$f(x) \longrightarrow \boxed{\mathfrak{F}} \xrightarrow{F(s)} \quad \text{then} \quad f(x-a) \longrightarrow \boxed{\mathfrak{F}}\ e^{-i2\pi sa} F(s)$$

shifting $(x-a)$ does not change spectrum $F(u)$
but adds linear phase (mult by $e^{-j2\pi ua}$)*,
Multiplying $F(u)$ by $e^{j2\pi ua}$ for
different a translates/shifts $f(x)$ by a.

$x' = x-a \quad dx = dx'$

$$F[f(x-a)] = F[f(x')] = \int_{-\infty}^{\infty} f(x') e^{-j2\pi s(x'+a)} dx'$$

$$e^{-j2\pi s(x'+a)} = \underbrace{e^{-j2\pi sa}}_{\text{constant}} e^{-j2\pi s x'}$$

$$\Rightarrow F[f(x-a)] = e^{-j2\pi sa} F(s)$$

$\Rightarrow$ FT is shift invariant

(* Adds linear phase $\theta = sa$ to original phase.
Or: adding linear phase filter produces transl. of signal.)

(following "Proof of..")

Part II: Convolution Theorem

$$f(t) \otimes g(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$

$\mathcal{F}$ - transform:

$$\mathcal{F}[f \otimes g] = \int_{-\infty}^{\infty} \Big[\underbrace{\int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau}_{\text{function of } t \text{ after evaluation of } \int \text{ over } \tau} \Big] e^{-j2\pi s t}\, dt$$

reverse order of integration

$$\int_{-\infty}^{\infty} f(\tau) \Big[\int_{-\infty}^{\infty} g(t-\tau)\, e^{-j2\pi s t}\, dt \Big] d\tau$$

(- $f(\tau)$ can be treated as constant when integrating over dt
- move bracket)

use Part I: $\int g(t-\tau) e^{-j2\pi s t} dt = \mathcal{F}[g(t-\tau)] = e^{-j2\pi s \tau} G(s)$

$$\int_{-\infty}^{\infty} f(\tau)\, e^{-j2\pi s \tau} G(s)\, d\tau$$

$$= \Big[\int_{-\infty}^{\infty} f(\tau)\, e^{-j2\pi s \tau}\, d\tau \Big] G(s)$$

$$= F(s) G(s) \Rightarrow \boxed{\mathcal{F}[f \otimes g] = F(s) G(s)}$$

Convolution in space / time domain is equivalent to multiplication in the frequency domain.