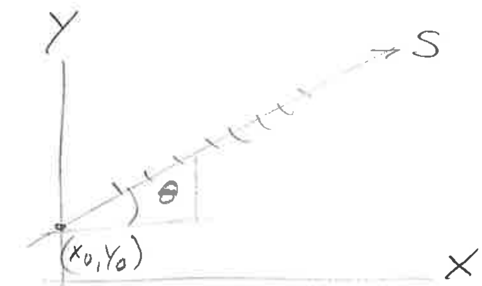
11/23/14 G. Gerig

"Snakes": Notes on length parametrization of curves

Parametrization with arc-length

Line: $\underline{v}(s) : \begin{bmatrix} x(s) \\ y(s) \end{bmatrix}$

Line :

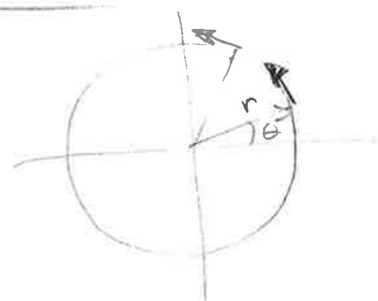


$$x(s) = x_0 + s \cdot \cos\theta$$
$$y(s) = y_0 + s \cdot \sin\theta$$

$$\underline{v}_s = \frac{\partial \underline{v}(s)}{\partial s} = \begin{pmatrix} \frac{\partial x(s)}{\partial s} \\ \frac{\partial y(s)}{\partial s} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

speed, velocity

Circle :



$$x = r \cdot \cos\theta \quad \left[0 \leq \theta \leq 2\pi \right]$$
$$y = r \cdot \sin\theta$$

velocity vector: $\frac{d\underline{v}(\theta)}{d\theta} = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \end{bmatrix}$

$$\|\underline{r}'(\theta)\| = \sqrt{r^2(\sin^2\theta + \cos^2\theta)} = r$$

- velocity is constant
- velocity vector is orthogonal to $\underline{r}$

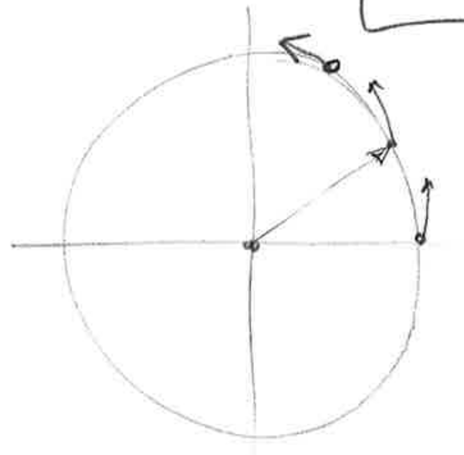
$$s(\theta) = \int_0^\theta \underbrace{\sqrt{|-r\sin u|^2 + |r\cos u|^2}}_{\text{velocity}}\, du$$

$$= \int_0^\theta \sqrt{r^2(\sin^2 u + \cos^2 u)}\, du =$$

$$\Rightarrow s(\theta) \quad r \int_0^\theta 1\, du = r \cdot \theta$$

$$\Rightarrow \theta = \frac{s(\theta)}{r} \Rightarrow \underline{\text{circle}}\ \underline{c}(s) = \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} r.\cos\left(\frac{s}{r}\right) \\ r.\sin\left(\frac{s}{r}\right) \end{bmatrix} \Bigg| \; 0 \leq s \leq 2\pi r$$

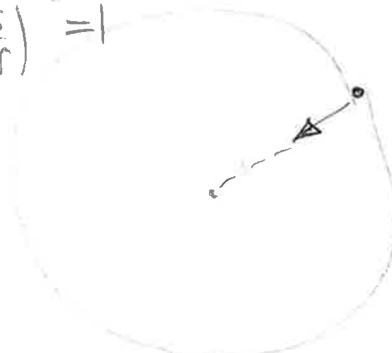
$$\underline{c}(s) = \begin{bmatrix} r\cos\left(\frac{s}{r}\right) \\ r\sin\left(\frac{s}{r}\right) \end{bmatrix}$$



$$\underline{c}_s(s) = \frac{d\,\underline{c}(s)}{ds} = \begin{bmatrix} -\sin\left(\frac{s}{r}\right) \\ \cos\left(\frac{s}{r}\right) \end{bmatrix}$$

$$\|c_s(s)\| = \sqrt{\sin^2\left(\frac{s}{r}\right) + \cos^2\left(\frac{s}{r}\right)} = 1$$

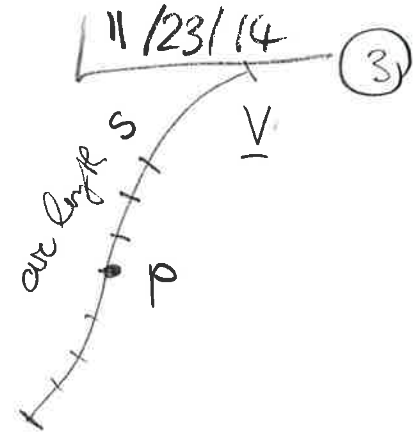
$$\underline{c}_{ss}(s) = \frac{d\,\underline{c}_s}{ds} = \begin{bmatrix} -\frac{1}{r}\cos\left(\frac{s}{r}\right) \\ -\frac{1}{r}\sin\left(\frac{s}{r}\right) \end{bmatrix}$$



- $\|\underline{c}_{ss}(s)\| = \frac{1}{r} \overset{*}{=} \kappa$
- $\underline{c}_{ss}$ points towards center for all s

curvature $\kappa = \frac{1}{r}$

General curves

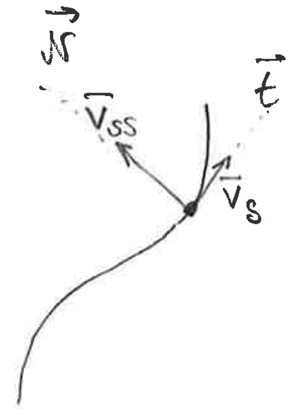


curve: $\underline{V}(s) = \left(x(p),\ y(p) \right)$

Velocity: $\underline{V}_p(p) = \left(x_p(p),\ y_p(p) \right)$

speed: $\left| \underline{V}_p(p) \right| = \sqrt{x_p(p)^2 + y_p(p)^2}$

(magnitude of velocity vector)



$$s = \int_0^p \left| \underline{V}_p(u) \right| du \implies \frac{ds}{dp} = \left| \underline{V}_p \right|$$

$$\underline{V}_s = \frac{dv}{dp} \frac{dp}{ds} = \frac{\frac{dv}{dp}}{\frac{ds}{dp}} = \frac{(x_p,\ y_p)}{\left| \underline{V}_p \right|}$$

Norm vector in tangent direction

$$\left\| \underline{V}_s \right\| = 1$$

chain rule

$$\underline{V}_{ss} = \frac{d\underline{V}_s}{dp} \cdot \frac{dp}{ds} = \underbrace{\frac{x_p y_{pp} - x_{pp} y_p}{\left(x_p^2 + y_p^2 \right)^{3/2}}}_{\kappa} \cdot \underbrace{\frac{(-y_p,\ x_p)}{\left(x_p^2 + y_p^2 \right)^{1/2}}}_{\vec{N}}$$

curvature

Normal pointing towards curvature center

$$\implies \boxed{\underline{V}_{ss} = \kappa \cdot \vec{N}}$$

$$\kappa = \frac{1}{r}$$