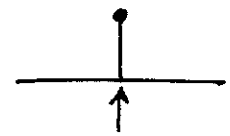
Canny Edge Detection

Concept: Image $\longrightarrow$ $\boxed{\mathbb{H}[\,]}$ $\longrightarrow$ Edge Map

$f \rightarrow \boxed{\mathbb{H}} \rightarrow g$



$g(x) = \mathbb{H}[f(x)]$ LSI system

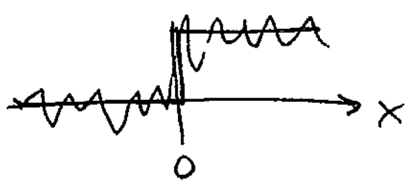
Criteria for optimality:

(i) low probability of error $\Rightarrow$ maximize SNR
(false negatives, false positives)

(ii) good localization (close to true edge)

(iii) only one response to a single step edge

Model: Noisy, ideal step edge centered at 0



$f(x) = m(x) + n(x)$
white, Gaussian noise

---

a) SNR

$f(\hat{x}) = m(\hat{x}) + n(\hat{x})$

$$\mathbb{H}[f(x)] = \int \underbrace{h(x-\hat{x})}_{\text{filter kernel}} \underbrace{f(\hat{x})}_{\text{signal}} \, d\hat{x}$$

- response to edge at position 0 : $\mathbb{H}[m](0) = \int_{-\infty}^{\infty} h(-\hat{x}) m(\hat{x}) \, d\hat{x}$

$$= H_m(0)$$

- response to noise: stochastic signal characterized by expectation

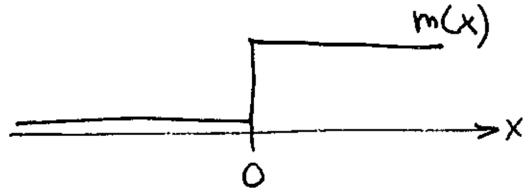
$$\sqrt{E\left[\int_{-\infty}^{\infty} h(\hat{x})\, n(x-\hat{x})\, d\hat{x}\right]^2}\,\Bigg|_{x=0}$$

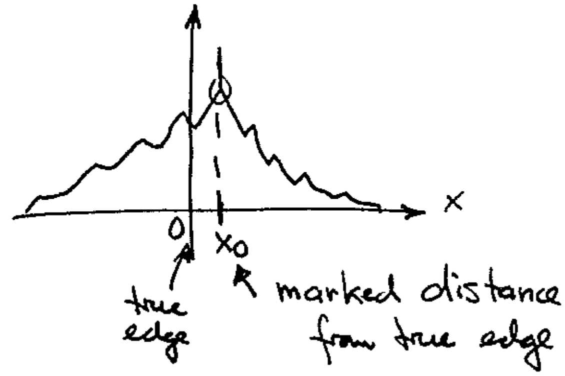
$$= \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\hat{x})\, h(\tilde{x})\, \underbrace{E\left[n(-\tilde{x})\, n(-\hat{x})\right]}_{\substack{\text{expected value of noise} \\ \text{variance: } \delta^2}}\, d\hat{x}\, d\tilde{x}}$$

$$= \sqrt{\delta^2 \int_{-\infty}^{\infty} h^2(\hat{x})\, d\hat{x}} = \delta \cdot \sqrt{\int_{-\infty}^{\infty} h^2(\hat{x})\, d\hat{x}}$$

$$\Rightarrow \underline{\underline{SNR}} = \frac{\mathbb{H}[m](0)}{\sqrt{E[\quad]^2}} = \frac{\text{edge response}}{\text{rms}_{\text{noise}}} = \underline{\underline{\frac{\int_{-\omega}^{\omega} h(-\hat{x})\, m(\hat{x})\, d\hat{x}}{\delta\sqrt{\int_{-\omega}^{\omega} h^2(\hat{x})\, d\hat{x}}}}}$$

b) Localization





1. Metric: $\dfrac{1}{\text{rms distance}} = \dfrac{1}{\text{rms}[x_0 - 0]}$

$$= \frac{1}{\sqrt{E[x_0^2]}}$$

2. Mark edges as local maxima of response of
$$h(x) \otimes m(x)$$
$$\Rightarrow \quad [h(x) \otimes m(x)]' = 0$$

Noisy edge: $e(x) = m(x) + n(x)$

$$\mathbb{H}[e(x)] = \mathbb{H}[m(x) + n(x)]$$
$$= \mathbb{H}[m(x)] + \mathbb{H}[n(x)] = H_m(x) + H_n(x)$$

Local maxima of $h(x) \otimes m(x)$ at point x_0: $\mathbb{H}'[m+n] = \mathbb{H}'[m] + \mathbb{H}'[n]$

$$\mathbb{H}'[m+n](x_0) = \mathbb{H}'[m](x_0) + \mathbb{H}'[n](x_0) = 0$$

$$H'_m(x_0) + H'_n(x_0) = 0$$

Taylor series expansion:

$$H'_m(x_0) = H'_m(0) + H''(0) \cdot x_0 + \mathcal{O}(x_0^2)$$

$$\mathbb{H}'[m](x_0) = \underbrace{\mathbb{H}'[m](0)}_{=0 \text{ (true edge position)}} + \mathbb{H}''[m](0) \cdot x_0 + \underbrace{\mathcal{O}(x_0^2)}_{\text{neglect}}$$

$$\Rightarrow \mathbb{H}'[m](x_0) \approx \mathbb{H}''[m](0) \cdot x_0$$

Combine: $\mathbb{H}'[m+n](x_0) = \mathbb{H}'[m](x_0) + h'[n](x_0) = 0$

$$\Rightarrow h''[m](0) \cdot x_0 + h'[n](x_0) = 0$$

$$\Rightarrow x_0 \approx -\frac{\mathbb{H}'[n](x_0)}{\mathbb{H}''[m](0)} = -\frac{\text{(response to noise at } x_0)'}{\text{(response to edge at } 0)''}$$

<u>localization criteria:</u>

$$\sqrt{E[x_0^2]}^{\,-1} = \sqrt{E\left[\frac{\mathbb{H}'[n](x_0)^2}{\mathbb{H}''[m](0)^2}\right]}^{\,-1}$$

$$= \left(\frac{\sqrt{E[\mathbb{H}'[n](x_0)]^2}}{\mathbb{H}''[m](0)}\right)^{-1}$$

- $\mathbb{H}[n](x) = \int_{-\omega}^{\omega} h(x-\hat{x}) \cdot n(\hat{x})\, d\hat{x}$

$$\mathbb{H}'[n](x) = \int_{-\omega}^{\omega} h'(x-\hat{x})\, n(\hat{x})\, d\hat{x} = \int_{-\omega}^{\omega} h'(\hat{x}) \cdot n(x-\hat{x})\, d\hat{x}$$

$$E[\mathbb{H}'[n](x)]^2 = \ldots\ldots\ldots$$

$$\ldots\ldots = \underline{\delta^2 \int_{-\omega}^{\omega} h'(\hat{x})^2\, d\hat{x}}$$

- $h''[m](x_0) = \frac{d}{dx}\left(\frac{d}{dx} \int_{-\omega}^{\omega} h(x-\hat{x}) \cdot m(\hat{x})\, d\hat{x}\right)$

$$= \ldots = \int_{-\omega}^{\omega} h'(\hat{x})\, m'(x-\hat{x})\, d\hat{x}$$

$$\mathbb{h}''[m](0) = \underline{\int_{-\omega}^{\omega} h'(\hat{x}) \cdot m'(-\hat{x})\, d\hat{x}}$$

Localization: $\underline{\underline{LOC}} = \frac{1}{\sqrt{E[x_0^2]}} = \frac{\mathbb{h}''[m](0)}{\sqrt{E[\mathbb{H}'[n](x_0)]^2}}$

$$= \underline{\underline{\frac{\left|\int_{-\omega}^{\omega} h'(\hat{x}) \cdot m'(-\hat{x})\, d\hat{x}\right|}{\delta \sqrt{\int_{-\omega}^{\omega} [h'(\hat{x})]^2\, d\hat{x}}}}}$$

Both criteria simultaneously: Maximize Product:

$$SNR \cdot LOC = MAX$$

$$SNR \cdot LOC = \frac{\left| \int h(\hat{x})\, m(-\hat{x})\, d\hat{x} \right| \left| \int h'(\hat{x})\, m'(-\hat{x})\, d\hat{x} \right|}{\sigma^2 \sqrt{\int_{-\omega}^{\omega} h^2(\hat{x})\, d\hat{x}} \cdot \sqrt{\int_{-\omega}^{\omega} [h'(\hat{x})]^2\, d\hat{x}^2}} = MAX$$

– denominator: constant scaling (only integration over kernel h[ ])

→ maximize numerator

Schwarz inequality:

$$\left| \int f(x) \cdot g(x)\, dx \right|^2 \leq \int f(x)^2 dx \cdot \int g(x)^2 dx$$

(only equal if $f(x) = \lambda \cdot g(x)$ or $f(x) \parallel g(x)$)

$$\Rightarrow \left| \int h(\hat{x})\, m(-\hat{x})\, d\hat{x} \right|^2 \leq \int h^2(\hat{x})\, d\hat{x} \int m^2(\hat{x})\, d\hat{x}$$

$$\left| \int h'(\hat{x})\, m'(-\hat{x})\, d\hat{x} \right|^2 \leq \int h'^2(\hat{x})\, d\hat{x} \int m'^2(\hat{x})\, d\hat{x}$$

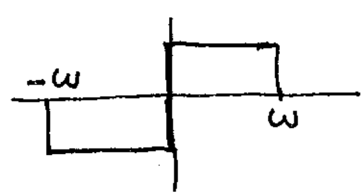
when is left side maximal?

equal only if: $\underset{\text{filter}}{h(x)} = \lambda \cdot \underset{\text{signal}}{m(x)} \quad (x \in [-\omega, \omega])$

First and second criteria:

SNR·LOC: Optimal matched filter: Edge Model itself

DOB filter
(difference of box)

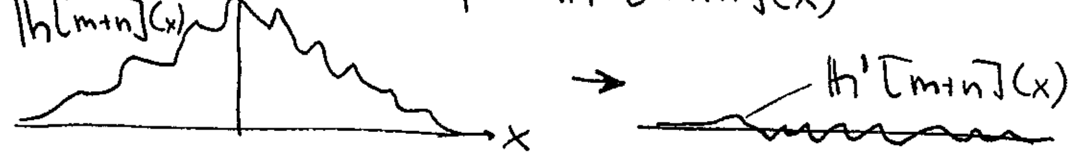


but: see Fig. 1c Canny paper: Many maxima in filter response ⇒ erroneous responses ⇒ apply 3rd criteria

c) Third criterion: Elimination of multiple responses

Requirement: unique, single response to edge

Mathematical formulation: Small number of zero-crossings for $h'[m+n](x)$



Rice-Theorem: Mean distance between zero-crossings (assuming Gaussian noise)

$$x_{ave} = \pi \left(\frac{-R(0)}{R''(0)} \right)^{1/2}$$

$R(0)$: autocorrelation function of g at 0

$$\Rightarrow R(0) = \int_{-\infty}^{\infty} g^2(x)\,dx \qquad R''(0) = \int_{-\infty}^{\infty} g'(x)\,dx$$

substitute: $g \rightarrow h'$

$$\text{MULT}: \quad x_{ave}(h) = \pi \left(\frac{\int h'^2(x)\,dx}{\int h''^2(x)\,dx} \right)^{1/2}$$

Criteria (a), (b) & (c): Maximize SNR·LOC subject to MULT

closed form solution: impossible

$\Rightarrow$ constrained numerical optimization

$\Rightarrow$ calculus of variation

maximize SNR·LOC

auxiliary constraints: MULT

zero DC component (zero output to const. input)

Results: Optimal Operators

p. 188 ridge profiles:

p. 193 edge profiles:

(Canny Fig. 4 p. 192: $\underset{SNR}{\Sigma} \cdot \underset{LOC}{\Lambda}$ max
r MULT close to 1)

Discussion: Detection (SNR) vs. Localization

- detection: broad filter has better SNR
- localization: narrow filter has better localization

⇒ uncertainty principle relating to detection and localization performance

⟹ ⟹ Optimal edge filter: Close to first derivative of Gaussian

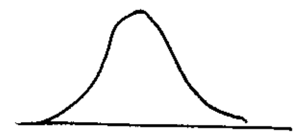
$$\frac{d}{dx} G(x; \mu, \delta)$$

$\Sigma \cdot \Lambda = 0{,}92$

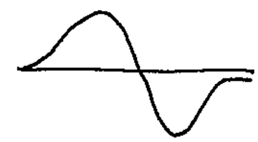
$r = 0{,}51$

Extension to 2-D

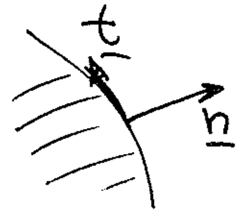
1-D: $G(\mu, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right)$



$\frac{d}{dx} G(\mu, \sigma) = -\frac{x}{\sqrt{2\pi}\,\sigma^3} \exp\left(-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right)$



2-D:



edge has orientation,
edge direction is tangent to contour

idea:

- edge detection <u>across</u> edge : $\vec{n}$
- projection function <u>along</u> edge : $\vec{t}$

<u>ideally</u>: projection function is Gaussian,
detection function is derivative of Gaussian (result Canny)

2-D Gaussian: $G(x, y; \bar{\mu}, \sigma) = c \cdot \exp\left(-\left(\frac{x^2+y^2}{2\sigma^2}\right)\right)$

Derivative in direction α; given by normal $\underline{n}$: $G_n = \frac{\partial G}{\partial n} = \underline{n} \cdot \nabla G$

$$= \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \begin{pmatrix} G_x \\ G_y \end{pmatrix} = \cos\alpha\, G_x + \sin\alpha\, G_y$$

Direction α: $\tan\alpha = \frac{G_x}{G_y}$

Estimation of $\underline{n}$: $\frac{\nabla(G \otimes I)}{|\nabla(G \otimes I)|}$ smoothed gradient direction

<u>Implementation</u>:
1. smoothing $G \otimes I$ (choose σ)
2. derivatives $\left(\frac{\partial}{\partial x}(G \otimes I), \frac{\partial}{\partial y}(G \otimes I)\right) = (G_x, G_y)$
3. $\underline{n} = \frac{\nabla(G \otimes I)}{|\nabla(G \otimes I)|}$
4. $\underbrace{G_n}_{\text{directional filter}} \otimes I = (\underline{n} \cdot \nabla G) \otimes I = \underline{n} \cdot \nabla(G \otimes I)$