Filtering Images in the Spatial Domain
Chapter 3b G&W

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Overview

- Correlation and convolution
- Linear filtering
  - Smoothing, kernels, models
  - Detection
  - Derivatives
- Nonlinear filtering
  - Median filtering
  - Bilateral filtering
  - Neighborhood statistics and nonlocal filtering

Cross Correlation

- Operation on image neighborhood and small …
  - “mask”, “filter”, “stencil”, “kernel”
- Linear operations within a moving window

Input image

100	130	104	99	…
87	95	103	150	…
50	36	150	104	…
20	47	205	77	…
.	.	.	.	
.	.	.	.	

Filter

0.0	0.1	0.0
0.1	0.6	0.1
0.0	0.1	0.0

Output image

.	.	.	.
.	.	.	.
.	55.8	134.8	.
.	.	.	.

```
  0.0*87 + 0.1*95 + 0.0*103
+ 0.1*50 + 0.6*36 + 0.1*150
+ 0.0*20 + 0.1*47 + 0.0*205 = 55.8 134.8
```

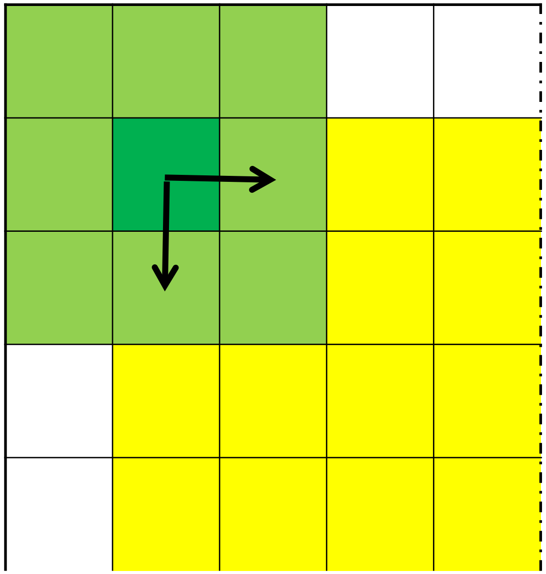
Cross Correlation

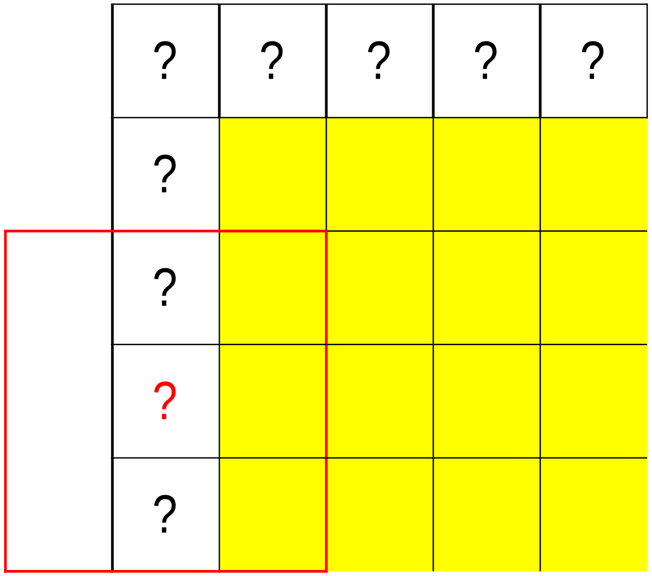
- 1D $$g(x) = \sum_{s=-a}^{a} w(s) f(x+s)$$
- 2D $$g(x,y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t) f(x+s, y+t)$$

$$w(s,t) = \boxed{\begin{matrix} w(-a,-b) & \cdots & & \cdots & w(a,-b) \\ \vdots & & & & \vdots \\ & \cdots & w(0,0) & \cdots & \\ \vdots & & & & \vdots \\ w(-a,b) & \cdots & & \cdots & w(a,b) \end{matrix}}$$

Correlation: Technical Details

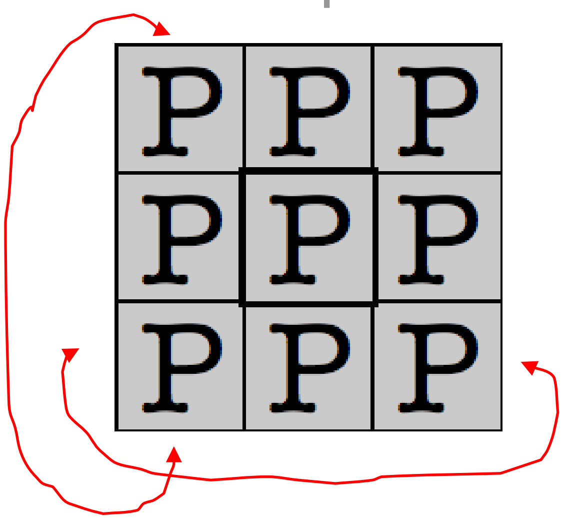
- How to filter boundary?

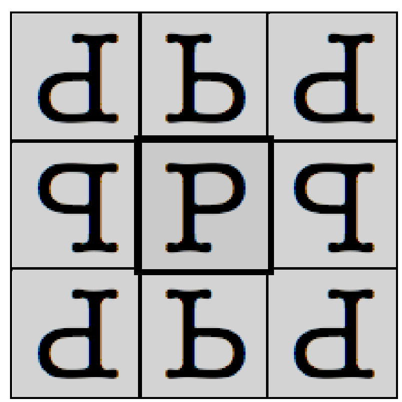




Correlation: Technical Details

- Boundary conditions
  - Boundary not filtered (keep it 0)
  - Pad image with amount (a,b)
    - Constant value or repeat edge values
  - Cyclical boundary conditions
    - Wrap or mirroring





Correlation: Technical Details

- Boundaries
  – Can also modify kernel – no longer correlation
- For analysis
  – Image domains infinite
  – Data compact (goes to zero far away from origin)

$$g(x,y) = \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} w(s,t) f(x+s, y+t)$$

Correlation: Properties

- Shift invariant

$$g = w \circ f \qquad\qquad g(x,y) = w(x,y) \circ f(x,y)$$

$$w(x,y) \circ f(x-x_0, y-y_0) = \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} w(s,t) f(x-x_0+s, y-y_0+t) = g(x-x_0, y-y_0)$$

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- Linear $w \circ (\alpha e + \beta f) = \alpha w \circ e + \beta w \circ f$

Compact notation

$$C_{wf} = w \circ f$$

Filters: Considerations

- Normalize
  - Sums to one
  - Sums to zero (some cases, see later)
- Symmetry
  - Left, right, up, down
  - Rotational
- Special case: auto correlation

$$C_{ff} = f \circ f$$

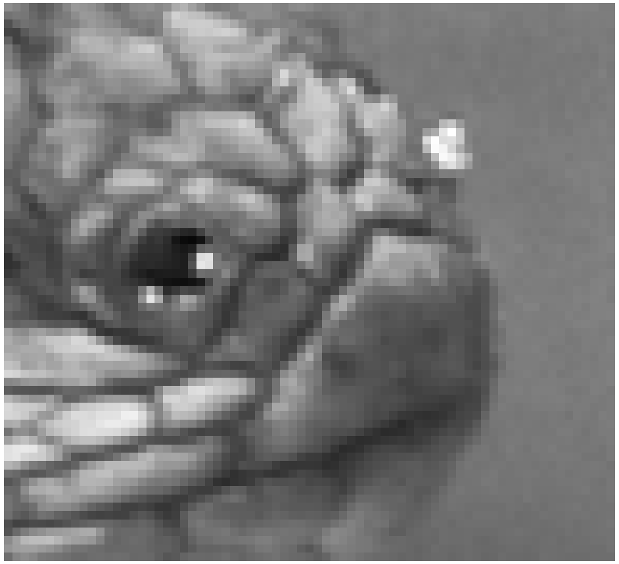
Examples 1

$$\begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}$$

$$1/9 * \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

Examples 1

$$1/9 * \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$



Examples 2

$$1/9 * \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

$$1/25 * \begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{matrix}$$

Smoothing and Noise

Noisy image

5x5 box filter

Other Filters

- Disk
  - Circularly symmetric, jagged in discrete case
- Gaussians
  - Circularly symmetric, smooth for large enough stdev
  - Must normalize in order to sum to one
- Derivatives – discrete/finite differences
  - Operators

Gaussian Kernel

$$\sigma = 1;\ \mathrm{Plot}\left[\frac{1}{\sqrt{2\pi\sigma^2}} E^{-\frac{x^2}{2\sigma^2}}, \{x, -4, 4\}, \mathrm{ImageSize} \to$$

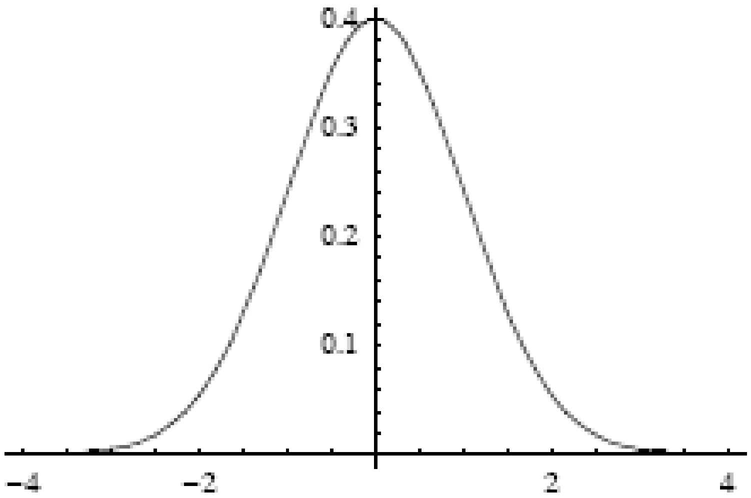


Figure 2.1 The Gaussian kernel with unit standard deviation in 1D.

Gaussian Kernel

$$G_{1D}(x;\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}},\ G_{2D}(x,\ y;\sigma) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}},\ G_{ND}(\vec{x};\sigma) = \frac{1}{\left(\sqrt{2\pi}\,\sigma\right)^N} e^{-\frac{|\vec{x}|^2}{2\sigma^2}}$$

Normalization to 1.0

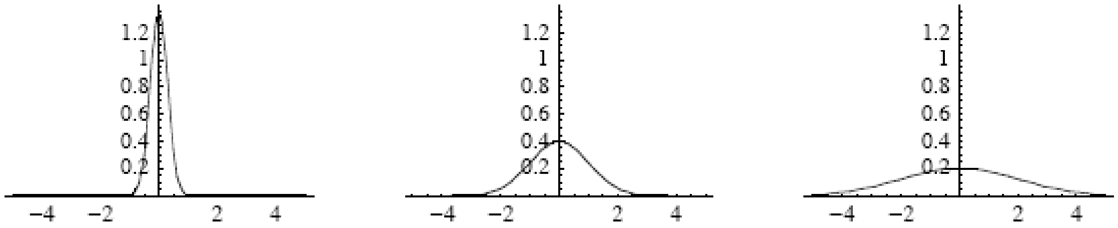
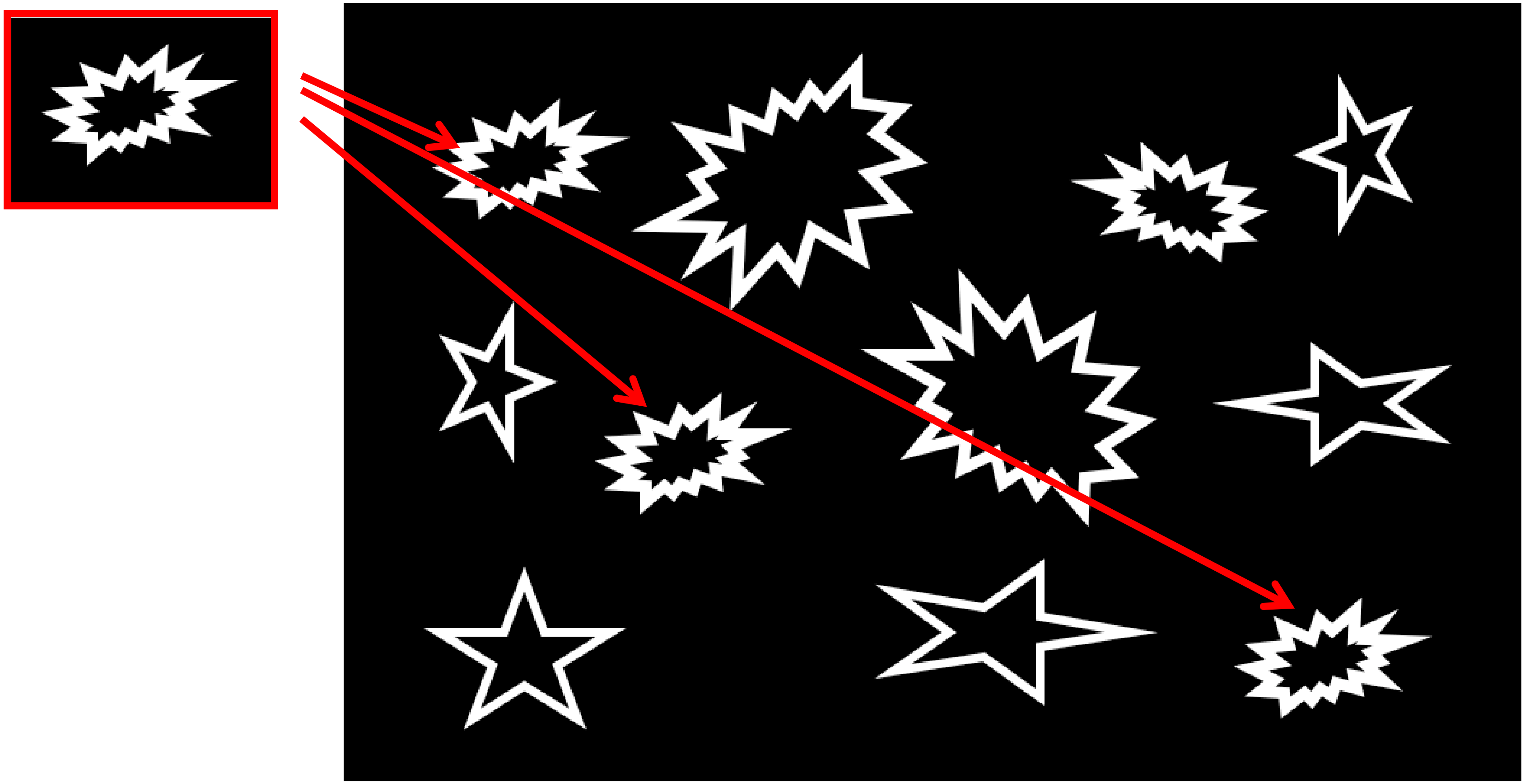


Figure 3.2 The Gaussian function at scales $\sigma = .3$, $\sigma = 1$ and $\sigma = 2$. The kernel is normalized, so the total area under the curve is always unity.

Pattern Matching



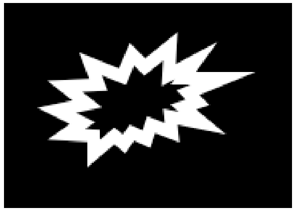
Pattern Matching/Detection

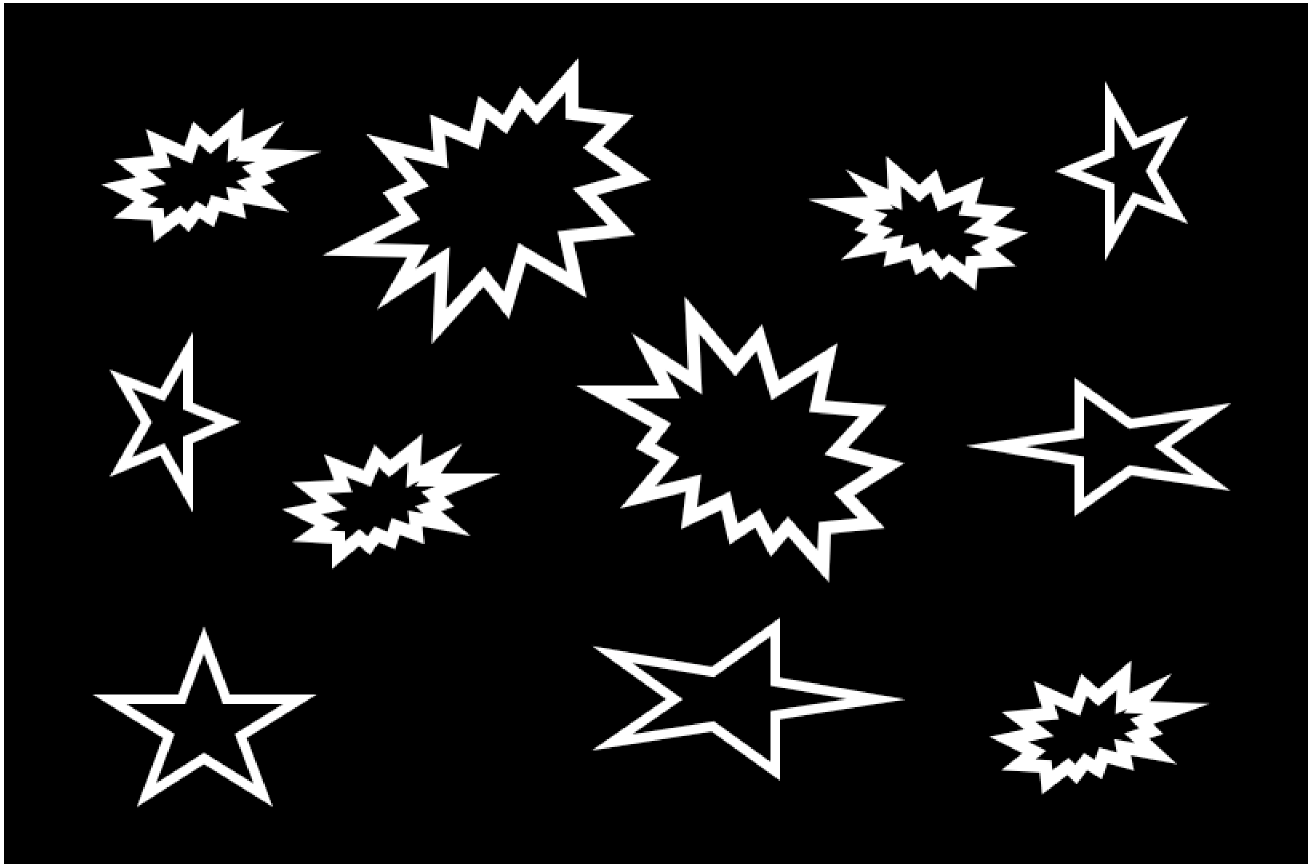
- The optimal (highest) response from a filter is the autocorrelation evaluated at position zero

$$\max_{\bar{x}} C_{ff}(\bar{x}) = C_{ff}(0) = \int f(\bar{s}) f(\bar{s}) d\bar{s}$$

- A filter responds best when it matches a pattern that looks itself
- Strategy
  – Detect objects in images by correlation with "matched" filter

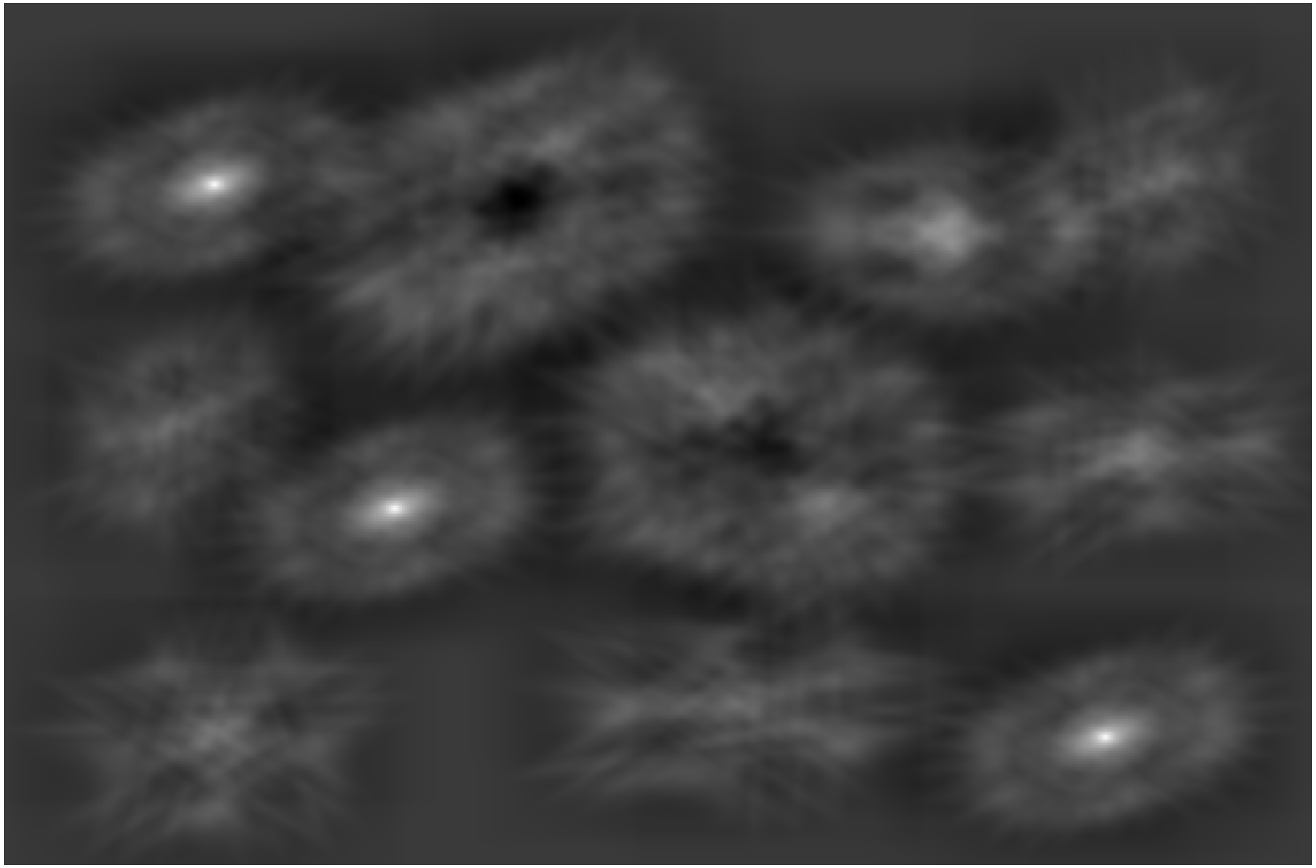
Matched Filter Example



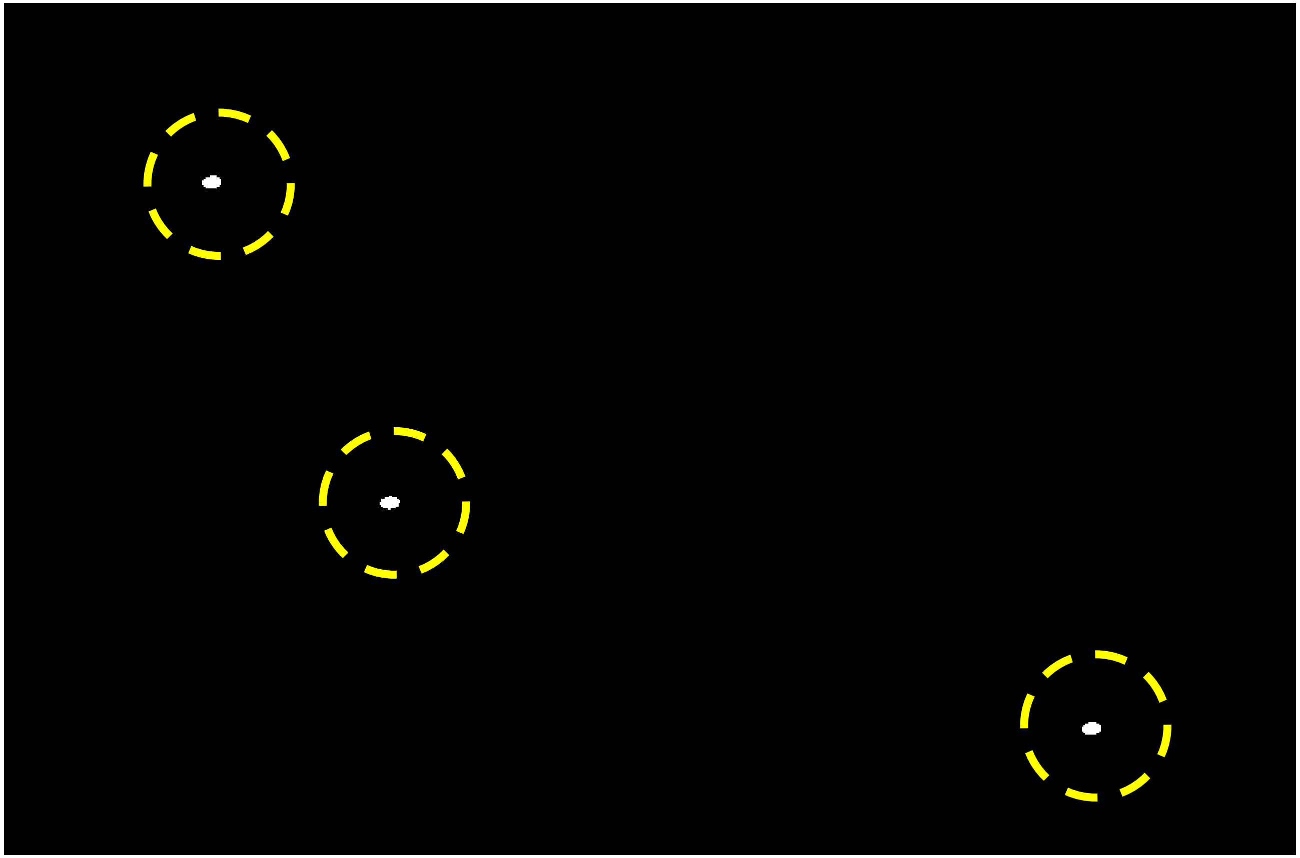


Trick: make sure kernel sums to zero

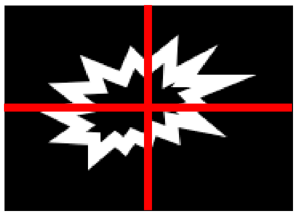
Matched Filter Example: Correlation of template with image

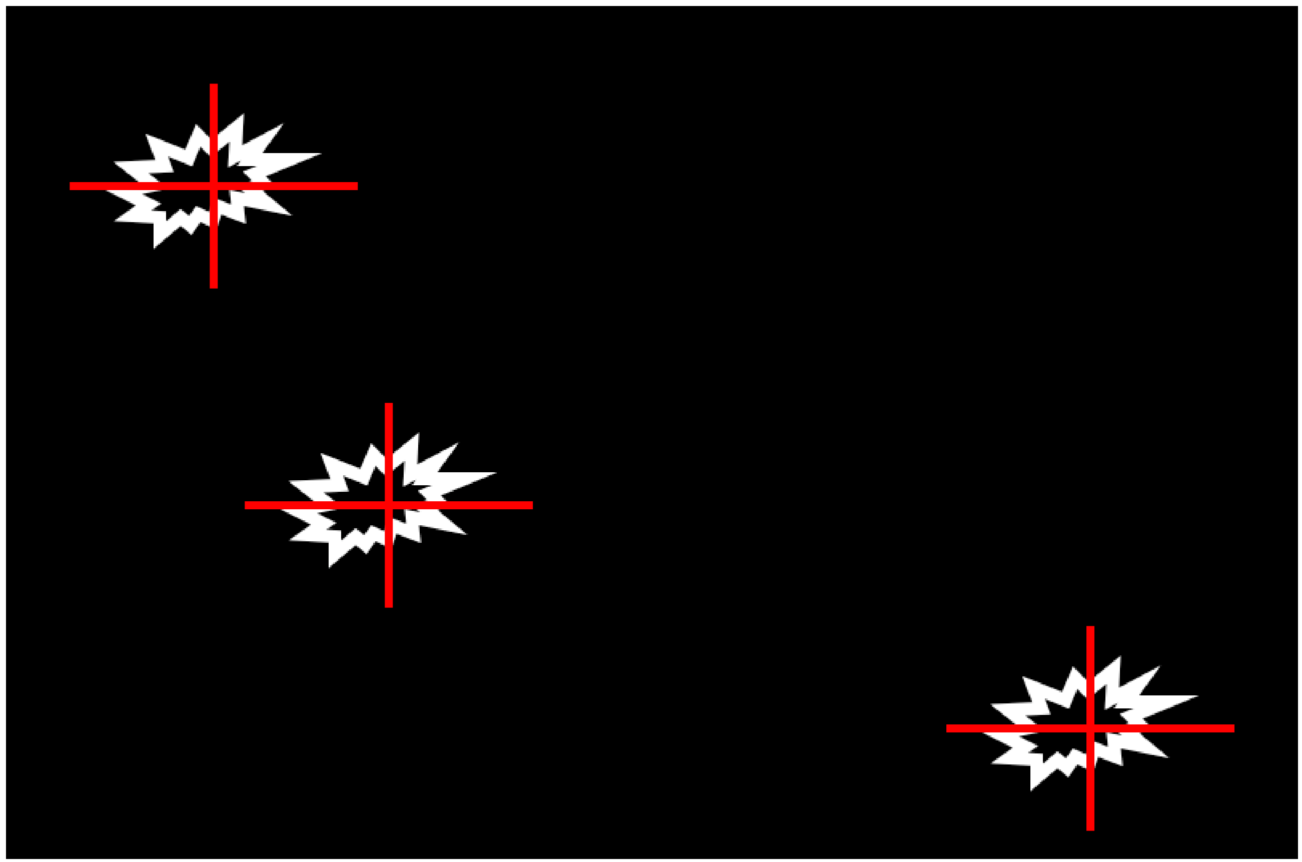


Matched Filter Example: Thresholding of correlation results

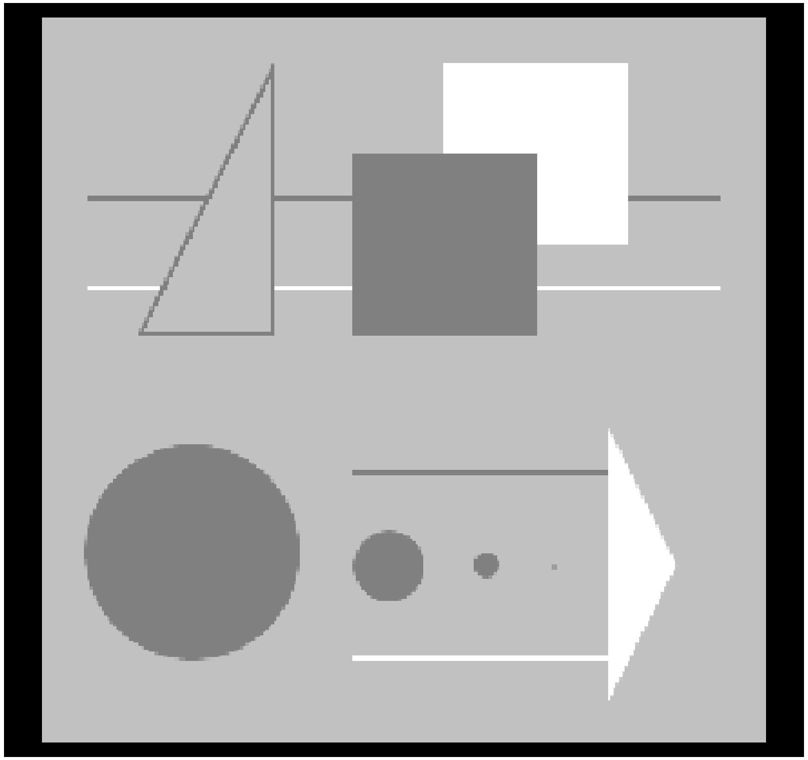


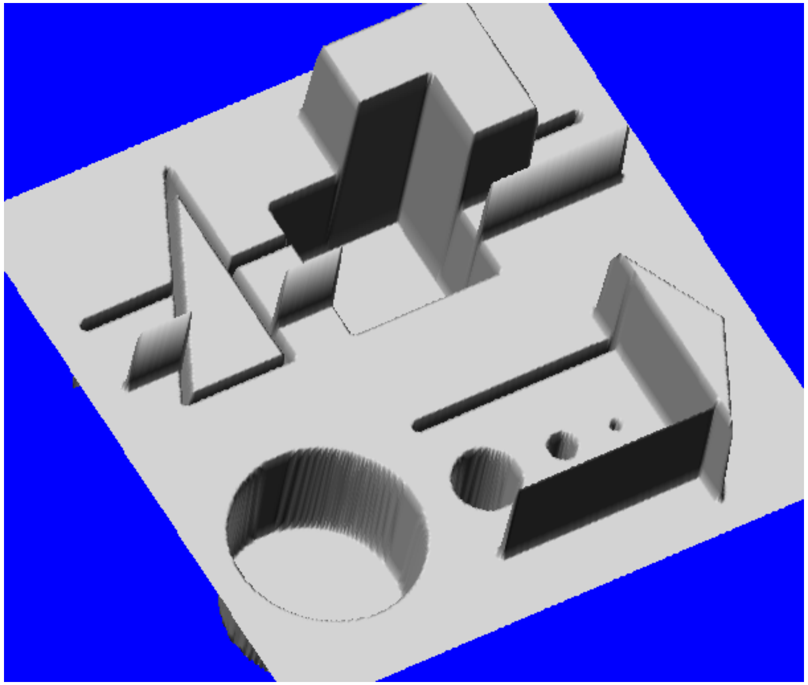
Matched Filter Example: High correlation → template found





Digital Images: Boundaries are “Lines” or “Discontinuities”





Example: Characterization of discontinuities?

Derivatives: Finite Differences

$$\frac{\partial f}{\partial x} \approx \frac{1}{2h}\left(f(x+1,y) - f(x-1,y)\right)$$

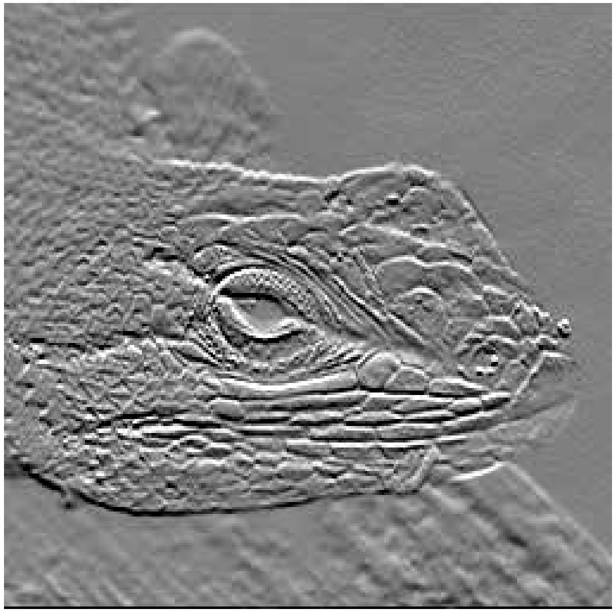
$$\frac{\partial f}{\partial x} \approx w_{dx} \circ f \qquad w_{dx} = \begin{array}{|c|c|c|}\hline -\frac{1}{2} & 0 & \frac{1}{2} \\ \hline \end{array}$$

$$\frac{\partial f}{\partial y} \approx w_{dy} \circ f \qquad w_{dy} = \begin{array}{|c|}\hline -\frac{1}{2} \\ \hline 0 \\ \hline \frac{1}{2} \\ \hline \end{array}$$

Derivative Example

$$\begin{matrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}$$



I GOT UP TO HERE ON 9/15/2010 (GUIDO)

Convolution

Java demo: http://www.jhu.edu/signals/convolve/

- Discrete

$$g(x,y) = w(x,y) * f(x,y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t) f(x-s, y-t)$$

- Continuous

$$g(x,y) = w(x,y) * f(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(s,t) f(x-s, y-t) ds dt$$

- Same as cross correlation <u>with kernel transposed</u> around each axis
- The two operations (correlation and convolution) are the same if the kernel is symmetric about axes

$$g = w \circ f = w^* * f$$

w^* reflection of w

Convolution: Properties

- Shift invariant, linear
- Commutative

$$f * g = g * f$$

- Associative

$$f * (g * h) = (f * g) * h$$

- Others (discussed later):
  - Derivatives, convolution theorem, spectrum…

Computing Convolution

- Compute time
  - MxM mask
  - NxN image

$O(M^2N^2)$

"for" loops are nested 4 deep

Computing Convolution

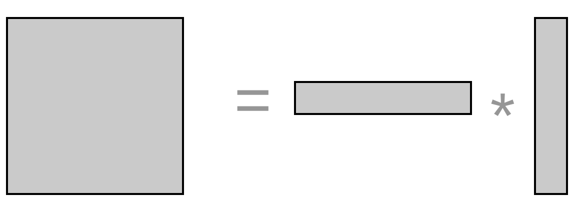
- Compute time
  - MxM mask
  - NxN image

$O(M^2N^2)$ "for" loops are nested 4 deep

- Special case: *separable*

$$w = \overbrace{w_x * w_y}^{\text{Two 1D kernels}}$$

$$w * f = \underbrace{(w_x * w_y) * f}_{O(M^2N^2)} = \underbrace{w_x * (w_y * f)}_{O(MN^2)}$$



Separable Kernels

- Examples
  - Box/rectangle
  - Bilinear interpolation
  - Combinations of partial derivatives
    - $d^2f/dxdy$
  - Gaussian
    - Only filter that is both circularly symmetric and separable
- Counter examples
  - Disk
  - Cone
  - Pyramid

Separability

$$g_{2D}(x, y; \sigma_1^2 + \sigma_2^2) \;=\; g_{1D}(x; \sigma_1^2) \otimes g_{1D}(y; \sigma_2^2)$$

```
DisplayTogetherArray[{Plot[gauss[x, σ = 1], {x, -3, 3}],
   Plot3D[gauss[x, σ = 1] gauss[y, σ = 1], {x, -3, 3}, {y, -3, 3}]},
  ImageSize -> 440];
```

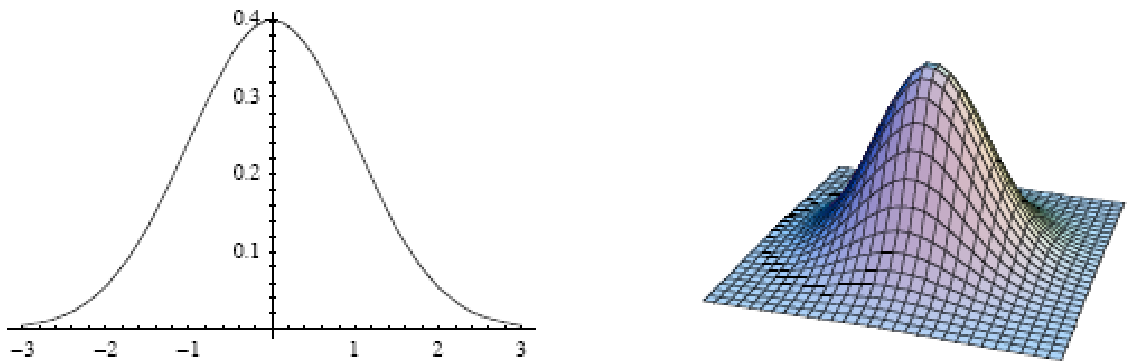


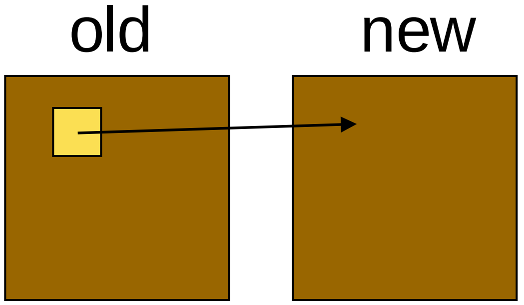
Figure 3.7 A product of Gaussian functions gives a higher dimensional Gaussian function. This is a consequence of the separability.

Nonlinear Methods For Filtering

- Median filtering
- Bilateral filtering
- Neighborhood statistics and nonlocal filtering

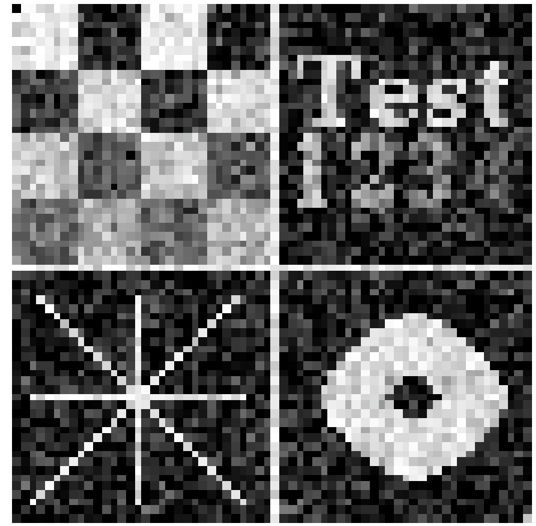
Median Filtering

- For each neighborhood in image
  - Sliding window
  - Usually odd size (symmetric) 5x5, 7x7,…
- Sort the greyscale values
- Set the center pixel to the median
- Important: use “Jacobi” updates
  - Separate input and output buffers
  - All statistics on the original image



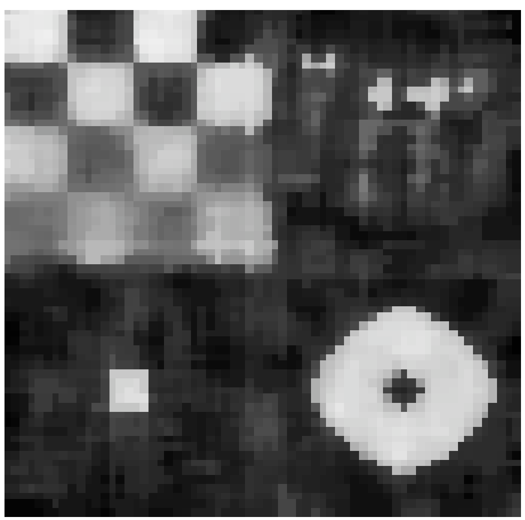
Median vs Gaussian

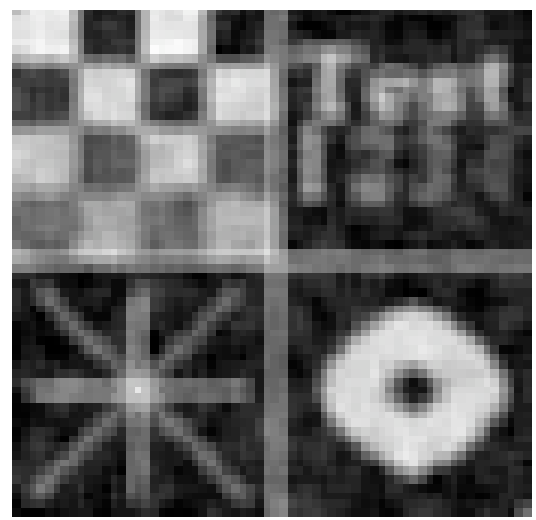
Original



+ Gaussian Noise

3x3 Median





3x3 Box

Median Filter

- Issues
  - Boundaries
    - Compute on pixels that fall within window
  - Computational efficiency
    - What is the best algorithm?
- Properties
  - Removes outliers (replacement noise – salt and pepper)
  - Window size controls size of structures
  - Preserves straight edges, but rounds corners and features

Replacement Noise

- Also: “shot noise”, “salt&pepper”
- Replace certain % of pixels with samples from pdf
- Best filtering strategy: filter to avoid outliers

Smoothing of S&P Noise

- It's not zero mean (locally)
- Averaging produces local biases

Smoothing of S&P Noise

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Median Filtering

Median 3x3

Median 5x5

Median Filtering

Median 3x3

Median 5x5

Median Filtering

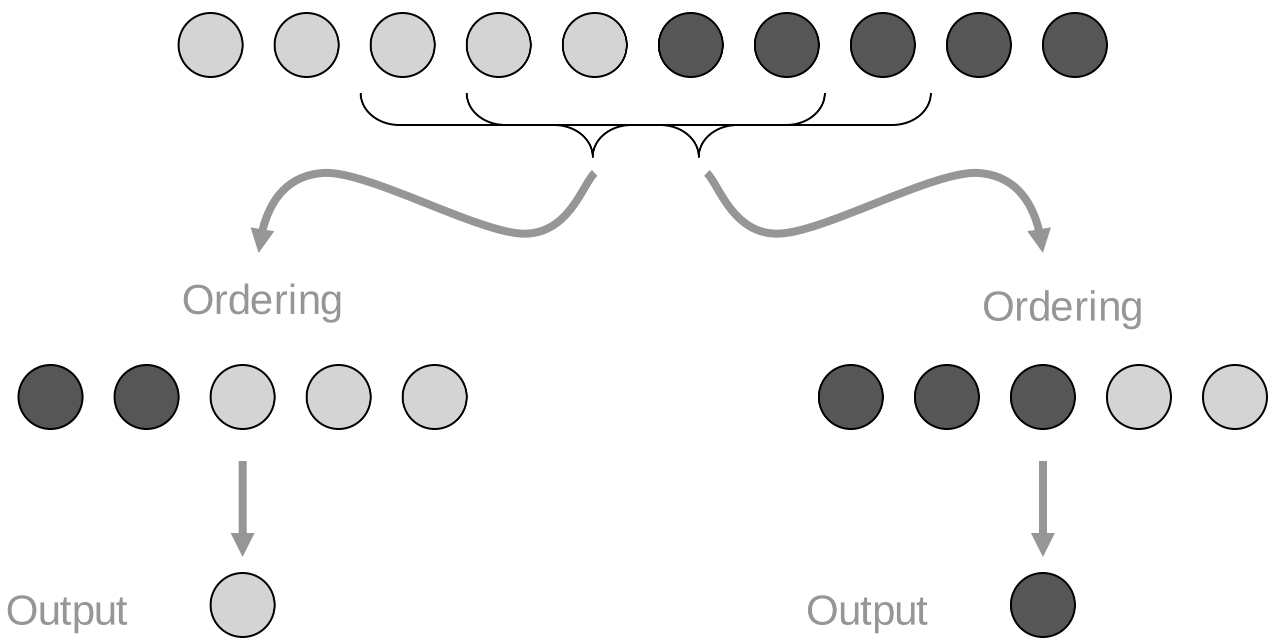
- Iterate

Median 3x3

2x Median 3x3

Median Filtering

- Image model: piecewise constant (flat)



Order Statistics

- Median is special case of order-statistics filters
- Instead of weights based on neighborhoods, weights are based on ordering of data

Neighborhood

$$X_1, X_2, \ldots, X_N$$

Ordering

$$X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(N)}$$

Filter

$$F(X_1, X_2, \ldots, X_N) = \alpha_1 X_{(1)} + \alpha_2 X_{(2)} + \ldots + \alpha_N X_{(N)}$$

Neighborhood average (box)

$$\alpha_i = 1/N$$

Median filter

$$\alpha_i = \begin{cases} 1 & i = (N+1)/2 \\ 0 & \text{otherwise} \end{cases}$$

Trimmed average (outlier removal)

$$\alpha_i = \begin{cases} 1/M & (N-M+1)/2 \leq i \leq (N+M+1)/2 \\ 0 & \text{otherwise} \end{cases}$$

Piecewise Flat Image Models

- Image piecewise flat -> average only within similar regions
- Problem: don’t know region boundaries

Piecewise-Flat Image Models

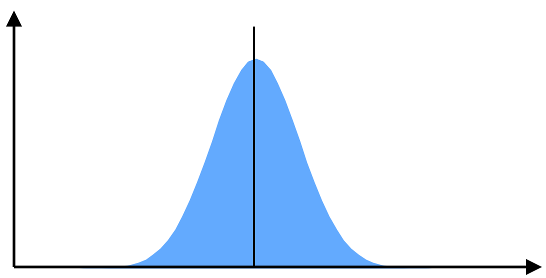
- Assign probabilities to other pixels in the image belonging to the same region
- Two considerations
  - Distance: far away pixels are less likely to be same region
  - Intensity: pixels with different intensities are less likely to be same region

Piecewise-Flat Images and Pixel Averaging

Distance (kernel/pdf)

$$G(\mathbf{x}_i - \mathbf{x}_j)$$

Prob pixel belongs to same region as i

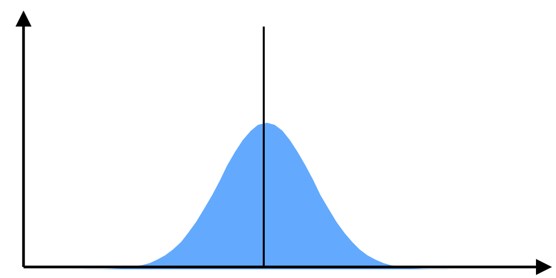


position

Distance (pdf)

$$H(f_i - f_j)$$

Prob pixel belongs to same region as i



intensity

Bilateral Filter

- Neighborhood – sliding window
- Weight contribution of neighbors according to:

$$f_i \leftarrow k_i^{-1} \sum_{j \in N} f_j G(\mathbf{x}_i - \mathbf{x}_j) H(f_i - f_j)$$

$$k_i = \sum_{j \in N} G(\mathbf{x}_i - \mathbf{x}_j) H(f_i - f_j)$$

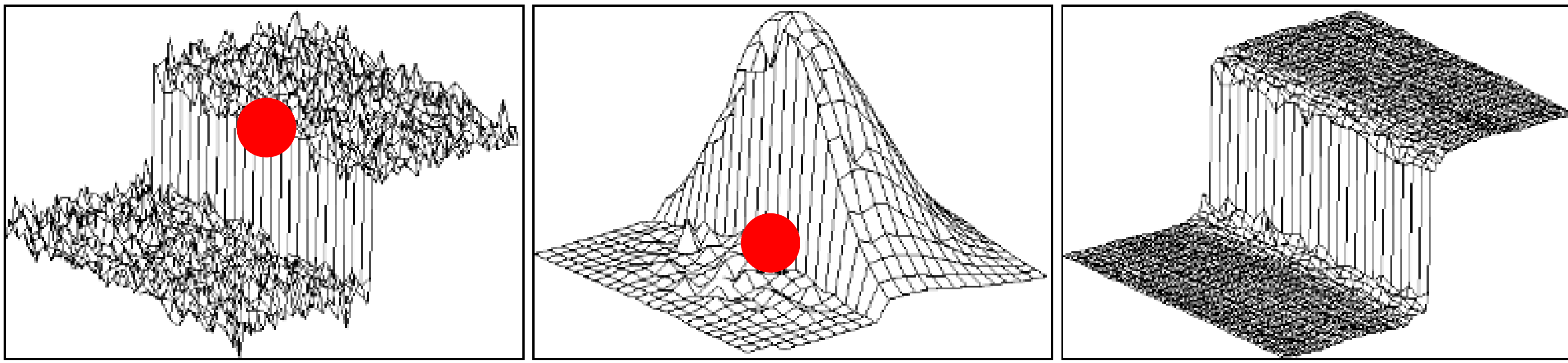
normalization: all weights add up to 1

- G is a Gaussian (or lowpass), as is H, N is neighborhood,
  - Often use $G(r_{ij})$ where r_{ij} is distance between pixels
  - Update must be normalized for the samples used in this (particular) summation
- Spatial Gaussian with extra weighting for intensity
  - Weighted average in neighborhood with downgrading of intensity outliers

Tomasi, Manduchi: http://en.wikipedia.org/wiki/Bilateral_filter
http://homepages.inf.ed.ac.uk/rbf/CVonline/LOCAL_COPIES/MANDUCHI1/Bilateral_Filtering.html

Bilateral Filter

Replaces the pixel value at **x** with an average of similar and nearby pixel values.



(a) (b) (c)

When the bilateral filter is centered, say, on a pixel on the bright side of the boundary, the similarity function *s* assumes values close to one for pixels on the same side, and values close to zero for pixels on the dark side. The similarity function is shown in figure 1(b) for a 23x23 filter support centered two pixels to the right of the step in figure 1(a).

Bilateral Filtering

Replaces the pixel value at **x** with an average of similar and nearby pixel values.

Gaussian Blurring

Bilateral

Bilateral Filtering

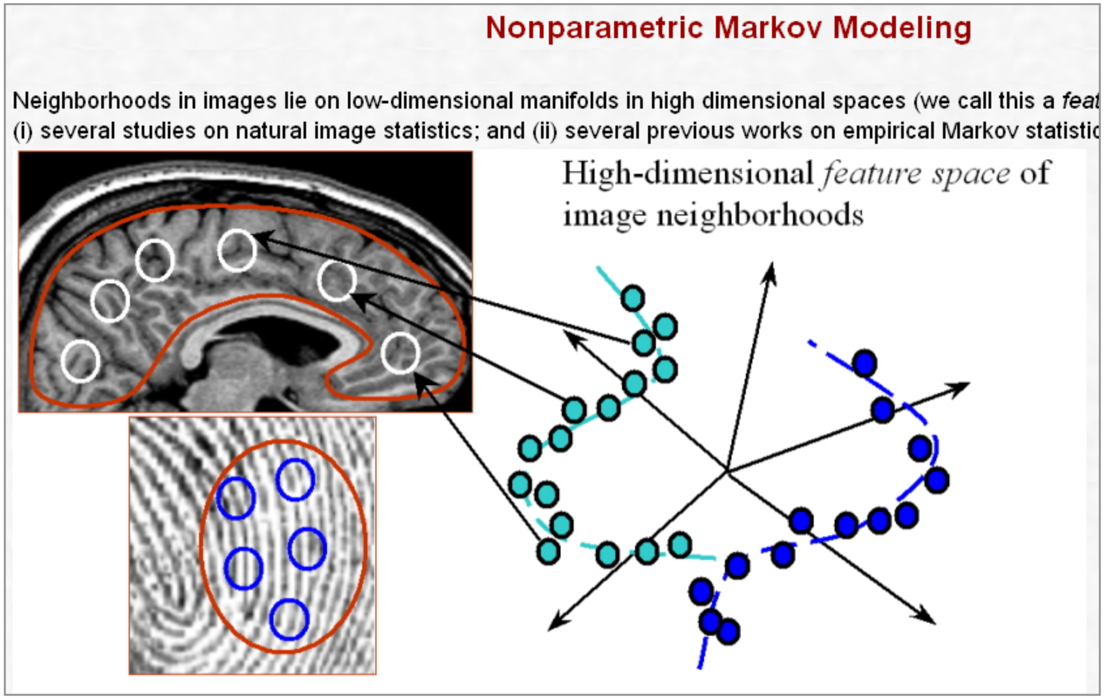
Gaussian Blurring

Bilateral

Nonlocal Averaging

- Recent algorithm
  - NL-means, Baudes et al., 2005
  - UINTA, Awate & Whitaker, 2005
- Different model
  - No need for piecewise-flat
  - Images consist of some set of pixels with similar neighborhoods → average several of those
    - Scattered around
      - General area of a pixel
      - All around
- Idea
  - Average sets of pixels with similar neighborhoods

UINTA: Unsupervised Information-Theoretic Adaptive Filtering : Excellent Introduction and Additional Readings (Suyash P. Awate)

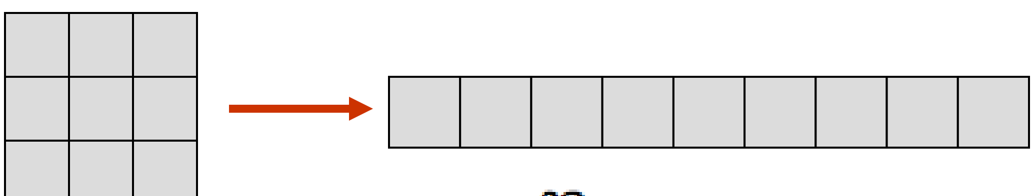


http://www.cs.utah.edu/~suyash/pubs/uinta/

Suyash P. Awate, Ross T. Whitaker
Unsupervised, Information-Theoretic, Adaptive Image Filtering with Applications to Image Restoration
IEEE Trans. Pattern Analysis & Machine Intelligence (TPAMI) 2006, Vol. 28, Num. 3, pp. 364-376

Nonlocal Averaging

- Strategy:
  - Average pixels to alleviate noise
  - Combine pixels with similar neighborhoods
- Formulation
  - $n_{i,j}$ – vector of pixels values, indexed by j, from neighborhood around pixel i



$= n_{i,j}$

n_i – vector

Nonlocal Averaging Formulation

- Distance between neighborhoods

$$d_{i,k} = d(n_i, n_k) = ||n_i - n_k|| = \left(\sum_{j=1}^{N} (n_{i,j} - n_{k,j})^2 \right)^{\frac{1}{2}}$$

- Kernel weights based on distances

$$w_{i,k} = K(d_{i,k}) = e^{-\frac{d_{i,k}^2}{2s^2}}$$

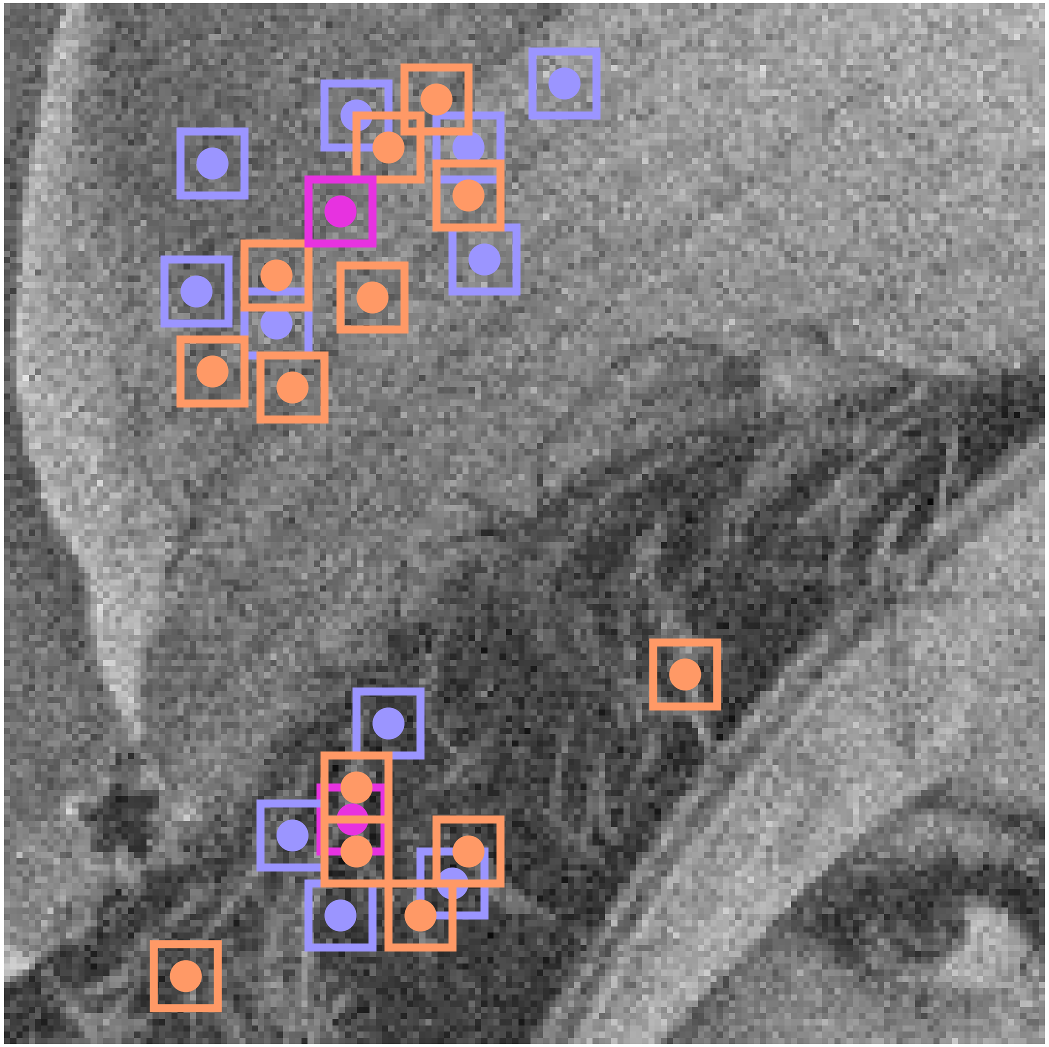
- Pixel values of k neighborhoods: f_k

Averaging Pixels Based on Weights

- For each pixel, i, choose a set of pixel locations k:
  - k = 1, …., M
  - Average them together based on neighborhood weights (prop. to intensity pattern difference)

$$g_i \leftarrow \frac{1}{\sum_{k=1}^{M} w_{i,k}} \sum_{k=1}^{M} w_{i,k} f_k$$

Nonlocal Averaging



Some Details

- Window sizes: good range is 5x5->11x11
- How to choose samples:
  - Random samples from around the image
    - UINTA, Awate&Whitaker
  - Block around pixel (bigger than window, e.g. 51x51)
    - NL-means
- Iterate
  - UNITA: smaller updates and iterate

NL-Means Algorithm

- For each pixel, p
  - Loop over set of pixels nearby
  - Compare the neighorhoods of those pixels to the neighborhood of p and construct a set of weights
  - Replace the value of p with a weighted combination of values of other pixels
- Repeat… but 1 iteration is pretty good

Results

Noisy image (range 0.0-1.0)

Bilateral filter (3.0, 0.1)

Results

Bilateral filter (3.0, 0.1)

NL means (7, 31, 1.0)

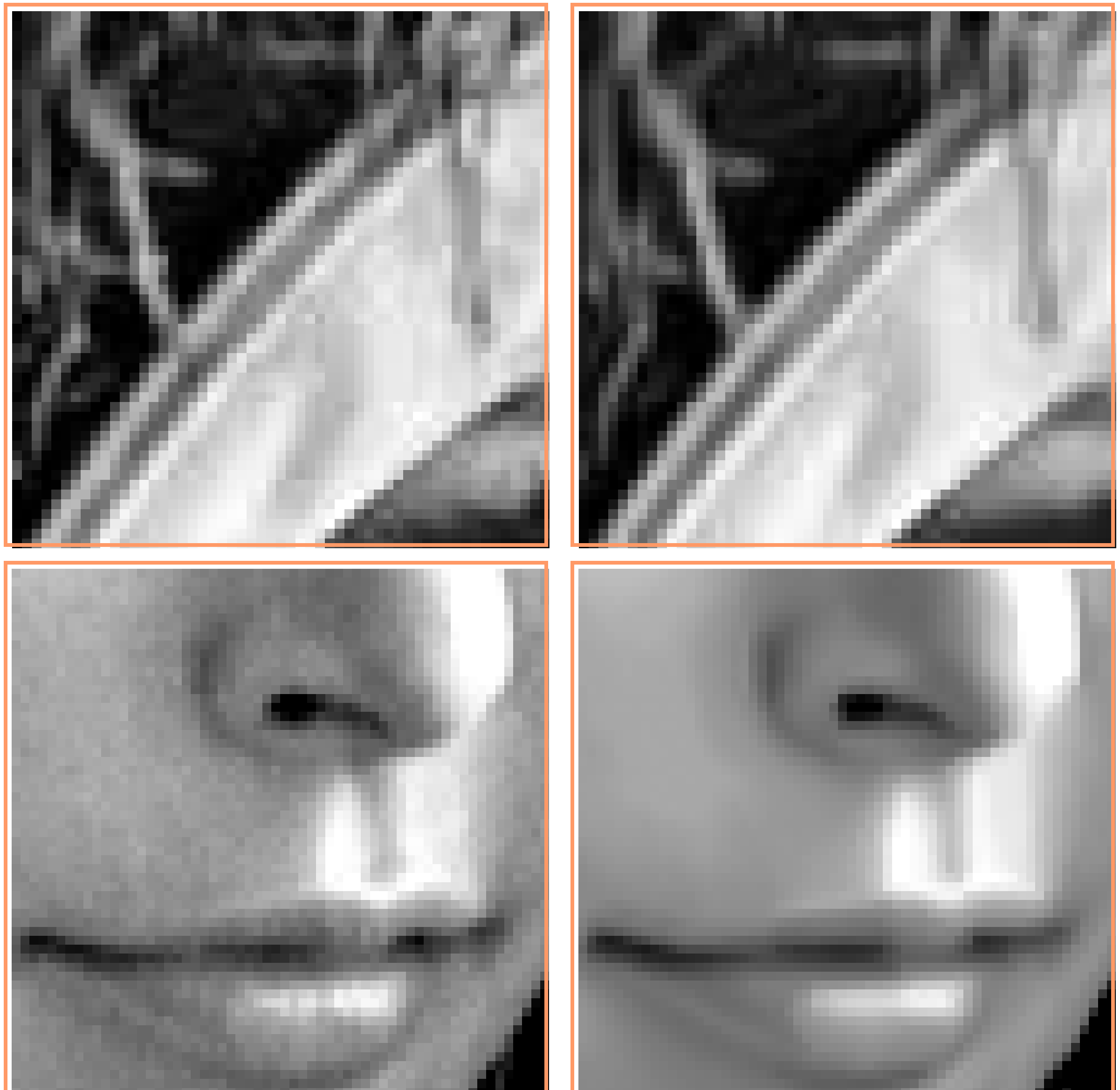
Results

Bilateral filter (3.0, 0.1)

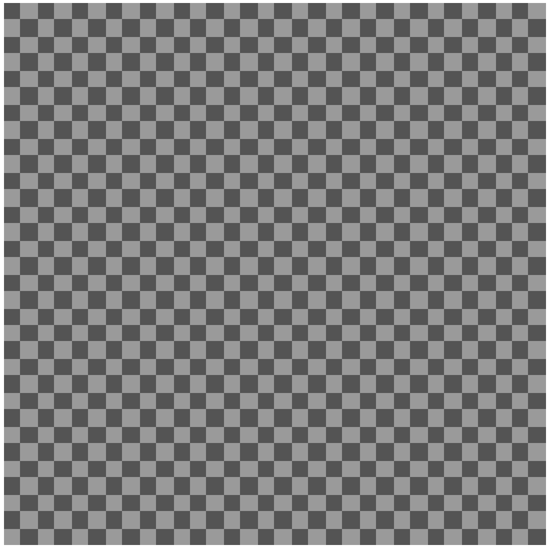
NL means (7, 31, 1.0)

Less Noisy Example

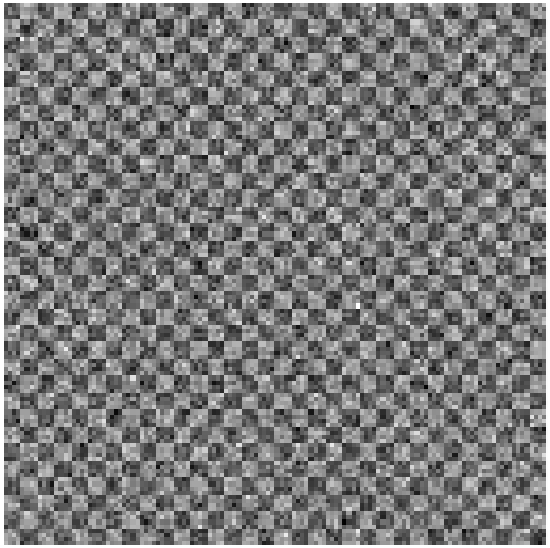
Less Noisy Example



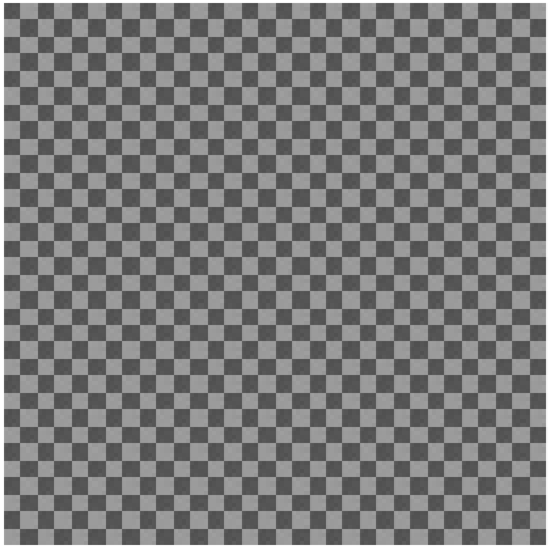
Results



Original

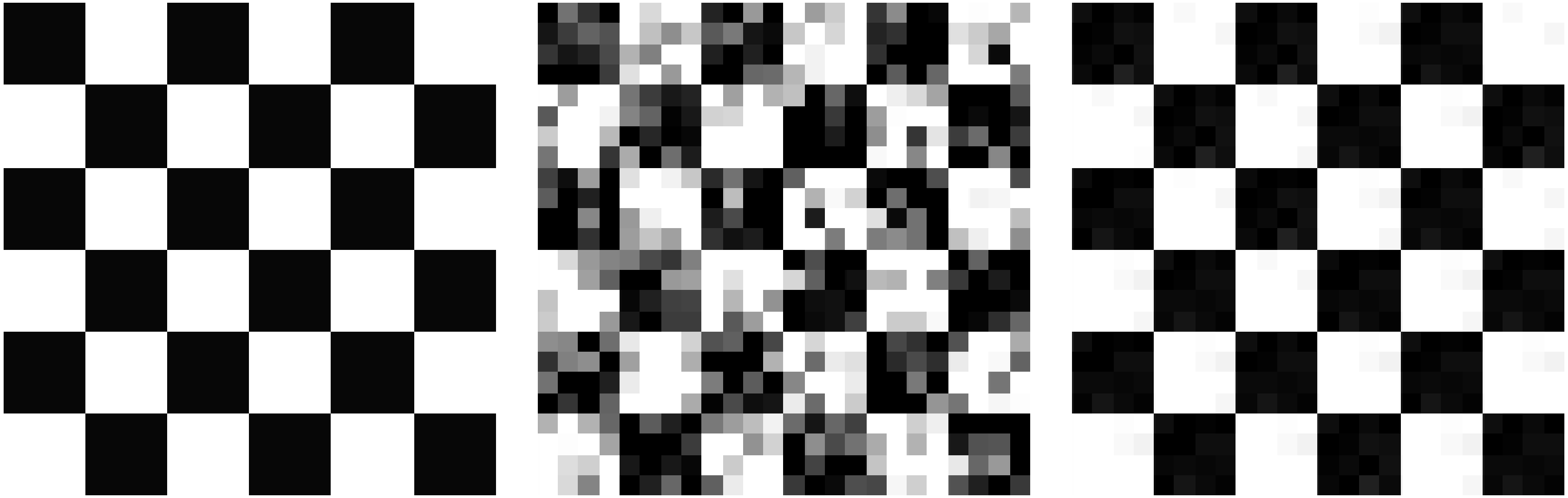


Noisy



Filtered

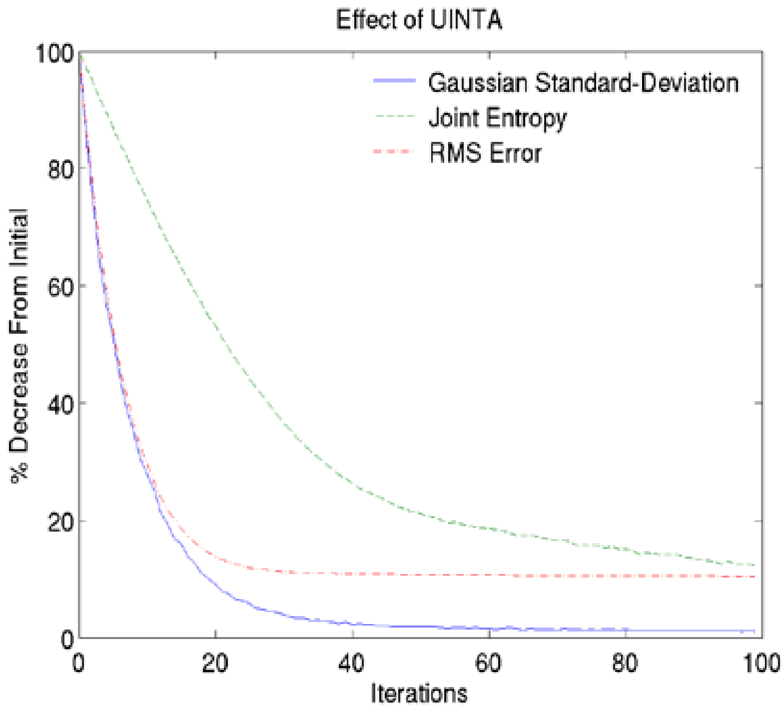
Checkerboard With Noise

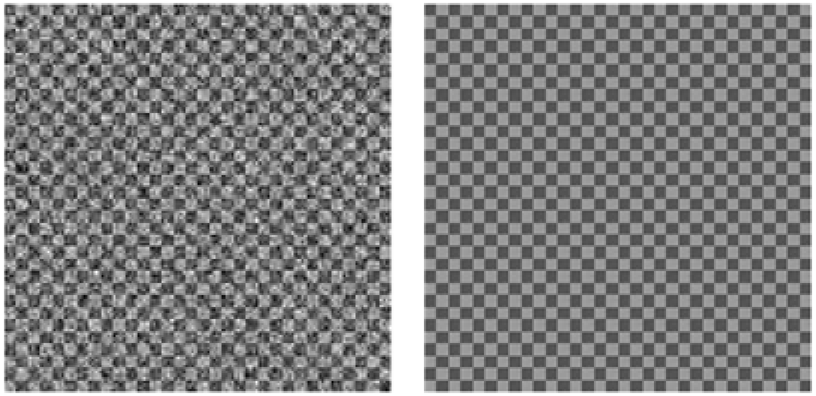


Original Noisy Filtered

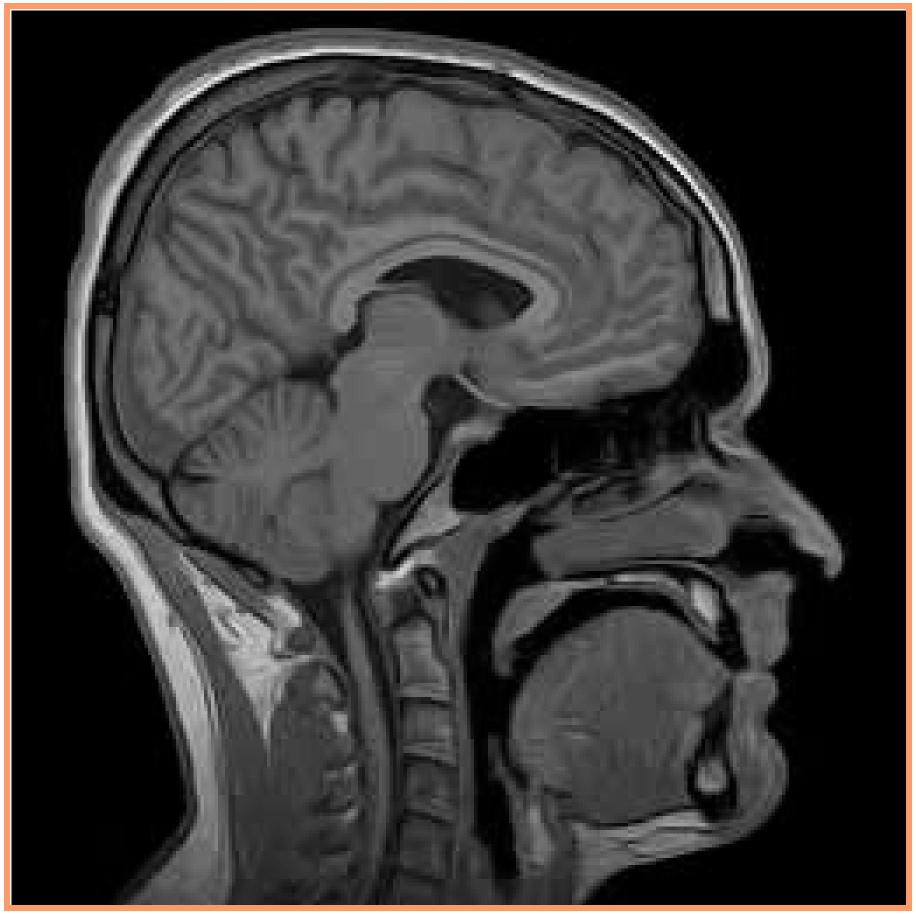
Quality of Denoising

- s, joint entropy, and RMS- error vs. number of iterations

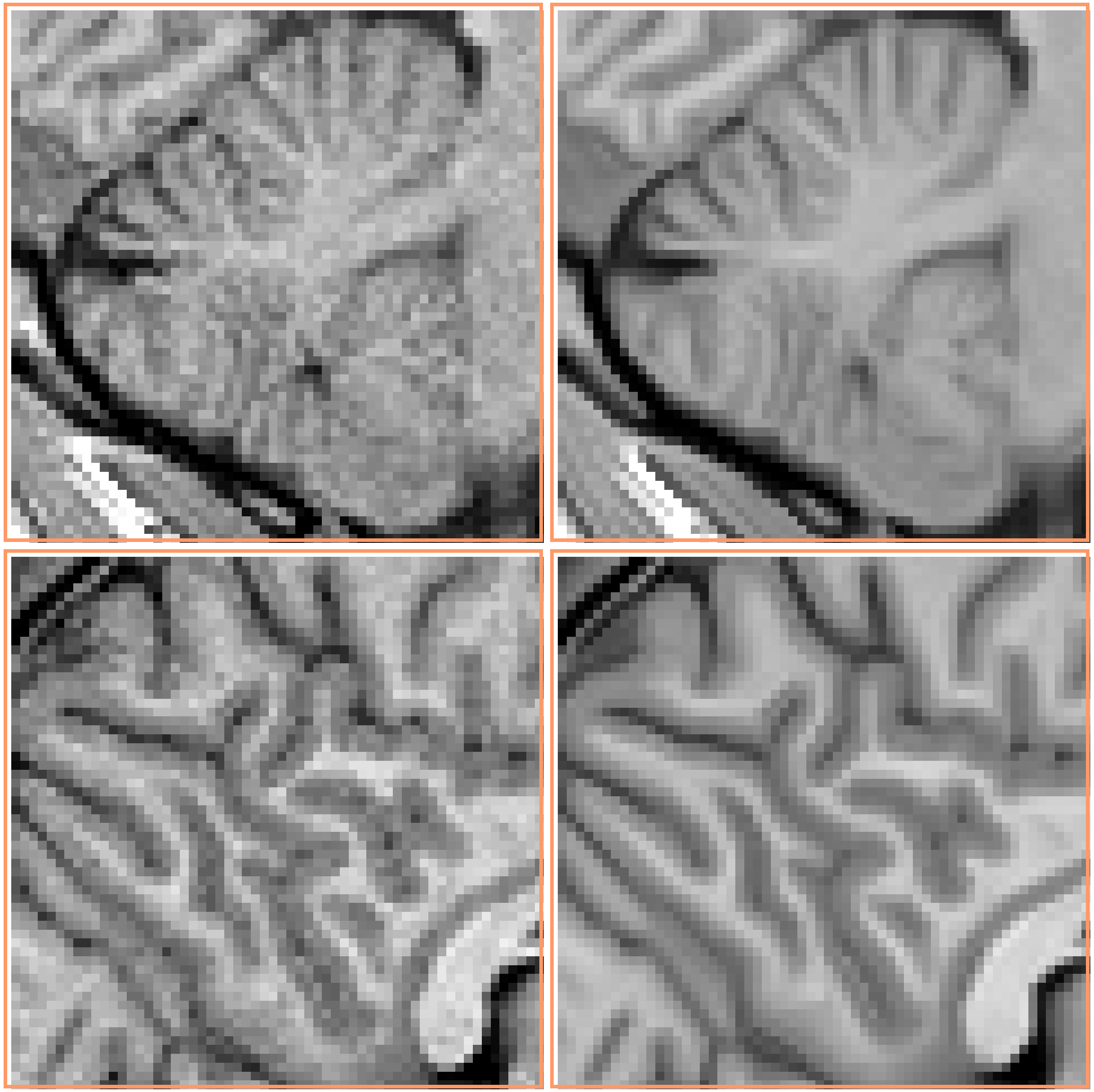




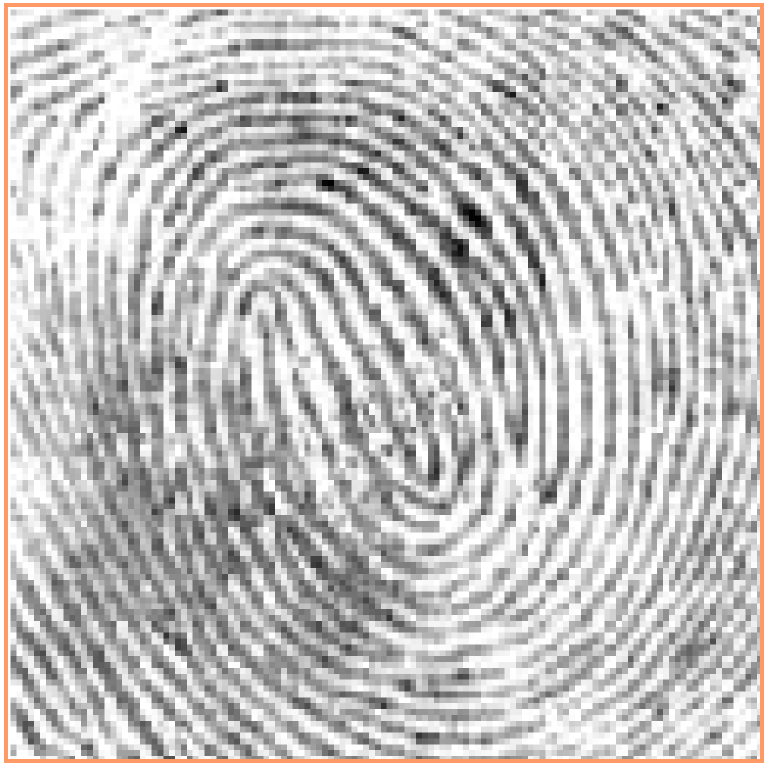
MRI Head



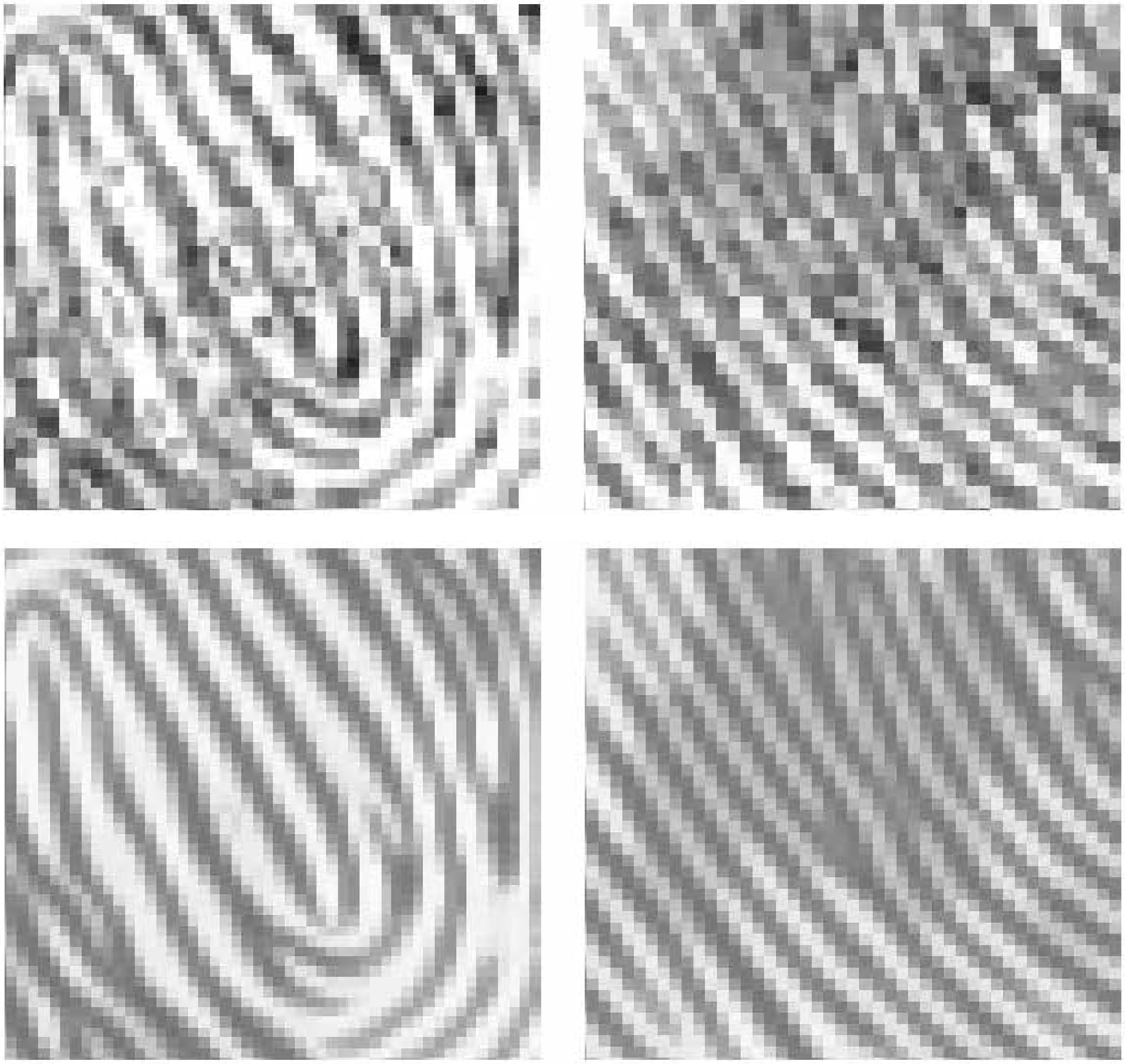
MRI Head



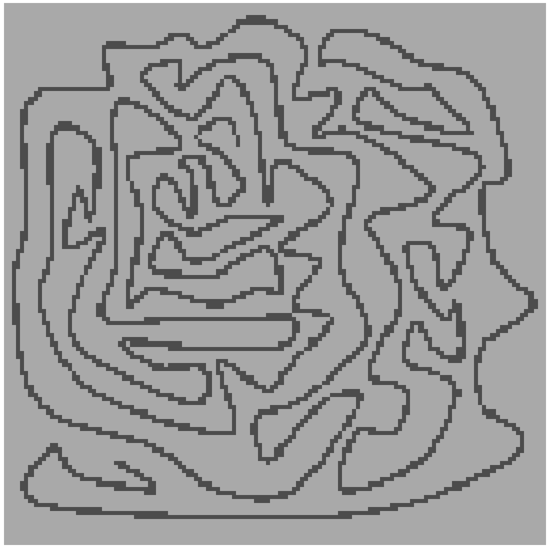
Fingerprint



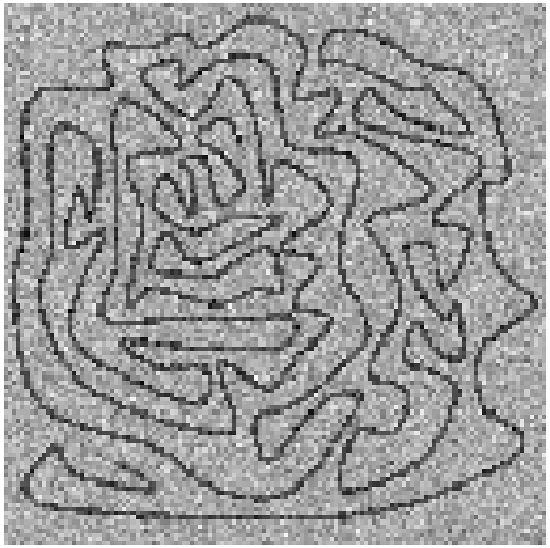
Fingerprint



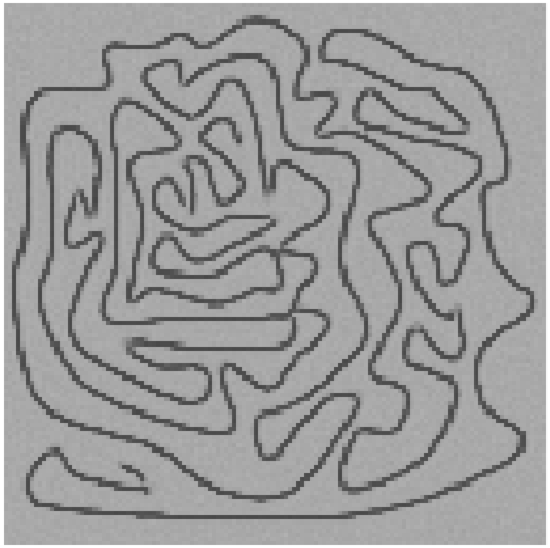
Results



Original

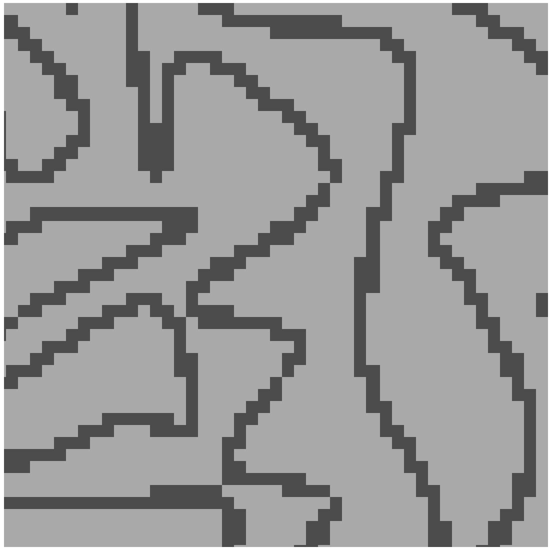


Noisy

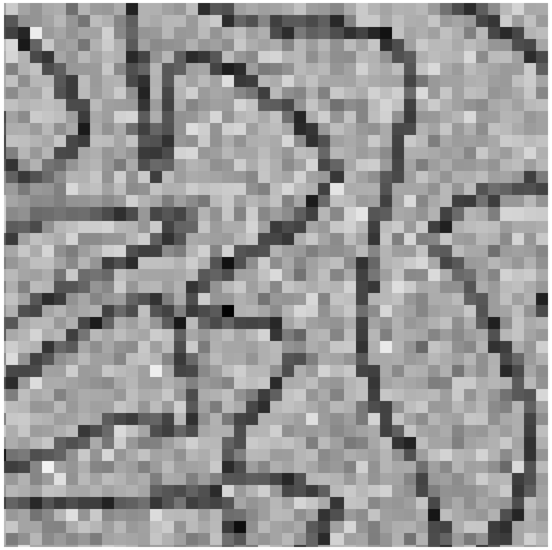


Filtered

Results



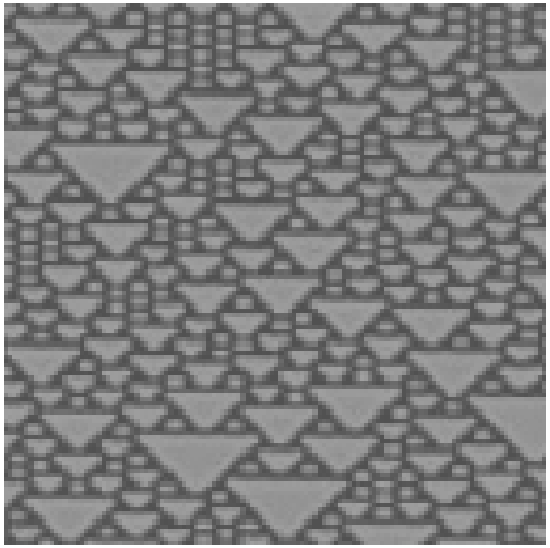
Original



Noisy

Filtered

Results



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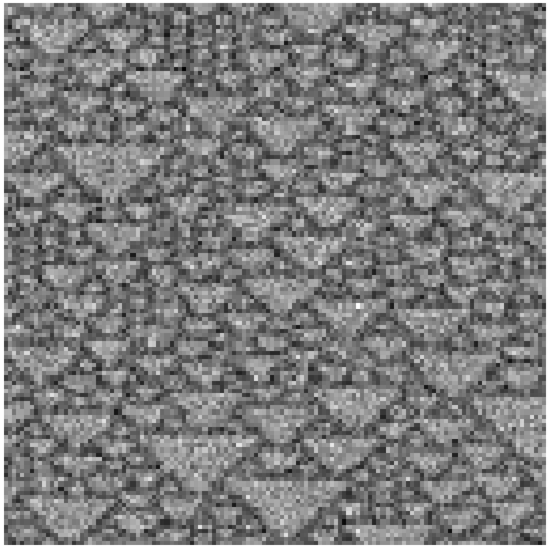

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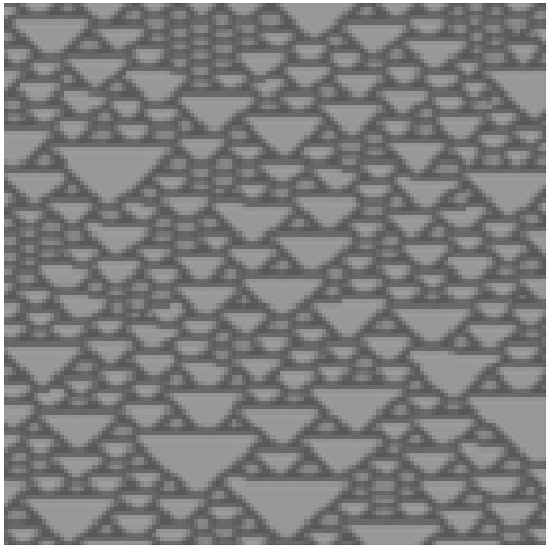

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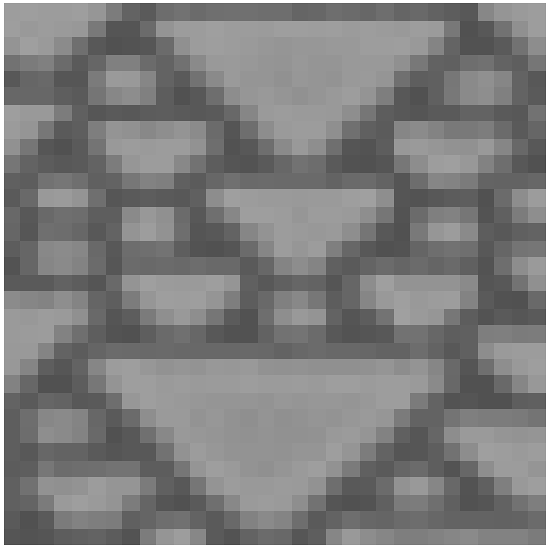
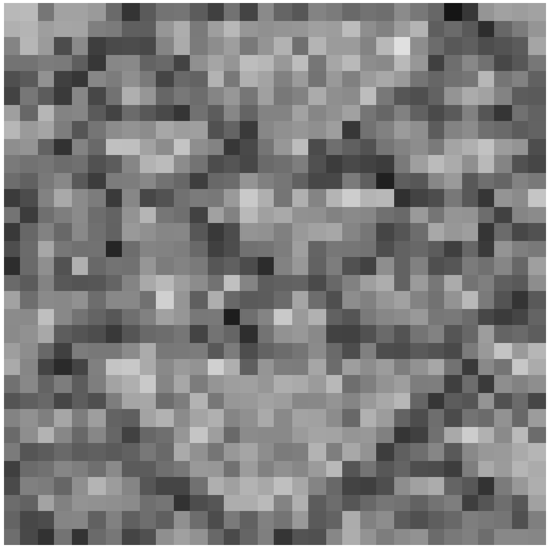
Fractal



Original



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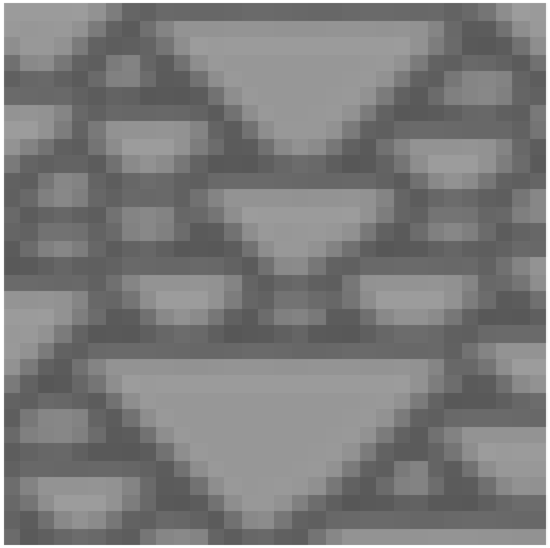
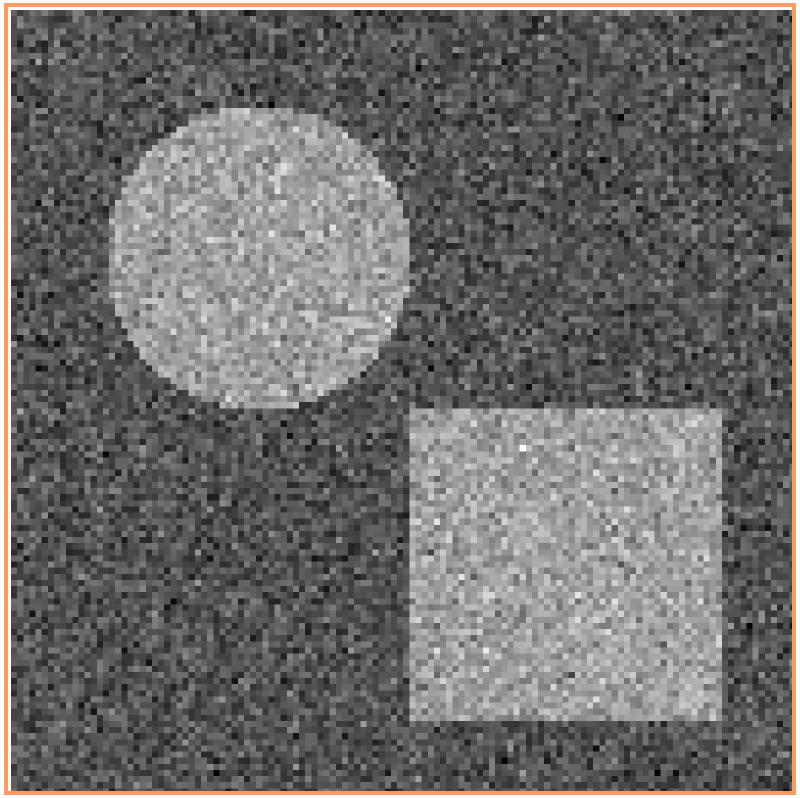
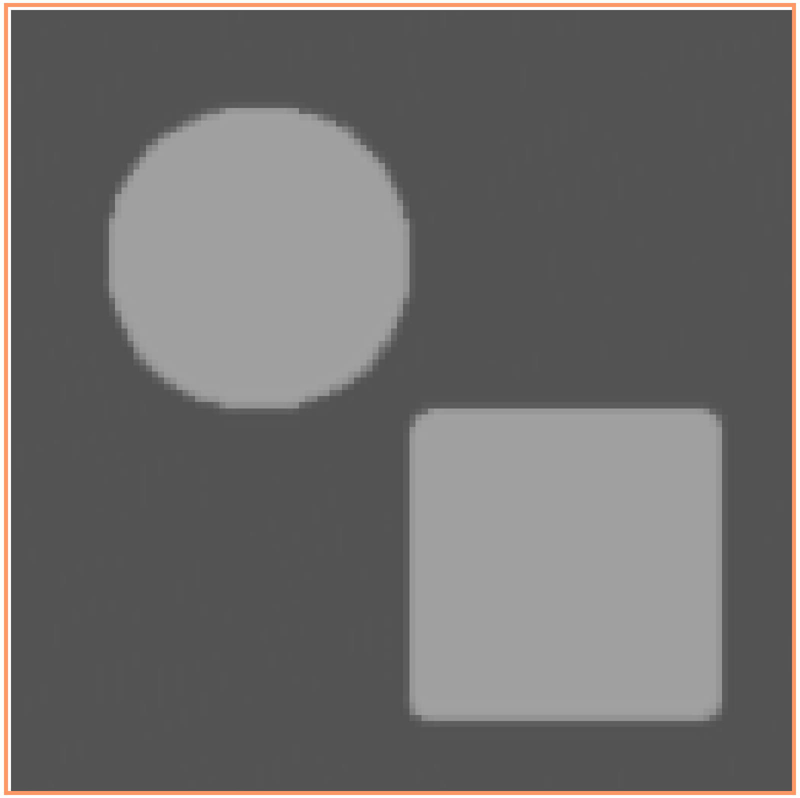


Filtered

Piecewise Constant

- Several 10s of Iterations
- Tends to obliterate rare events (e.g. corners)





Texture, Structure