Seminal paper: Kass, Witkin, Terzopoulos '87

Approach:
- Energy-minimizing spline guided by external constraint forces and restricted by internal properties.
- Dynamic, elastic model of curve shape driven by potential energy of image contour → find minimum of energy functional.

\[ V(s) \]

\[ \frac{\partial V(s)}{\partial s} = \left( \frac{\partial x(s)}{\partial s}, \frac{\partial y(s)}{\partial s} \right) \]

curve parametrization

Energy formulation:

\[ E_{\text{snake}} = \int E_{\text{int}}(V(s)) ds + \int E_{\text{image}}(V(s)) ds + \sum E_{\text{constraints}} \]

1. physical properties of deformable model
2. influence of image
3. constraints

Goal: find solution that minimizes \( E \)
\( E_{\text{int}} (v(s)) \): internal energy of snake due to bending and stretching

\[ \tilde{v} = v + \delta v \]
\[ \delta v = a t + b n \]

\( t \): tangent direction
\( n \): normal \((\perp t)\)

\[ \dot{v} = \frac{dv}{ds} = v_s (s) \]
\[ n = \frac{dt}{ds} = \frac{d^2v}{ds^2} = V_{ss} (s) \]

\[ \int E_{\text{int}} (v(s)) ds = \int (d(s) |v_s|^2 + \beta (s) |V_{ss}|^2) ds \]

\( \Rightarrow E_{\text{int}} = d(s) E_{\text{continuity}} + \beta (s) E_{\text{smoothness}} \)

(Kass, Witkin, Terzopoulos: \( d(s) = d = \text{const} \), \( \beta (s) = \beta = \text{const} \))
Cases:

a) straight line with regular sampling:
   \[ |V_0|^2 = \Phi \quad |V_{su}|^2 = \Phi \]

b) \( \beta (s_i) = \Phi \) : ignore smoothness at \( s_i \)
   \( \Rightarrow \) tangent discontinuity at \( s'_i \)
   \( \Rightarrow \) develops corners

c) \( L (s_i) = \Phi \) : ignore continuity term at \( s_i \)
   \( \Rightarrow \) local stretching or compression
   \( \Rightarrow \) develops irregular sampling

Discretization:

In practice, curve \( C \) represented by discrete set of ordered points: \( \{ \bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n \} \).

Continuity:

\[ |V_0|^2 = \left| \frac{dv(s)}{ds} \right|^2 \approx \| \bar{p}_i - \bar{p}_{i-1} \|^2 \]

Euclidean distance between points

Implementation: to ensure homogeneous sampling over iterations (equal spacing):

\[ \bar{d} = \| \bar{p}_i - \bar{p}_{i-1} \|^2 \quad \text{(keep distance close to average distance)} \]

where \( \bar{d} = \frac{1}{n} \leq \| \bar{p}_i - \bar{p}_{i-1} \| \) (average distance between points)
b) $E_{\text{smoothness}}$

$$\|V_{ss}\|^2 \approx \| P_{i-1} - 2P_i + P_{i+1}\|^2$$

(approximation of local curvature if points equally spaced)

2) **External Energy** $E_{\text{image}}$

External forces due to image contours attract snake close to contours:

Image $(x,y) \rightarrow$ Potential Surface $E_{\text{image}}(x,y) = P(y(s))$

Desirable: $E_{\text{image}}(x,y)$ small close to "good" image features, large otherwise

Common choices:
- Edges: $-\|\nabla G \ast I\|$(negative gradient magnitude of blurred image)
- Dark lines: $(I(x,y) \ast G_b)$ (smoothed image presenting dark line structure)
- etc.
Combine integration along contour

\[ E_{\text{snake}} = \int_0^L E_{\text{snake}}(v(s)) \, ds \]

\[ = \int_0^L \left( E_{\text{int}}(v(s)) + E_{\text{image}}(v(s)) + E_{\text{constraints}}(v(s)) \right) ds \]

- continuity, smoothness
- image forces, potential
- additional external forces (springs, volcos)

\[ = \int_0^L \left( d(s) |v_s|^2 + \beta(s) |v_s|^2 + P(v(s)) + E_{\text{constraints}}(v(s)) \right) ds \]

Problem to solve: We are looking for curve \( v(s) \), \( s \in [0, \ldots, L] \) such that \( E_{\text{snake}} \) is minimal.

\[ \Rightarrow \text{Energy minimization problem.} \]

Please note that we usually only get a local minimum and not the global minimum.
(see paper Kaes, Witkin, Teszopoulos '87)

**Springs:**

\[ E_{spring} = -K_{spring} (P_{image} - P_{snake})^2 \]

(spring gets lower energy if point is moving towards the spring anchor point)

**Volcanoes:**

Local repulsion force by deformation of \( P(v(s)) \):

\[ E_{volcano} = -\pi \text{volcano} \cdot \frac{1}{r^2} \]

(useful for pushing a snake out of a local minimum)
Solutions to energy minimization

(A) Greedy algorithm: Iteratively minimize each point
   ⇒ assume local minimizations will lead to
global minimum.

Source: Trucco/Verri, Introductory Techniques for
3-D Computer Vision, chapter 5

Idea: For each point \( p_i \), find local
   new minimum within a 3x3 or 5x5 local
   neighborhood.
   - visit each point along curve,
     find new minimum, move point, then go
to next point OR
   - calculate new best location for
     each point but then move all
     points together before next
     iteration
   - stop when # points that moved < threshold

(B) Variational Calculus
   Euler Lagrange differential equation:

→ see Bryan S. Morse, lecture 21, for
description (page 3-5)
→ see also Kass, Witkin, Terzopoulos '87
Sketching the solution (following Morse):

minimize \[ E_{\text{snake}} \approx \sum_{i=1}^{n} E_{\text{snake}}(\bar{V}(s_i)) \]

discrete points all points along contour

\[ \Rightarrow \nabla E_{\text{snake}} = 0 \]

\[ \nabla E_{\text{snake}} \approx \nabla \sum_{i=1}^{n} E_{\text{snake}}(\bar{V}(s_i)) \]

\[ = \sum_{i=1}^{n} \nabla E_{\text{snake}}(\bar{V}(s_i)) \]

\[ \nabla E_{\text{snake}}(\bar{V}(s_i)) = \nabla \left[ E_{\text{int}}(\bar{V}(s_i)) + E_{\text{image}}(\bar{V}(s_i)) + E_{\text{con}}(\cdot) \right] \]

\[ \nabla \text{ depend only on image } \Rightarrow \text{ precalculate derivatives} \]

see Kass et al.

\[ \nabla E_{\text{int}}(\bar{V}(s)) = \ldots \]

set \( l(s), \beta(s) \) as constants

\[ = \lambda \frac{\partial^2 \bar{V}}{\partial s^2} + \beta \frac{\partial^4 \bar{V}}{\partial s^4} \]

\[ \text{together: } \lambda \bar{V}_{ss} + \beta \bar{V}_{ssss} + \nabla E_{\text{external}} = 0 \]

\[ \text{components: } \lambda \bar{X}_{ss} + \beta \bar{X}_{ssss} + \frac{\partial E_{\text{ext}}}{\partial x} = 0 \]

\[ \lambda \bar{Y}_{ss} + \beta \bar{Y}_{ssss} + \frac{\partial E_{\text{ext}}}{\partial y} = 0 \]

(Two independent Euler-Lagrange equations)