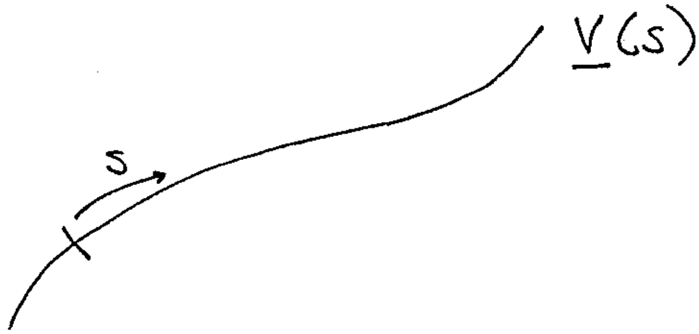
Snakes: Active Contour Models

Seminal paper: Kass, Witkin, Terzopoulos '87

Approach:
- Energy-minimizing spline guided by external constraint forces and restricted by internal properties.
- Dynamic, elastic model of curve shape driven by potential energy of image contour → find minimum of energy functional.



s: arc length

$\underline{V}(s) = \begin{pmatrix} x(s) \\ y(s) \end{pmatrix}$

curve parametrization

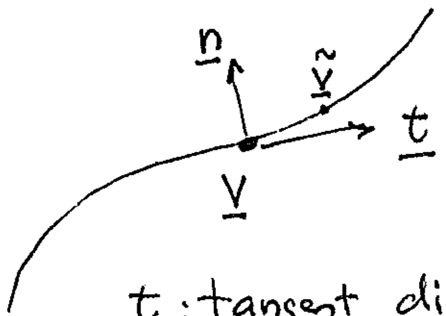
Energy formulation:

$$E_{snake} = \underbrace{\int E_{int}(\underline{V}(s))\,ds}_{\text{physical properties of deformable model}} + \underbrace{\int E_{image}(\underline{V}(s))\,ds}_{\text{influence of image}} + \sum E_{constraints}$$

(1) (2) (3)

Goal: find solution that minimizes E

① $\underline{E_{int}(\underline{v}(s))}$: internal energy of snake due to bending and stretching



$$\tilde{\underline{v}} = \underline{v} + \delta \underline{v}$$

$$\delta \underline{v} = \underbrace{a\,\underline{t}} + \underbrace{b\,\underline{n}}$$

$\underline{t}$: tangent direction
$\underline{n}$: normal ($\perp \underline{t}$)

$$\delta \underline{v} = a\,\underline{t} + b\,\underline{n}$$

speed, velocity $\underline{t} = \dfrac{d\underline{v}}{ds} = \underline{v}_s(s)$

$\underline{n} = \dfrac{d\underline{t}}{ds} = \dfrac{d^2\underline{v}}{ds^2} = \underline{v}_{ss}(s)$

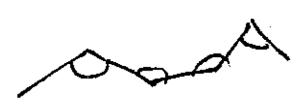
change of tangent vector α curvature

$$\Rightarrow \int E_{int}(\underline{v}(s))\,ds = \int \left(\alpha(s)\,|\underline{v}_s|^2 + \beta(s)\,|\underline{v}_{ss}|^2\right) ds$$

rigidity, stretching

bending

tension, elasticity, acts like membrane



rigidity, stiffness, acts like thin plate

$$\Rightarrow E_{int} = \alpha(s)\,E_{continuity} + \beta(s)\,E_{smoothness}$$

(Kass, Witkin, Terzopoulos: $\alpha(s) = \alpha = \text{const}$, $\beta(s) = \beta = \text{const}$)

Cases:

a) straight line with regular sampling:

$$|\underline{V}_s|^2 = \emptyset \qquad |\underline{V}_{ss}|^2 = \emptyset$$

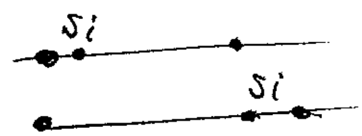
b) $\beta(s_i) = 0$: ignore smoothness at s_i

$\Rightarrow$ tangent discontinuity at s_i

$\Rightarrow$ develops corners

c) $\alpha(s_i) = 0$: ignore continuity term at s_i

$\Rightarrow$ local stretching or compression



$\Rightarrow$ develops irregular sampling

Discretization:

In practice: Curve C represented by discrete set of ordered points : $\{\bar{p}_1, \bar{p}_2, \dots \bar{p}_n\}$.

$E_{continuity}$:

a) $$|\underline{V}_s|^2 = \left|\frac{d\underline{v}(s)}{ds}\right|^2 \approx \|\underline{p}_i - \underline{p}_{i-1}\|^2$$

Euclidean distance btw. points

implementation: to ensure homogeneous sampling over iterations (equal spacing):

$$\bar{d} - \|\underline{p}_i - \underline{p}_{i-1}\|^2$$ (keep distance close to average distance)

where $\bar{d} = \frac{1}{n}\sum_n \|\underline{p}_i - \underline{p}_{i-1}\|$ (average distance between points)

b) $E_{smoothness}$:

$$\|\underline{v}_{ss}\|^2 \approx \|\underline{p}_{i-1} - 2\underline{p}_i + \underline{p}_{i+1}\|^2$$

(approximation of local curvature if points equally spaced)

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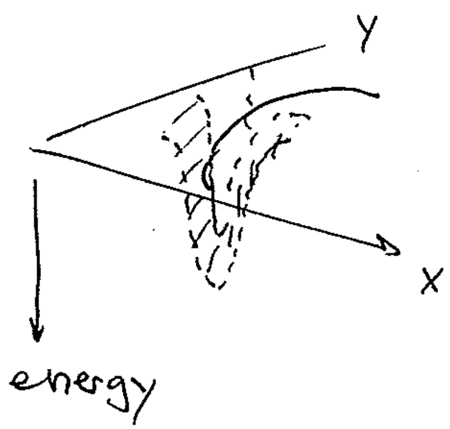
(2) External Energy E_{image}

External forces due to image contours attract snake close to contours :



Image$(x,y) \longrightarrow$ Potential Surface $E_{image}(x,y)$

$\Rightarrow P(\underline{v}(s))$

Desirable: $E_{image}(x,y)$ small close to "good" image features, large otherwise

Common choices for $P(\underline{v}(s))$:

- Edges: $-\|\nabla G_\sigma \otimes I\|$ (negative gradient magnitude of blurred image)
- Dark lines: $(I(x,y) \otimes G_\sigma)$ (Smoothed image presenting dark line structures)
- etc.

⇒ Combine

integration along contour

$$E_{snake} = \int_0^L E_{snake}(\underline{v}(s))\, ds$$

$$= \int_0^L \Big(\underbrace{E_{int}(\underline{v}(s))}_{\text{continuity, smoothness}} + \underbrace{E_{image}(\underline{v}(s))}_{\text{image forces, potential}} + \underbrace{E_{constraints}(\underline{v}(s))}_{\text{additional external forces (springs, volcanos)}} \Big)\, ds$$

$$= \int_0^L \Big(\alpha(s)\,|v_s|^2 + \beta(s)\,|\underline{v}_{ss}|^2 + P(\underline{v}(s)) + E_{constraints}(\underline{v}(s)) \Big)\, ds$$

Problem to solve: We are looking for curves $\underline{v}(s)$, $s \in [0,..,L]$ such that E_{snake} is minimal.

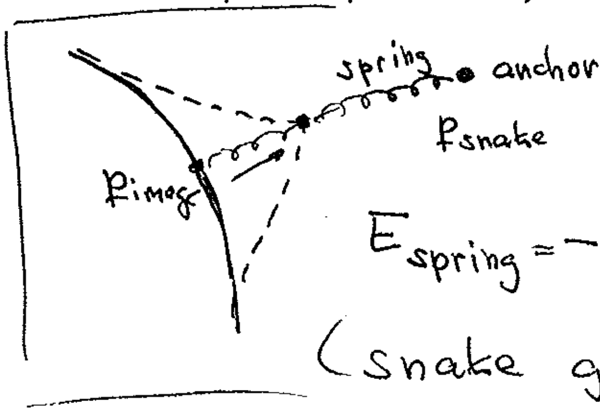
⇒ Energy minimization problem.

Please note that we usually only get a local minimum and not the global minimum.

3 External Constraints

(see paper Kass, Witkin, Terzopoulos '87)

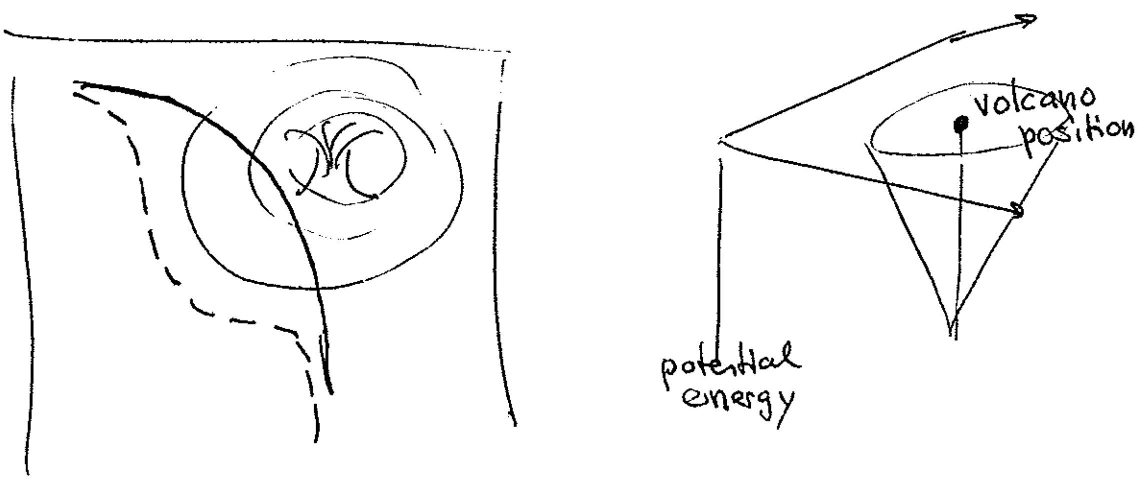
Springs:



$$E_{spring} = -K_{spring}(p_{image} - p_{snake})^2$$

(snake gets lower energy if point is moving towards the spring anchor point)

Volcano's:



local repulsion force by deformation of $P(\underline{v}(s))$:

$$E_{volcano}(\underline{x}) = -\mathcal{K}_{volcano} \cdot \frac{1}{r^2}$$

(useful for pushing a snake out of a local minimum)

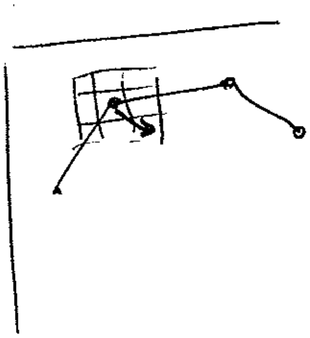
Solutions to energy minimization

(A) Greedy algorithm: Iteratively minimize each point

⇒ assume local minimizations will lead to global minimum.

Source: → Trucco/Verri, Introductory Techniques for 3-D Computer Vision, chapter 5

Idea:
- For each point p_i, find new local minimum within 3×3 or 5×5 local neighborhood.



- -visit each point along curve, find new minimum, move point, then go to next point OR
- -calculate new best location for each point but then move all points together befor next iteration
- stop when #points that moved < threshold

(B) Variational Calculus

Euler Lagrange differential equation:

→ see Bryan S. Morse, lecture 21, for description (pages 3-5)

→ see also Kass, Witkin, Terzopoulos '87 sketching the solution (following Morse):

minimize $E_{snake} \approx \underbrace{\sum_{i=1}^{n}}_{\text{all points along contour}} E_{snake}\,\bar{v}(s_i)$

discrete points

$\Rightarrow \nabla E_{snake} = 0$

$$\nabla E_{snake} \approx \nabla \sum_{i=1}^{n} E_{snake}(\bar{v}(s_i))$$
$$= \sum_{1}^{n} \nabla E_{snake}(\bar{v}(s_i))$$

$$\nabla E_{snake}(\bar{v}(s_i)) = \nabla \left[E_{int}(\bar{v}(s_i)) + \underbrace{E_{image}(\bar{v}(s_i)) + E_{con}(\,)}_{\nabla \text{ depend only on image} \Rightarrow \text{precalculate derivatives}} \right]$$

see Kass et al.

$\nabla E_{int}(\bar{v}(s_i)) = \ldots$

set $\alpha(s), \beta(s)$ as constants

$$= \alpha \frac{\partial^2 \bar{v}}{\partial s^2} + \beta \frac{\partial^4 \bar{v}}{\partial s^4}$$

together: $\alpha \bar{v}_{ss} + \beta \bar{v}_{ssss} + \nabla E_{external} = 0$

components:
$$\alpha x_{ss} + \beta x_{ssss} + \frac{\partial E_{ext}}{\partial x} = 0$$
$$\alpha y_{ss} + \beta y_{ssss} + \frac{\partial E_{ext}}{\partial y} = 0$$

(Two independent Euler Lagrange equations)