$\boxed{\text{SNAKES}}$ G. Gerig, Nov. 2010

Euler Lagrange DE: variational calculus: CS 6640 (4)

$$E_{snake}: \int \alpha \cdot \left|\frac{dv}{ds}\right|^2 + \beta \left|\frac{d^2 v}{ds^2}\right|^2 + P(\underline{v}(s))\, ds = \min$$

(1) (2) (3)

Variational calculus: concept

$\underline{v}_0(s)$ curve in minimal energy state

$\underline{v}(s) = \underline{v}_0(s) + \delta\underline{v}$ curve with minimal perturbation (very small deviation)

[Euler: for $\delta\underline{v} = 0$: $\underline{v}(s) \rightarrow \underline{v}_0(s)$ minimal]

(1) $$\frac{d\underline{v}}{ds} \cdot \frac{d\underline{v}}{ds} = \frac{d}{ds}(\underline{v}_0(s) + \delta\underline{v}) \cdot \frac{d}{ds}(\underline{v}_0(s) + \delta\underline{v})$$

$$= \left(\frac{d\underline{v}_0}{ds}\right)^2 + 2\frac{dv_0}{ds} \cdot \frac{d}{ds} \delta\underline{v} + \cancel{\frac{d}{ds}\delta v \cdot \frac{d}{ds}\delta v}$$

$O(\delta v^2)$ very small deviation

(2) : ..

(3) $P(\underline{v}(s)) = P(\underline{v}_0(s) + \delta\underline{v}) \underset{\substack{\uparrow \\ \text{Taylor} \\ \text{series} \\ \text{exp.}}}{=} P(\underline{v}_0) + \nabla_{\underline{v}} P \cdot \delta\underline{v} + \ldots$

Extremal condition:

$$\int_0^1 \left(\alpha \cdot \frac{d\underline{v}_0}{ds} \cdot \frac{d}{ds}\delta\underline{v} + \beta \ldots + \nabla P \cdot \delta\underline{v}\right) ds = \emptyset$$

$$= \underbrace{\int_0^1 \left(\alpha \cdot \frac{d\underline{v}_0}{ds} \cdot \frac{d}{ds}\delta\underline{v}\right) ds}_{\alpha \cdot \frac{d\underline{v}_0}{ds} \cdot \delta\underline{v} \Big|_0^1 - \int \delta\underline{v} \frac{d}{ds}\left(\alpha \cdot \frac{dv_0}{ds}\right) ds} + \int_0^1 \beta \ldots ds + \int_0^1 \nabla P \cdot \delta\underline{v}\, ds = 0$$

$$\Longrightarrow \int \left\{ -\frac{d}{ds}\left(\alpha \frac{d\underline{v}_0(s)}{ds}\right) + \frac{d^2}{ds^2}\left(\beta \cdot \frac{d^2\underline{v}_0(s)}{ds^2}\right) + \nabla P \right\} \delta\underline{v}\, ds$$

$\boxed{\text{Euler Lagrange DE 4th order}}$

Euler Lagrange DE 4th order: initial value: $v(0)$, $v'(0)$, $v''(0)$, $v'''(0)$

boundary value: $v(0)$, $v'(0)$, $v(1)$, $v'(1)$

$\Rightarrow$ write in coordinates:

$$\alpha\, x_{ss} + \beta x_{ssss} + \frac{\partial E_{ext}}{\partial x} = \emptyset$$

$$\alpha\, y_{ss} + \beta y_{ssss} + \frac{\partial E_{ext}}{\partial y} = \emptyset$$

$$\boxed{-\alpha \underline{v}_{ss} + \beta \underline{v}_{ssss} + \nabla P = (}$$

solution analytically, or with problems in discrete

discretization:

$$\boxed{-\alpha \underline{v}_{ss} + \beta \underline{v}_{ssss} + \nabla P = \emptyset}$$

finite differences:

$$\underline{v}_s = \frac{1}{\Delta}\left\{\underline{v}(s+\tfrac{\Delta}{2}) - \underline{v}(s-\tfrac{\Delta}{2})\right\}$$

$$\underline{v}_{ss} = \frac{1}{\Delta^2}\left\{\underline{v}(s+\Delta) - 2\underline{v}(s) + \underline{v}(s-\Delta)\right\}$$

$$\underline{v}_{ssss} = \frac{1}{\Delta^4}\left\{\underline{v}(s-2\Delta) - 4\underline{v}(s-\Delta) + 6\underline{v}(s) - \dots\right\}$$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & -1 & & & \\ & & 1 & & -2 & & 1 & & \\ & 1 & & -3 & & 3 & & - & \\ 1 & & -4 & & 6 & & -4 & & \end{array}$$

DE: $\alpha\, \underline{v}_{ss}$: $\frac{1}{\Delta^2}\begin{pmatrix}1 & -2 & 1\end{pmatrix}\begin{pmatrix}\underline{v}(s-\Delta)\\ \underline{v}(s)\\ \underline{v}(s+\Delta)\end{pmatrix}$

$$\Rightarrow \frac{1}{\Delta^2}\begin{pmatrix}1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & & \ddots & \\ & & 1 & -2 & 1\end{pmatrix}\begin{pmatrix}\underline{v}(-1)\\ \underline{v}(0)\\ \underline{v}(1)\\ \vdots\\ \underline{v}(n+1)\end{pmatrix} = -\begin{pmatrix}\nabla P(\underline{v}(-1))\\ \vdots\\ \nabla P(\underline{v}(n+1))\end{pmatrix}$$

$$+\frac{1}{\Delta^4}\begin{pmatrix}1 & -4 & 6 & -4 & 1 & \\ & 1 & -4 & 6 & -4 & 1\\ & & & \ddots & & \\ & 1 & -4 & 6 & -4 & 1\end{pmatrix}\begin{pmatrix}\underline{v}(-1)\\ \underline{v}(0)\\ \vdots\\ \underline{v}(n+1)\end{pmatrix}$$

structure: $\underline{v}(i) = \begin{matrix} x(i)\\ y(i)\end{matrix}$

Structure

$$A \cdot x + P_x(x,y)$$
$$A \cdot y + P_y(x,y)$$

A: penta diag. matrix

$$\begin{pmatrix} x(i) \\ y(i) \end{pmatrix} = \underline{v}(i)$$

$$\Rightarrow \quad K \cdot \underline{v}^{[t]} = -P_v \Big|_{\underline{v}^{[t-1]}}$$

$$K \cdot x^{[t]} = -P_x \Big|_{x^{[t-1]}}$$
$$K \cdot y^{[t]} = -P_y \Big|_{y^{[t-1]}}$$