2.0 Generalization of Hough Transform (HT) 2010

straight line: \( x \cos \theta + y \sin \theta - g = 0 \)

\[ \bar{x} = (x, y) \quad \bar{a} = (g, \theta) = \text{parameter vector} \]

\( \mathcal{F}(\bar{x}, \bar{a}) = \phi \)

\( \bar{x} \): image point

\( \bar{a} \): parameter vector

Transformation:

\( \forall \bar{x}_c \in \text{contours} : (\bar{x}_c \rightarrow \{ \bar{a} | f(\bar{x}_c, \bar{a}) = \phi \}) \)

Incrementation:

\( \forall \bar{a}_j \in \{ \bar{a} \} : \text{acc}(a_j) = \text{acc}(a_j) + \text{inc} \)

Example: Detection of circles

\( (x-x_0)^2 + (y-y_0)^2 - r^2 = \phi \)

Symmetry of points/parameters:

- fix center \((x_0, y_0)\) and radius \(r\):
  - image points \((x, y)\) describe circle centered at \((x_0, y_0)\).

- fix an image point \((x, y)\) and choose a radius \(r\): possible centers \((x_0, y_0)\) describe circle centered at \((x, y)\).
Put it together:

- Image Space
- Parameter Space

Select $r$

If $r$ is not known:

- Image point $\bar{X}_i$ transforms into right cone in a 3D parameter space $(x_0, y_0, r)$.
- Each point $\bar{X}_i$: accumulation of votes of cells intersected by right cones.

Correct $r$:
- High density of votes indicates center
- Smaller $r$: no accumulation of center
2.1 Extension to arbitrary 2D shapes

**Example circle:**

![Circle Diagram]

Model given by set of discrete vectors

**Algorithm:**

- Construct model \( m = \{ \vec{m}_k, k=1...n \} \) of discrete model points

- Incertitation: for each image contour point \( \hat{x}_i \) in parameter space, map vector backwards and increment parameter cells:

\[
\forall k \in \text{model} : \{ \hat{x}_i - \vec{m}_k \} \Rightarrow \text{acc}(\hat{x}_i - \vec{m}_k)++
\]

**Formal:**

\[
\forall \hat{x}_i \left[ \forall \vec{m}_k (\text{acc}(\hat{x}_i - \vec{m}_k)++) \right]
\]
HT Algorithm for arbitrary 2D curves

1. Define discrete model curve, choose center, represent model as set of vectors from center to curve points:
\[ \{ \bar{m}_k; k=0 \ldots m-1 \} \]

2. Incrementation of accumulator:
\[ \forall \bar{x}_i : [\forall \bar{m}_k (acc(\bar{x}_i - \bar{m}_k)++) \]

\( \bar{x}_i \): image points that are part of contours/edges (after Canny edge detection and non maximum suppression)

3. For rotation and scaling:
   transform discrete model and start new incrementation in new accumulator buffer

4. Find accumulator cells with high number of votes \( \Rightarrow \) centers of likely structures
Comparison of HT and Template Matching

HT is efficient implementation of a general template matching strategy.

a) Matched Filtering:

\[
F(x, y) = \sum_{i} \sum_{j} T(i, k) I(x-j, y-k)
\]

T: template
F: sum of all products over template size

Computational expense:

\[(x \cdot y) \cdot (j \cdot k)\]

b) Hough Transform:

\[
\forall x_i \in \text{edges} \cdot \forall \hat{m}_k
\]

\# edge pixels \# template vectors
Models with local edge/gradient direction

\[ \{ (\tilde{m}_k, \theta_k); k=0 \ldots m-1 \} \]

each model vector has edge orientation 
gradient at tip location 
as additional attribute

\[ \Rightarrow \text{edge gradient and model orientation have to match?} \]

Incorporation:

\[ \begin{align*}
&\bullet \text{each image edge point with gradient orientation } (\tilde{x}_i; \theta_i): \\
&\bullet \text{accumulate not whole model but only model vectors with same gradient orientation:} \\
&(\tilde{x}_i, \theta_i) \leftrightarrow (\tilde{m}_k, \theta_k), \theta_k = \theta_i \\
&\forall (\tilde{x}_i, \theta_i) \left[ \forall (\tilde{m}_k, \theta_k | \theta_k = \theta_i) \text{ acc}(\tilde{x}_i - \tilde{m}_k + t) \right]
\end{align*} \]

leads to concept presented by Ballard 1981:
Generalized HT (GHT)
Generalized Hough Transform (GHT)

(see original paper Ballard 1981)

Basic idea: sort model vectors \((\vec{m}_k, \Theta_k)\) as a function of the associated contour normal \(\Theta_k\)

\[
\begin{array}{c|c|c|c}
\Theta_1 & \vec{m}_{11}, \vec{m}_{12}, \vec{m}_{13} & \cdots \\
\Theta_2 & \vec{m}_{21} & \cdots \\
\Theta_3 & \vec{m}_{31}, \vec{m}_{32} & \cdots \\
\vdots & \vec{m}_{m1}, \vec{m}_{m2} & \cdots \\
\Theta_m & & \\
\end{array}
\]

Operation with GHT:

a) construct discrete model curve, chose reference centers, store model vectors and associated contour normal (gradient direction)

b) generate \(R\)-table by sorting the \((\vec{m}_k, \Theta_k)\) vectors by angle and putting them into a list with discrete bins of angles

c) \(\forall (\vec{x}, \Theta) \in \text{image contours:}\)

- index \(R\)-table at \(\Theta_i\)
- get model vectors \(\{\vec{m}_k\}\)
- increment accumulator at positions \((\vec{x} - \vec{m}_k), k=1 \ldots c\)

Properties of \(R\)-table

- scaling: \(\vec{m}^{(s)}_{\Theta_i} = \vec{m}_{\Theta_i} \cdot s\)
- rotation: \(\vec{m}^{(r)}_{\Theta_i} = \vec{m}_{\Theta_i} + \Delta\)
- \(\vec{m}_{\Theta_i} = \vec{m}_{\Theta_i} \cdot [R(\Delta)]\)