Mathematical Morphology: Binary Morphology

CS 650: Computer Vision

Mathematical Morphology

- The analysis of signals and images based on shape (Morphology = “study of shape”)
- Uses a set-theoretic approach to modify shapes based on local operators
- Many operations are similar to convolution but use set operations
- Useful for
  - enhancing structural properties
  - segmentation
  - quantitative description

Set Operations on Binary Images

- $A$ the image (“on” pixels)
- $A^c$ the compliment of the image (inverse)
- $A \cup B$ the union of images $A$ and $B$
- $A \cap B$ the intersection of images $A$ and $B$
- $A - B = A \cap B^c$ the difference between $A$ and $B$
  (the pixels in $A$ that aren’t in $B$)
- $\#A$ the cardinality of $A$ (area of the object(s))

Translation

- The image $A$ translated by movement vector $t$ is $A_t = \{c \mid c = a + t$ for some $a \in A\}$
- Literally, pick up each pixel in $A$ and move it by the movement vector $t$

Dilation

- Everywhere the structuring element $B$ overlaps the shape:
  $A \oplus B = \{c \mid c = a + b$ for some $a \in A$ and $b \in B\}$
- Unioned copies of the shifted image:
  $A \oplus B = \bigcup_{t \in B} A_t$
- Unioned copies of the shifted $B$:
  $A \oplus B = \bigcup_{t \in A} B_t$
Dilation

Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company’s software may recognize a date using “00” as 1900 rather than the year 2000.

![Dilation Diagram]

Properties of Dilation

▶ Commutative:
\[ A \oplus B = B \oplus A \]

▶ Associative:
\[ (A \oplus B) \oplus C = A \oplus (B \oplus C) \]

Erosion

▶ All the positions where the structuring element B fits entirely inside the shape:
\[ A \ominus B = \{ x | x + b \in A \text{ for every } b \in B \} \]

▶ Intersected copies of the shifted image
\[ A \ominus B = \bigcap_{t \in B} A - t \]

![Erosion Diagram]

Properties of Erosion

▶ Not commutative:
\[ A \ominus B \neq B \ominus A \]

▶ Not associative:
\[ (A \ominus B) \ominus C \neq A \ominus (B \ominus C) \]
Duality of Erosion and Dilation

- **Dual** operations are ones that can be defined in terms of each other
- Duals are **not** inverse operations
- Dilation and erosion are dual operations:
  \[
  (A \ominus B)^c = A^c \ominus \tilde{B} \\
  (A \oplus B)^c = A^c \oplus \tilde{B}
  \]
  where $\tilde{B}$ denotes the reflection of $B$
- Intuition: dilating the foreground is the same as eroding the background—the structuring just element flips

Example Application: Finding Boundary Pixels

- Can find all of the boundary pixels by dilating the object and subtracting the original:
  \[
  \text{Bound}_{\text{ext}}(A) = (A \oplus B) - A
  \]
  where $B$ is a $3 \times 3$ structuring element containing all 1s
- Or by eroding the original and subtracting that from the original:
  \[
  \text{Bound}_{\text{int}}(A) = A - (A \ominus B)
  \]

Hit-and-Miss

- Hit-and-miss operators find target pixel configurations:
  \[
  A \otimes (J, K) = (A \ominus J) \cap (A^c \ominus K)
  \]
  - $J$ is the target that must be “hit” (foreground)
  - $K$ is the target that must be “missed” (background)
- Example:
  
  $$
  \begin{array}{cccc}
  0 & 0 & 0 & 0 \\
  0 & 1 & 1 & 0 \\
  1 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  \end{array}
  \rightarrow
  \begin{array}{cccc}
  x & 0 & 0 & 0 \\
  1 & 1 & 0 & x \\
  x & 1 & x & 0 \\
  \end{array}
  $$

Opening and Closing

- An **opening** operation is an erosion followed by a dilation:
  \[
  A \circ B = (A \ominus B) \oplus B
  \]
- Used to remove small objects, protrusions, connections

- A **closing** operation is a dilation followed by an erosion:
  \[
  A \bullet B = (A \oplus B) \ominus B
  \]
- Used to remove small holes, gaps, etc.
Duality of Opening and Closing

- Opening and closing are also duals:
  \[(A \circ B)^c = A^c \bullet \bar{B}\]

- Intuition: opening to remove small foreground objects is the same as closing small holes in the background

Example: Opening and Closing for Noise Removal

Idempotence of Opening and Closing

- An operation is *idempotent* if applying it once is as good as applying it many times
  
  Opening and closing are idempotent

\[(A \circ B) \circ B = A \circ B\]

\[(A \bullet B) \bullet B = A \bullet B\]