Computer Vision

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Geometric Camera Calibration

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Series outline

- Cameras and lenses
- Geometric camera models
- Geometric camera calibration
- Stereopsis
Lecture outline

- The calibration problem
- Least-square technique
- Calibration from points
- Radial distortion
- A note on calibration patterns
**Camera calibration**

*Camera calibration* is determining the *intrinsic* and *extrinsic* parameters of the camera.

The are three coordinate systems involved: image, camera, and world.

**Key idea:** to write the *projection equations* linking the known coordinates of a set of 3-D points and their projections, and solve for the camera parameters.
Projection matrix

\[
M = \begin{pmatrix}
\alpha r_1^T - \alpha \cot \theta r_2^T + u_0 r_3^T & \alpha t_x - \alpha \cot \theta t_y + u_0 t_z \\
\frac{\beta}{\sin \theta} r_2^T + v_0 r_3^T & \frac{\beta}{\sin \theta} t_y + v_0 t_z \\
r_3^T & t_z
\end{pmatrix}
\]

Replacing \( M \) by \( \lambda M \) in

\[
\begin{align*}
u &= \frac{m_1 \cdot P}{m_3 \cdot P} \\
v &= \frac{m_2 \cdot P}{m_3 \cdot P}
\end{align*}
\]

does not change \( u \) and \( v \).

M is only defined up to scale in this setting.
The calibration problem

Given $n$ points $P_1, \ldots, P_n$ with known positions and their images $p_1, \ldots, p_n$

Find $i$ and $e$ such that

$$
\sum_{i=1}^{n} \left[ \left( u_i - \frac{m_1(i, e) \cdot P_i}{m_3(i, e) \cdot P_i} \right)^2 + \left( v_i - \frac{m_2(i, e) \cdot P_i}{m_3(i, e) \cdot P_i} \right)^2 \right] \text{ is minimized}
$$
**Linear systems**

Square system:
- Unique solution
- Gaussian elimination

Rectangular system:
- Underconstrained: Infinity of solutions
- Overconstrained: no solution

Minimize $|Ax-b|^2$
How do you solve overconstrained linear equations?

• Define $E = |\mathbf{e}|^2 = \mathbf{e} \cdot \mathbf{e}$ with

$$
\mathbf{e} = A\mathbf{x} - \mathbf{b} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \mathbf{b}
$$

$$
= x_1 \mathbf{c}_1 + x_2 \mathbf{c}_2 + \cdots + x_n \mathbf{c}_n - \mathbf{b}
$$

• At a minimum,

$$
\frac{\partial E}{\partial x_i} = \left( \frac{\partial \mathbf{e}}{\partial x_i} \right) \cdot \mathbf{e} + \mathbf{e} \cdot \left( \frac{\partial \mathbf{e}}{\partial x_i} \right) = 2 \left( \frac{\partial \mathbf{e}}{\partial x_i} \right) \cdot \mathbf{e}
$$

$$
= 2 \frac{\partial}{\partial x_i} (x_1 \mathbf{c}_1 + \cdots + x_n \mathbf{c}_n - \mathbf{b}) \cdot \mathbf{e} = 2 \mathbf{c}_i \cdot \mathbf{e}
$$

$$
= 2 \mathbf{c}_i^T (A\mathbf{x} - \mathbf{b}) = 0
$$

• or

$$
0 = \begin{bmatrix} \mathbf{c}_1^T \\ \vdots \\ \mathbf{c}_n^T \end{bmatrix} (A\mathbf{x} - \mathbf{b}) = A^T (A\mathbf{x} - \mathbf{b}) \Rightarrow A^T A\mathbf{x} = A^T \mathbf{b},
$$

where $\mathbf{x} = A^T \mathbf{b}$ and $A^\dagger = (A^T A)^{-1} A^T$ is the pseudoinverse of $A$!
Homogeneous linear equations

Square system:
- Unique solution = 0
- Unless $\text{Det}(A) = 0$

Rectangular system:
- $0$ is always a solution

Minimize $|Ax|^2$ under the constraint $|x|^2 = 1$
How do you solve overconstrained homogeneous linear equations?

$$E = |Ux|^2 = x^T(U^TU)x$$

- Orthonormal basis of eigenvectors: $e_1, \ldots, e_q$.
- Associated eigenvalues: $0 \leq \lambda_1 \leq \ldots \leq \lambda_q$.
- Any vector can be written as
  $$x = \mu_1 e_1 + \ldots + \mu_q e_q$$
  for some $\mu_i$ ($i = 1, \ldots, q$) such that $\mu_1^2 + \ldots + \mu_q^2 = 1$.

$$E(x) - E(e_1) = x^T(U^TU)x - e_1^T(U^TU)e_1$$
$$= \lambda_1^2 \mu_1^2 + \ldots + \lambda_q^2 \mu_q^2 - \lambda_1^2$$
$$\geq \lambda_1^2(\mu_1^2 + \ldots + \mu_q^2 - 1) = 0$$

The solution is the eigenvector $e_1$ with least eigenvalue of $U^TU$. 
Example: Line fitting

Problem: minimize

\[ E(a, b, d) = \sum_{i=1}^{n} (ax_i + by_i - d)^2 \]

with respect to \((a, b, d)\).

- Minimize \( E \) with respect to \( d \):

\[
\frac{\partial E}{\partial d} = 0 \implies d = \frac{\sum_{i=1}^{n} ax_i + by_i}{n} = \overline{a}x + \overline{b}y
\]

- Minimize \( E \) with respect to \( a, b \):

\[
E = \sum_{i=1}^{n} [a(x_i - \overline{x}) + b(y_i - \overline{y})]^2 = |\mathcal{U}n|^2
\]

where

\[
\mathcal{U} = \begin{pmatrix} x_1 - \overline{x} & y_1 - \overline{y} \\ \vdots & \vdots \\ x_n - \overline{x} & y_n - \overline{y} \end{pmatrix}
\]

and

\[
\mathcal{U}^T \mathcal{U} = \begin{pmatrix} \sum_{i=1}^{n} x_i^2 - n\overline{x}^2 & \sum_{i=1}^{n} x_i y_i - n\overline{x}\overline{y} \\ \sum_{i=1}^{n} x_i y_i - n\overline{x}\overline{y} & \sum_{i=1}^{n} y_i^2 - n\overline{y}^2 \end{pmatrix}
\]
Estimation of the projection matrix

Given \( n \) points \( P_1, \ldots, P_n \) with known positions and their images \( p_1, \ldots, p_n \)

\[
\begin{pmatrix} u_i \\ v_i \end{pmatrix} = \begin{pmatrix} m_1 \cdot P_i \\ m_3 \cdot P_i \\ m_2 \cdot P_i \\ m_3 \cdot P_i \end{pmatrix} \iff \begin{pmatrix} m_1 - u_i m_3 \\ m_2 - v_i m_3 \end{pmatrix} P_i = 0
\]

The constraints associated with the \( n \) points yield a system of \( 2n \) homogeneous linear equations in the 12 coefficients of the matrix \( M \),

\[
\mathcal{P} m = 0 \text{ with } \mathcal{P} \overset{\text{def}}{=} \begin{pmatrix} P_1^T & 0^T & -u_1 P_1^T \\ 0^T & P_1^T & -v_1 P_1^T \\ \vdots & \vdots & \vdots \\ P_n^T & 0^T & -u_n P_n^T \\ 0^T & P_n^T & -v_n P_n^T \end{pmatrix} \text{ and } m \overset{\text{def}}{=} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = 0
\]

When \( n \geq 6 \), homogeneous linear least-square can be used to compute the value of the unit vector \( m \) (hence the matrix \( M \)) that minimizes \( |Pm|^2 \) as the solution of an eigenvalue problem. The solution is the eigenvector with least eigenvalue of \( P^T P \).
Estimation of the intrinsic and extrinsic parameters

Once $M$ is known, you still got to recover the intrinsic and extrinsic parameters!

This is a decomposition problem, NOT an estimation problem.

$$\rho \mathcal{M} = \begin{pmatrix}
\alpha r_1^T - \alpha \cot \theta r_2^T + u_0 r_3^T & \alpha t_x - \alpha \cot \theta t_y + u_0 t_z \\
\beta \frac{r_2^T}{\sin \theta} + v_0 r_3^T & \beta \frac{t_y + v_0 t_z}{\sin \theta} \\
r_3^T & t_z
\end{pmatrix}$$

- Intrinsic parameters
- Extrinsic parameters
Estimation of the intrinsic and extrinsic parameters

Write $M = (A, b)$, therefore

$$\rho (A \ b) = \mathcal{K}(\mathcal{R}, t) \iff \rho \left( \begin{pmatrix} a_x^T \\ a_y^T \\ a_z^T \end{pmatrix} \right) = \begin{pmatrix} \alpha r_1^T - \alpha \cot \theta r_2^T + u_0 r_3^T \\ \frac{\beta}{\sin \theta} r_2^T + u_0 r_3^T \\ r_3^T \end{pmatrix}$$

Using the fact that the rows of a rotation matrix have unit length and are perpendicular to each other yields

$$\begin{cases} \rho = \varepsilon / |a_3|, \\ r_3 = \rho a_3, \\ u_0 = \rho^2 (a_1 \cdot a_3), \\ v_0 = \rho^2 (a_2 \cdot a_3), \\ \end{cases}$$

where $\varepsilon = \pm 1$.

Since $\theta$ is always in the neighborhood of $\pi/2$ with a positive sine, we have

$$\begin{cases} \rho^2 (a_1 \times a_3) = -\alpha r_2 - \alpha \cot \theta r_1, \\ \rho^2 (a_2 \times a_3) = \frac{\beta}{\sin \theta} r_1, \\ \end{cases}$$

and

$$\begin{cases} \rho^2 |a_1 \times a_3| = \frac{\alpha}{\sin \theta}, \\ \rho^2 |a_2 \times a_3| = \frac{\beta}{\sin \theta}. \\ \end{cases}$$

Thus,

$$\begin{cases} \cos \theta = -\frac{(a_1 \times a_3) \cdot (a_2 \times a_3)}{|a_1 \times a_3||a_2 \times a_3|}, \\ \alpha = \rho^2 |a_1 \times a_3| \sin \theta, \\ \beta = \rho^2 |a_2 \times a_3| \sin \theta, \\ \end{cases}$$

and

$$\begin{cases} r_1 = \frac{\rho^2 \sin \theta}{\beta} (a_2 \times a_3) = \frac{1}{|a_2 \times a_3|} (a_2 \times a_3), \\ r_2 = r_3 \times r_1. \\ \end{cases}$$

Note that there are two possible choices for the matrix $\mathcal{R}$ depending on the value of $\varepsilon$. 

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Computer Vision

Geometric Camera Calibration
Estimation of the intrinsic and extrinsic parameters

The translation parameters can now be recovered by writing $Kt = \rho b$, and hence $t = \rho K^{-1}b$. In practical situations, the sign of $t_z$ is often known in advance (this corresponds to knowing whether the origin of the world coordinate system is in front or behind the camera), which allows the choice of a unique solution for the calibration parameters.
Taking radial distortion into account

Assuming that the image centre is known \((u_0 = v_0 = 0)\), model the projection process as:

\[
p = \frac{1}{z} \begin{pmatrix} 1/\lambda & 0 & 0 \\ 0 & 1/\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} M P
\]

where \(\lambda\) is a polynomial function of the squared distance \(d^2\) between the image centre and the image point \(p\).

It is sufficient to use low-degree polynomial:

\[
\lambda = 1 + \sum_{p=1}^{q} \kappa_p d^{2p}, \quad \text{with } q \leq 3 \text{ and the distortion coefficients } \kappa_p \ (p = 1, \ldots, q)
\]

\[
d^2 = \hat{u}^2 + \hat{v}^2
\]

This yields highly nonlinear constraints on the \(q + 11\) camera parameters.
The accuracy of the calibration depends on the accuracy of the measurements of the calibration pattern.
Line intersection and point sorting

- Extract and link edges using Canny;
- Fit lines to edges using orthogonal regression;
- Intersect lines.
References

- “Geometric Frame Work for Vision – Lecture Notes”. A. Zisserman, University of Oxford