Computer Vision

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Geometric Camera Calibration

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Series outline

- Cameras and lenses
- Geometric camera models
- Geometric camera calibration
- Stereopsis
Lecture outline

- The calibration problem
- Least-square technique
- Calibration from points
- Radial distortion
- A note on calibration patterns
Camera calibration

Camera calibration is determining the intrinsic and extrinsic parameters of the camera.

The are three coordinate systems involved: image, camera, and world.

Key idea: to write the projection equations linking the known coordinates of a set of 3-D points and their projections, and solve for the camera parameters.
Projection matrix

\[ M = \begin{pmatrix}
\alpha r_1^T - \alpha \cot \theta r_2^T + u_0 r_3^T & \alpha t_x - \alpha \cot \theta t_y + u_0 t_z \\
\frac{\beta}{\sin \theta} r_2^T + v_0 r_3^T & \beta t_y + v_0 t_z \\
r_3^T & t_z
\end{pmatrix} \]

Replacing \( M \) by \( \lambda M \) in

\[
\begin{align*}
u &= \frac{m_1 \cdot P}{m_3 \cdot P} \\
v &= \frac{m_2 \cdot P}{m_3 \cdot P}
\end{align*}
\]

does not change \( u \) and \( v \).

\( M \) is only defined up to scale in this setting.
The calibration problem

Given \( n \) points \( P_1, \ldots, P_n \) with known positions and their images \( p_1, \ldots, p_n \)

Find \( i \) and \( e \) such that

\[
\sum_{i=1}^{n} \left[ \left( u_i - \frac{m_1(i, e) \cdot P_i}{m_3(i, e) \cdot P_i} \right)^2 + \left( v_i - \frac{m_2(i, e) \cdot P_i}{m_3(i, e) \cdot P_i} \right)^2 \right] \text{ is minimized}
\]
Linear systems

Square system:
- Unique solution
- Gaussian elimination

Rectangular system:
- underconstrained: Infinity of solutions
- Overconstrained: no solution

Minimize $|Ax-b|^2$
How do you solve overconstrained linear equations?

- Define $E = |e|^2 = e \cdot e$ with

$$e = Ax - b = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - b$$

$$= x_1c_1 + x_2c_2 + \cdots x_nc_n - b$$

- At a minimum,

$$\frac{\partial E}{\partial x_i} = \frac{\partial e}{\partial x_i} \cdot e + e \cdot \frac{\partial e}{\partial x_i} = 2 \frac{\partial e}{\partial x_i} \cdot e$$

$$= 2 \frac{\partial}{\partial x_i} (x_1c_1 + \cdots + x_nc_n - b) \cdot e = 2c_i \cdot e$$

$$= 2c_i^T(Ax - b) = 0$$

- or

$$0 = \begin{bmatrix} c_1^T \\ \vdots \\ c_n^T \end{bmatrix} (Ax - b) = A^T(Ax - b) \Rightarrow A^T A \mathbf{x} = A^T \mathbf{b},$$

where $\mathbf{x} = A^T \mathbf{b}$ and $A^T = (A^T A)^{-1} A^T$ is the pseudoinverse of $A$!
Homogeneous linear equations

Square system:
- Unique solution = 0
- Unless $\det(A) = 0$

Rectangular system:
- $0$ is always a solution

Minimize $|Ax|^2$ under the constraint $|x|^2 = 1$
How do you solve overconstrained homogeneous linear equations?

\[ E = |Ux|^2 = x^T(U^TU)x \]

- Orthonormal basis of eigenvectors: \( e_1, \ldots, e_q \).
- Associated eigenvalues: \( 0 \leq \lambda_1 \leq \ldots \leq \lambda_q \).
- Any vector can be written as

\[ x = \mu_1 e_1 + \ldots + \mu_q e_q \]

for some \( \mu_i \) (\( i = 1, \ldots, q \)) such that \( \mu_1^2 + \ldots + \mu_q^2 = 1 \).

\[
E(x) - E(e_1) = x^T(U^TU)x - e_1^T(U^TU)e_1 \\
= \lambda_1^2 \mu_1^2 + \ldots + \lambda_q^2 \mu_q^2 - \lambda_1^2 \\
\geq \lambda_1^2(\mu_1^2 + \ldots + \mu_q^2 - 1) = 0
\]

The solution is the eigenvector \( e_1 \) with least eigenvalue of \( U^TU \).
Example: Line fitting

Problem: minimize

\[ E(a, b, d) = \sum_{i=1}^{n} (a x_i + b y_i - d)^2 \]

with respect to \((a, b, d)\).

- Minimize \(E\) with respect to \(d\):
  \[
  \frac{\partial E}{\partial d} = 0 \implies d = \frac{1}{n} \sum_{i=1}^{n} a x_i + b y_i = \bar{x} a + b \bar{y}
  \]

- Minimize \(E\) with respect to \(a, b\):

  where \(U = \begin{pmatrix} x_1 - \bar{x} & y_1 - \bar{y} \\ \vdots & \vdots \\ x_n - \bar{x} & y_n - \bar{y} \end{pmatrix}\) and

  \[
  U^T U = \begin{pmatrix} \sum_{i=1}^{n} x_i^2 - n \bar{x}^2 & \sum_{i=1}^{n} x_i y_i - n \bar{x} \bar{y} \\ \sum_{i=1}^{n} x_i y_i - n \bar{x} \bar{y} & \sum_{i=1}^{n} y_i^2 - n \bar{y}^2 \end{pmatrix}
  \]
Estimation of the projection matrix

Given \( n \) points \( P_1, \ldots, P_n \) with known positions and their images \( p_1, \ldots, p_n \),

\[
\begin{pmatrix} u_i \\ v_i \end{pmatrix} = \begin{pmatrix} m_1 \cdot P_i \\ m_3 \cdot P_i \\ m_2 \cdot P_i \\ m_3 \cdot P_i \end{pmatrix} \iff \left( m_1 - u_i m_3 \right) P_i = 0
\]

The constraints associated with the \( n \) points yield a system of \( 2n \) homogeneous linear equations in the 12 coefficients of the matrix \( M \),

\[
\mathcal{P}m = 0 \quad \text{with} \quad \mathcal{P} \overset{\text{def}}{=} \begin{pmatrix} P_1^T & 0^T & -u_1P_1^T \\ 0^T & P_1^T & -v_1P_1^T \\ \vdots & \vdots & \vdots \\ P_n^T & 0^T & -u_nP_n^T \\ 0^T & P_n^T & -v_nP_n^T \end{pmatrix} \quad \text{and} \quad m \overset{\text{def}}{=} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = 0
\]

When \( n \geq 6 \), homogeneous linear least-square can be used to compute the value of the unit vector \( m \) (hence the matrix \( M \)) that minimizes \(|Pm|^2\) as the solution of an eigenvalue problem. The solution is the eigenvector with least eigenvalue of \( P^TP \).
Estimation of the intrinsic and extrinsic parameters

Once $\mathcal{M}$ is known, you still got to recover the intrinsic and extrinsic parameters!

This is a decomposition problem, NOT an estimation problem.

\[ \rho \mathcal{M} = \begin{pmatrix} \alpha r_1^T - \alpha \cot \theta r_2^T + u_0 r_3^T & \alpha t_x - \alpha \cot \theta t_y + u_0 t_z \\ \frac{\beta}{\sin \theta} r_2^T + v_0 r_3^T & \frac{\beta}{\sin \theta} t_y + v_0 t_z \\ r_3^T & t_z \end{pmatrix} \]

- Intrinsic parameters
- Extrinsic parameters
Estimation of the intrinsic and extrinsic parameters

Write \( M = (A, b) \), therefore

\[
\rho(A, b) = \mathcal{K}(R, t) \iff \rho \begin{pmatrix}
a_1^T \\
a_2^T \\
a_3^T
\end{pmatrix} = \begin{pmatrix}
\alpha r_1 - \alpha \cot \theta r_1^T + u_0 r_1^T \\
\beta \\
r_3^T
\end{pmatrix}
\]

Using the fact that the rows of a rotation matrix have unit length and are perpendicular to each other yields

\[
\begin{aligned}
\rho &= \varepsilon / |a_3|, \\
r_3 &= \rho a_3, \\
u_0 &= \rho^2 (a_1 \cdot a_3), \\
v_0 &= \rho^2 (a_2 \cdot a_3), \\
\end{aligned}
\]

where \( \varepsilon = \mp 1 \).

Since \( \theta \) is always in the neighborhood of \( \pi / 2 \) with a positive sine, we have

\[
\begin{aligned}
\rho^2 (a_1 \times a_3) &= -\alpha r_2 - \alpha \cot \theta r_1, \\
\rho^2 (a_2 \times a_3) &= \frac{\beta}{\sin \theta} r_1, \\
\rho^2 |a_1 \times a_3| &= \frac{\alpha}{\sin \theta}, \\
\rho^2 |a_2 \times a_3| &= \frac{\beta}{\sin \theta}. \\
\end{aligned}
\]

Thus,

\[
\begin{aligned}
\cos \theta &= \frac{(a_1 \times a_3) \cdot (a_2 \times a_3)}{|a_1 \times a_3| |a_2 \times a_3|}, \\
\alpha &= \rho^2 |a_1 \times a_3| \sin \theta, \\
\beta &= \rho^2 |a_2 \times a_3| \sin \theta, \\
r_1 &= \frac{\rho^2 \sin \theta}{\beta} (a_2 \times a_3) = \frac{1}{|a_2 \times a_3|} (a_2 \times a_3), \\
r_2 &= r_3 \times r_1. \\
\end{aligned}
\]

Note that there are two possible choices for the matrix \( R \) depending on the value of \( \varepsilon \).
Estimation of the intrinsic and extrinsic parameters

The translation parameters can now be recovered by writing $Kt = \rho b$, and hence $t = \rho K^{-1}b$. In practical situations, the sign of $t_z$ is often known in advance (this corresponds to knowing whether the origin of the world coordinate system is in front or behind the camera), which allows the choice of a unique solution for the calibration parameters.
Taking radial distortion into account

Assuming that the image centre is known \((u_0 = v_0 = 0)\), model the projection process as:

\[
p = \frac{1}{z} \begin{pmatrix} 1/\lambda & 0 & 0 \\ 0 & 1/\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \, M \, P
\]

where \(\lambda\) is a polynomial function of the squared distance \(d^2\) between the image centre and the image point \(p\).

It is sufficient to use low-degree polynomial:

\[
\lambda = 1 + \sum_{p=1}^{q} \kappa_p d^{2p}, \quad \text{with} \; q \leq 3 \; \text{and the distortion coefficients} \; \kappa_p \; (p = 1, \ldots, q)
\]

\[
d^2 = \hat{u}^2 + \hat{v}^2
\]

This yields highly nonlinear constraints on the \(q + 11\) camera parameters.
The accuracy of the calibration depends on the accuracy of the measurements of the calibration pattern.
Line intersection and point sorting

- Extract and link edges using Canny;
- Fit lines to edges using orthogonal regression;
- Intersect lines.
References

- “Geometric Frame Work for Vision – Lecture Notes”. A. Zisserman, University of Oxford