Project 2: Diffeomorphic Image Registration I

In this project you will implement geodesic shooting for diffeomorphic image registration in 2D. Along with a written report, you should turn in all source code that you write. In this first part, you will implement only the methods needed to shoot a geodesic forward in time. In the second part (coming later), you will implement a gradient descent methods for doing image matching.

Methods: You will need to implement functions that perform the following operations:

1. **Lie Algebra Operations.** You will need to have functions to compute the L and K operators, ad, and $\mathrm{ad}^\dagger$.
2. **Geodesic Shooting.** Given an initial velocity (vector field), v_0, solve the geodesic equation in the Lie algebra. In the end, you should have a time-varying velocity field v_t, for $t \in [0, 1]$.
3. **Integrate a Velocity Flow.** Given a time-varying velocity field, v_t, integrate the inverse diffeomorphism using the equation:
$$\frac{d\phi_t^{-1}}{dt} = -D\phi_t^{-1} v_t.$$
4. **Transform an Image.** Given an image I and a diffeomorphism ϕ_1^{-1}, transform the image by $I \circ \phi_1^{-1}$. Use bilinear interpolation to resample the transformed image.

Experiments: For these experiments, you will use images and momenta fields provided here:

`http://www.sci.utah.edu/~fletcher/CS7640/hw2/`

1. **Random Initial Velocities.** Generate a momentum vector field that is independent Gaussian values: $m \sim N(0, \lambda)$. Use your kernel operator to convert this into an initial velocity vector field: $v = Km$. Shoot a geodesic using this initial velocity.
2. **Provided Initial Velocities.** Download the provided initial momenta. Again, convert them to initial velocities using your kernel operator, and shoot the resulting geodesics.

Report: You should submit a report (either as html or pdf) describing your work. Be sure to include the following:

- Repeat both the shooting experiments with varying values for the metric parameters (the α and k in $L = (1 - \alpha\Delta)^k$). What is the effect of different parameters? What range seems to give valid deformations?
- Shoot the geodesic for each experiment. Deform the given images by the resulting diffeomorphism, ϕ_t^{-1}, for several times in the interval $0 \le t \le 1$. (You should generate a smooth deformation of the initial image.)

- **Optional:** Plot a deforming grid over the image. This is a good way to check if your transformation is staying diffeomorphic (the grid lines should stay smooth and not get crossed up.) If you are using R or Matlab, grid lines can easily be displayed as the contours of your ϕ^{-1} coordinates.
- Make sure that your geodesic equation (roughly) preserves the magnitude of the velocity. That is, demonstrate that $\|v_t\|$ is constant in t up to numerical integration error. Be sure to use the Riemannian metric to compute the velocity norm.
- **Optional:** You can implement time integrals with a first-order Euler integration, or you might try a more advanced numerical integrator, such as Runge-Kutta.