THE UNIVERSITY OF UTAH

CS-5630

Scientific Visualization

Basics of Vector Field Topology

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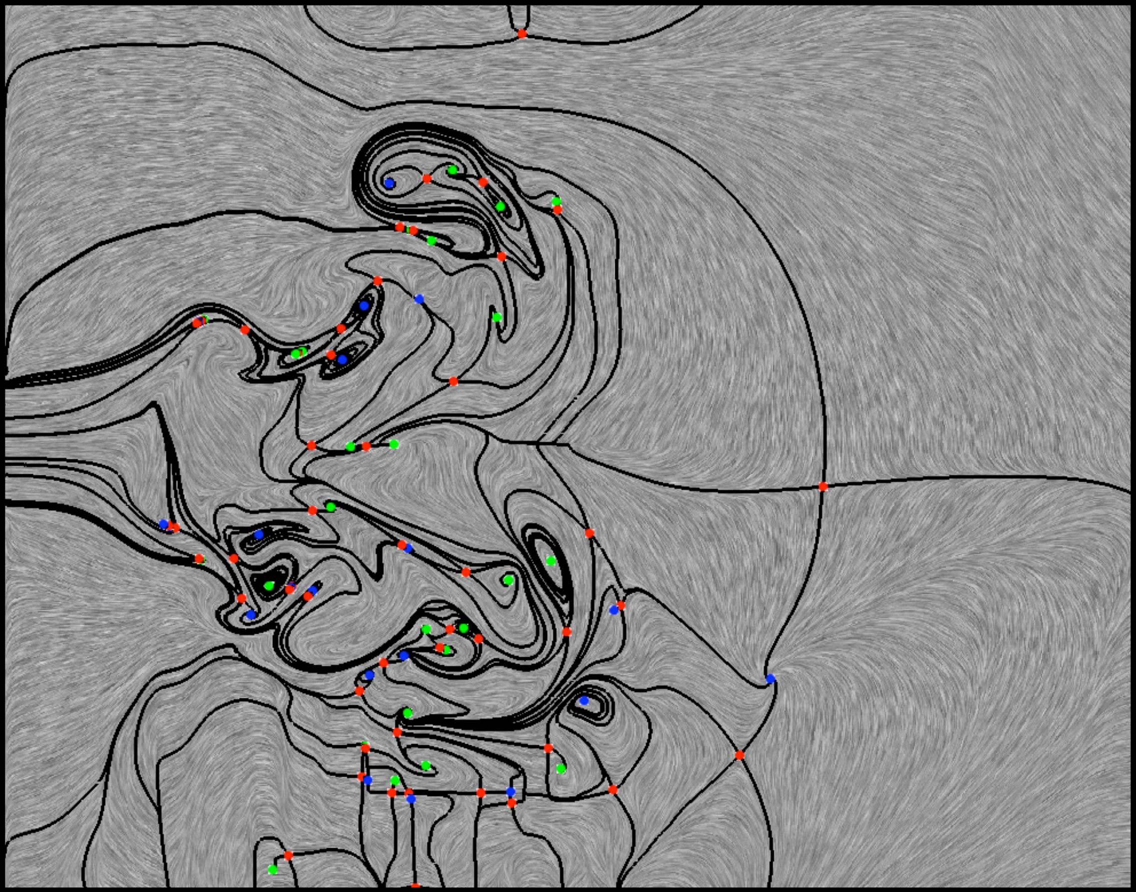
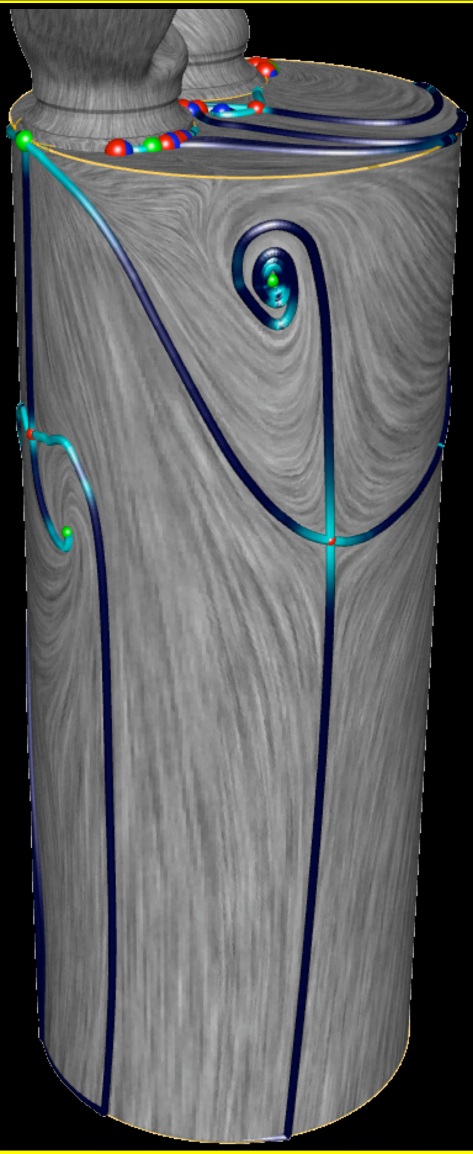
Examples





Motivation

- Abstract representation of flow field
- Characterization of global flow structures
- Basic idea (steady case):
  - *Interpret flow in terms of streamlines*
  - *Classify them w.r.t. their limit sets*
  - *Determine regions of homogenous behavior*
- Graph depiction
- Fast computation

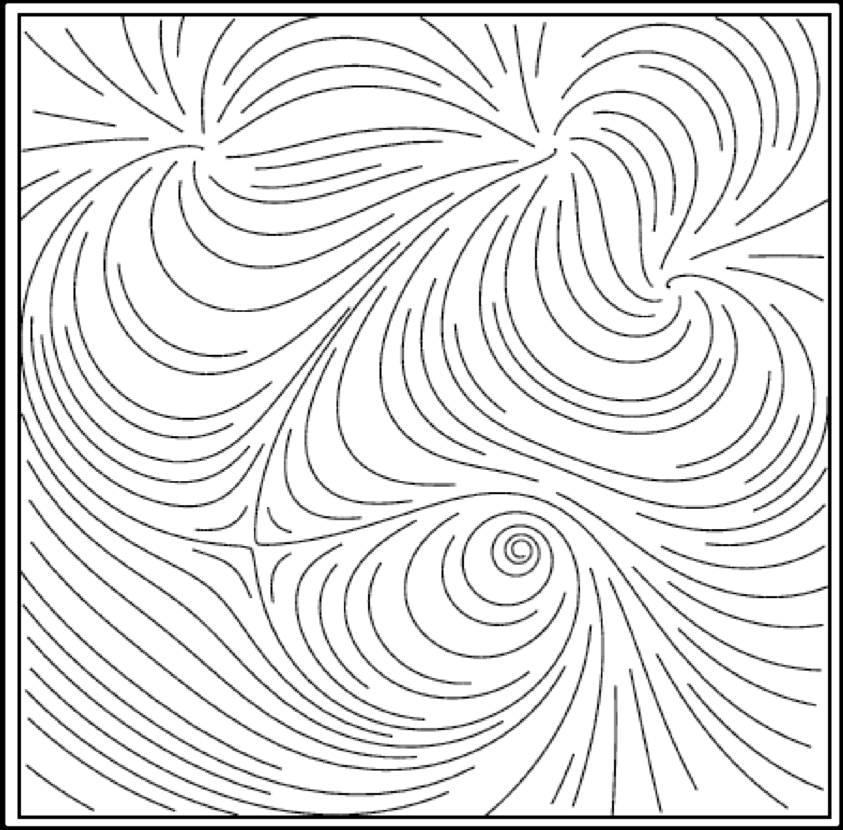
Limit Sets and Basins

- Limit sets of a point $\mathbf{x} \in \mathbb{R}^n$
  - $\omega(\mathbf{x})$: ***omega limit set of x*** = *point (or curve) reached after **forward** integration by streamline seeded at x*
  - $\alpha(\mathbf{x})$: ***alpha limit set of x*** = *point (or curve) reached after **backward** integration by streamline seeded at x*

- Sources (α) and sinks (ω) of the flow
- Basin: region of influence of a limit set

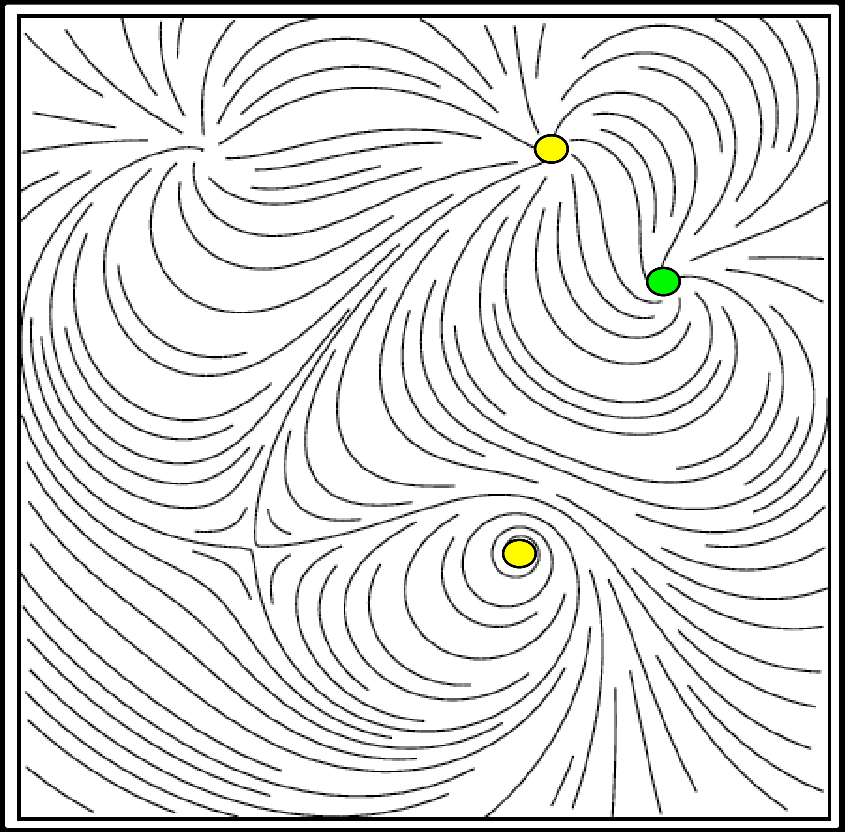
Limit Sets and Basins

- Phase portrait



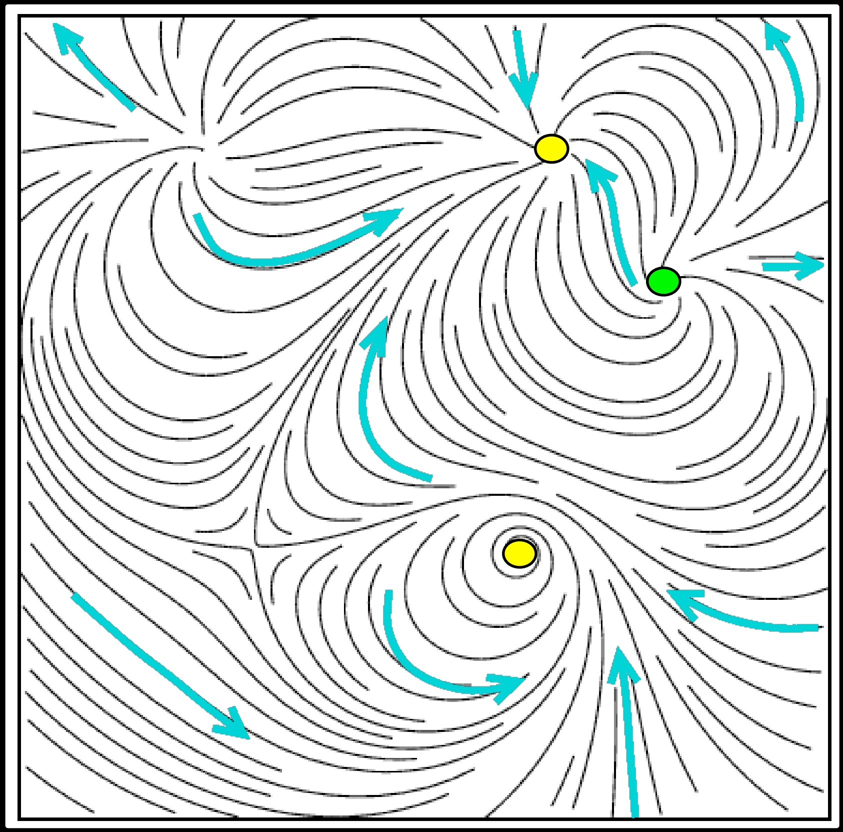
Limit Sets and Basins

- Limit sets



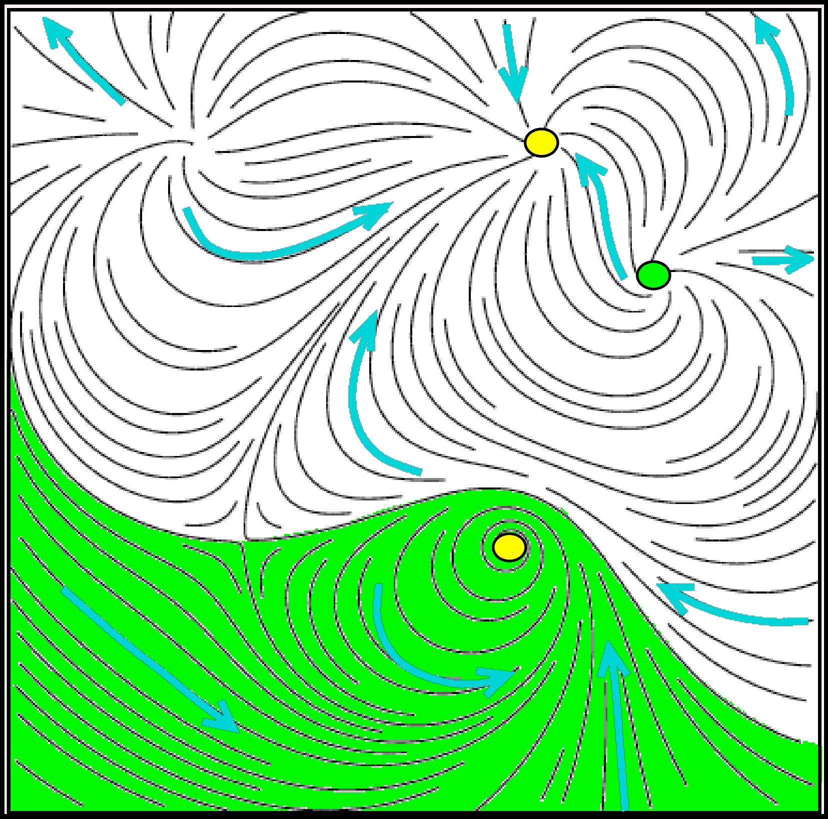
Limit Sets and Basins

- Flow direction



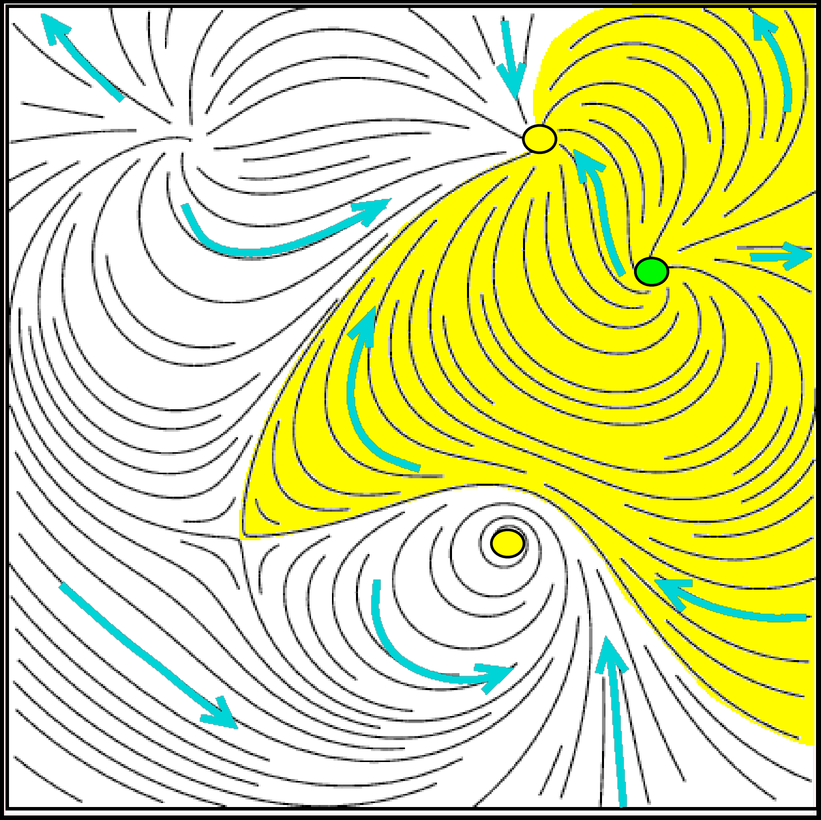
Limit Sets and Basins

- ω-basin of sink



Limit Sets and Basins

- α-basin of source

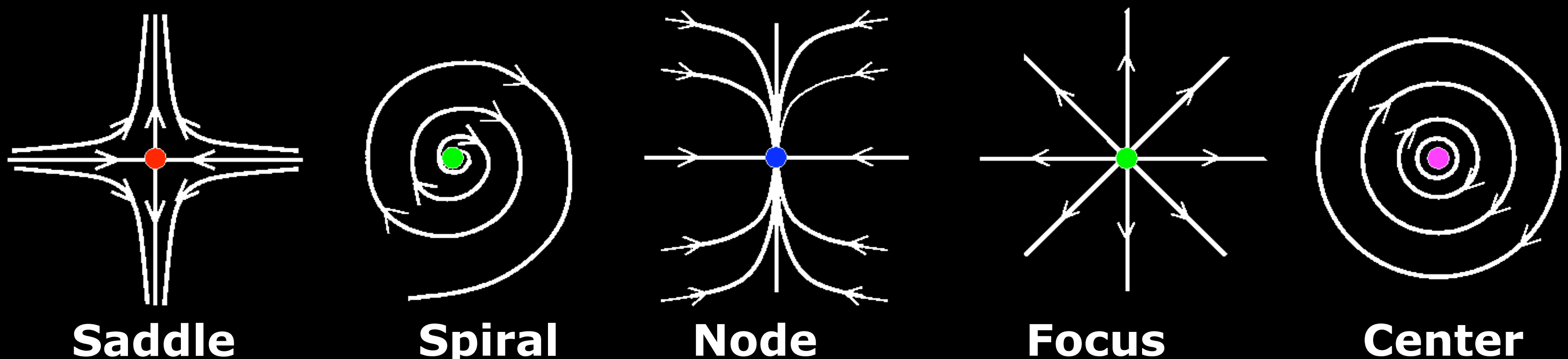


Critical Points

- Equilibrium
  - $\vec{v}(\mathbf{x}_0) = \vec{0}$
  - *Streamline reduced to a single point*
- Remarks
  - *Asymptotic flow convergence / divergence*
  - *Streamlines "intersect" at critical points*
- Type of critical point determines local flow pattern around it

Critical Points

- Jacobian has full rank
  - *No zero eigenvalue*
- Major cases

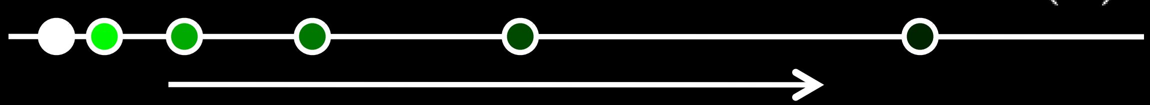


- Hyperbolic / non-hyperbolic

Critical Points

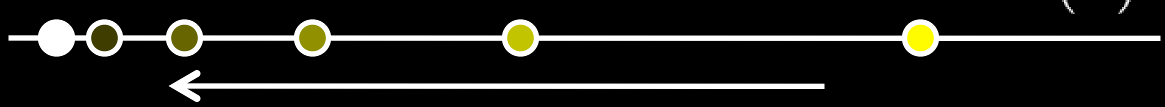
- Type determined by Jacobian's eigenvalues:
  - *Positive real part: repelling (source)*

$$\vec{v}(\mathbf{x}) = k\mathbf{x},\ \ k > 0$$



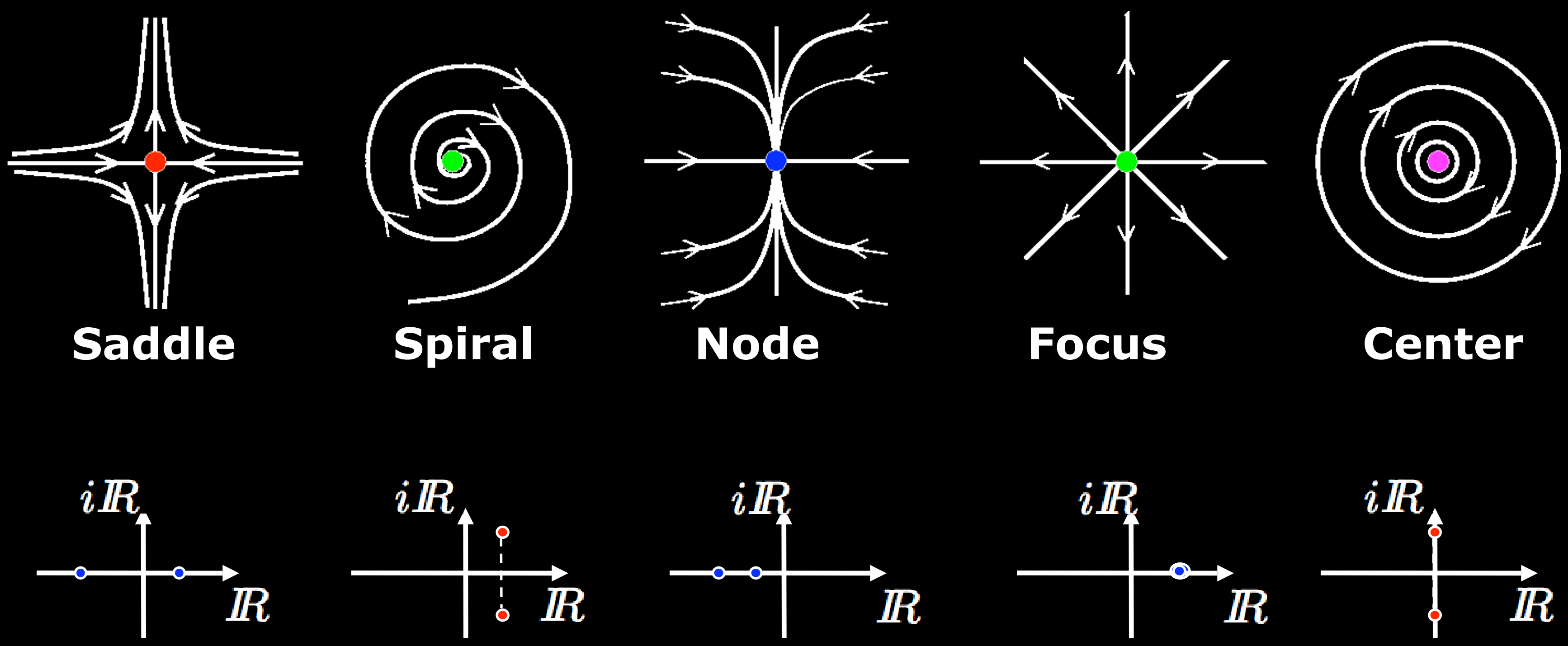
  - *Negative real part: attracting (sink)*

$$\vec{v}(\mathbf{x}) = k\mathbf{x},\ \ k < 0$$

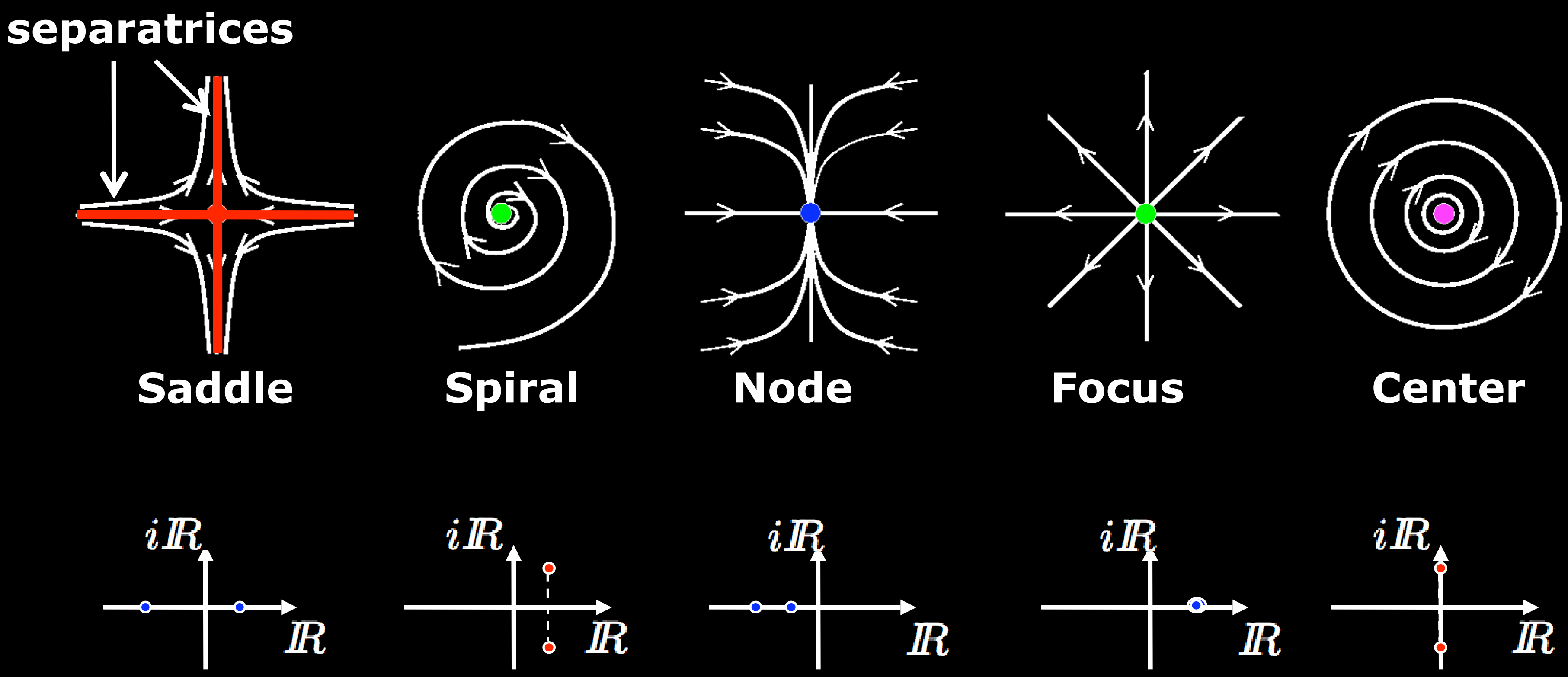


  - *Complex: rotation*

Critical Points

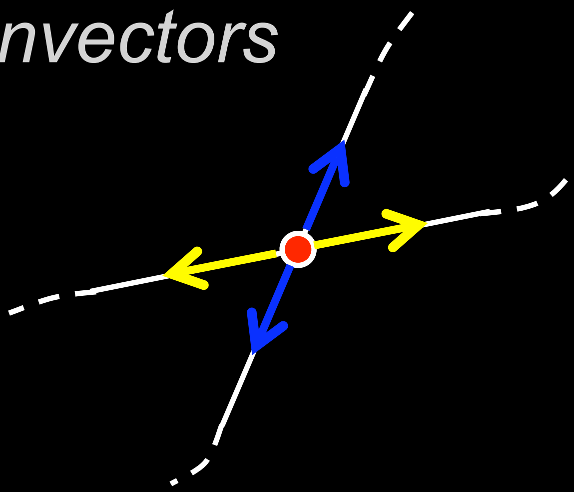


Critical Points



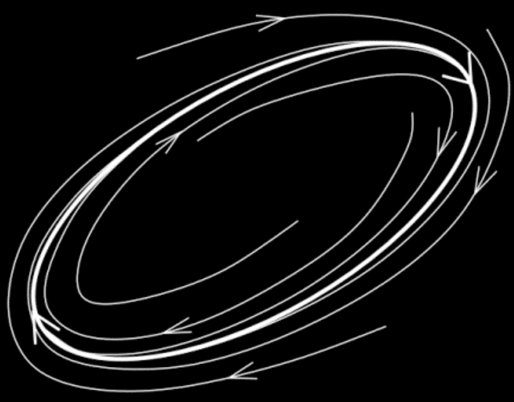
Critical Point Extraction

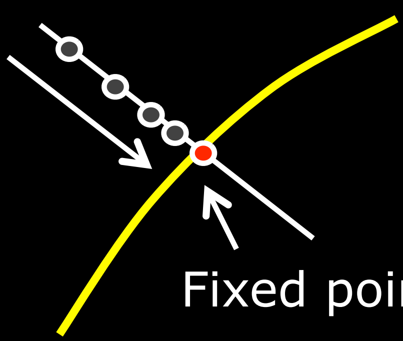
- Cell-wise analysis
  - *Solve linear / quadratic equation to determine position of critical point in cell*
  - *Compute Jacobian at that position*
  - *Compute eigenvalues*
  - *If type is saddle, compute eigenvectors*



Closed Orbits

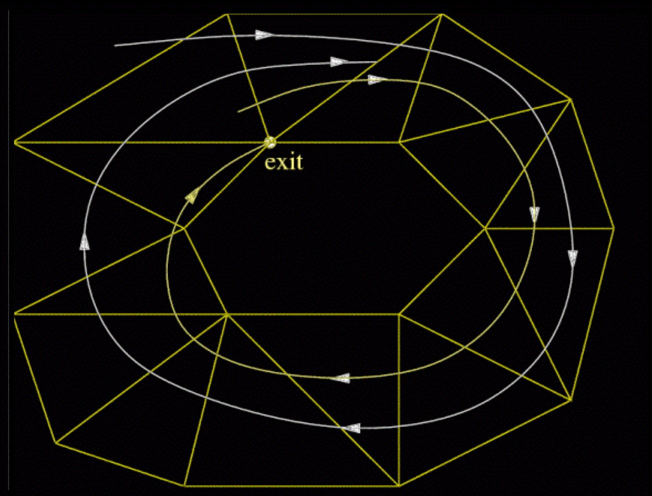
- Curve-type limit set
- Sink / source behavior
- Poincaré map:
  - *Defined over cross section*
  - *Map each position to next intersection with cross section along flow*
  - *Discrete map*
  - *Cycle intersects at fixed point*
  - *Hyperbolic / non-hyperbolic*

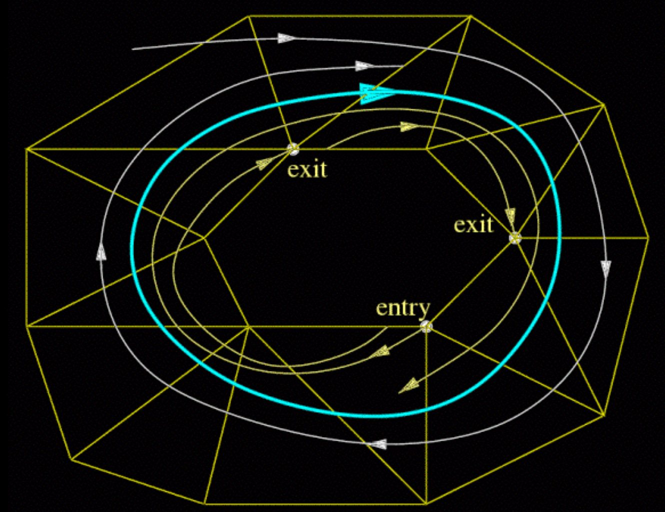




Closed Orbit Extraction

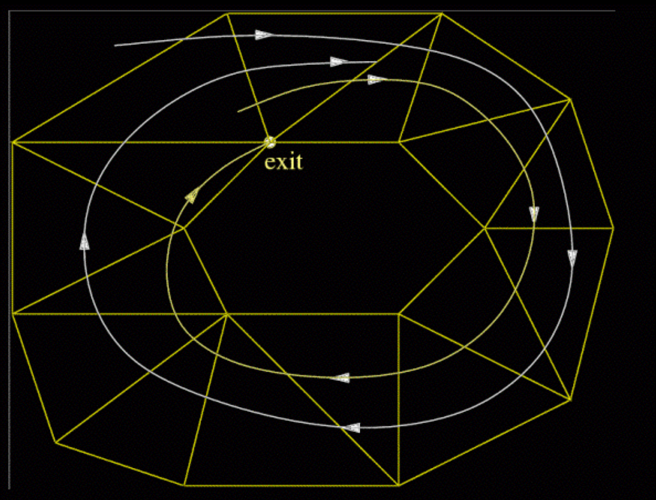
- Poincaré-Bendixson theorem:
  - *If a region contains a limit set and no critical point, it contains a closed orbit*

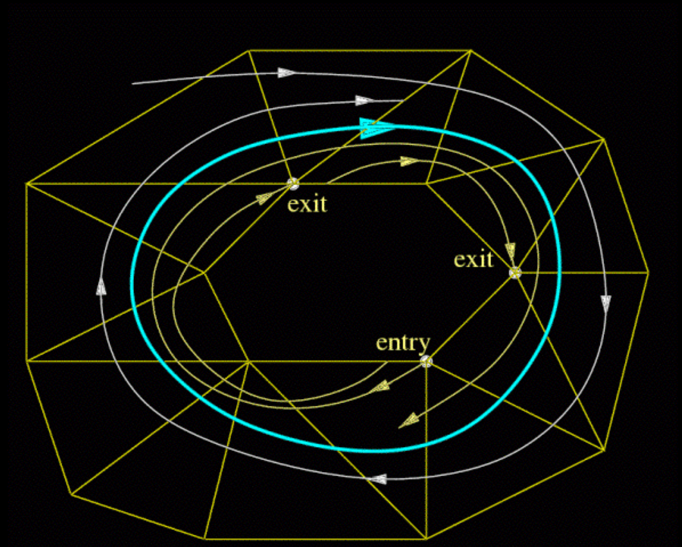




Closed Orbit Extraction

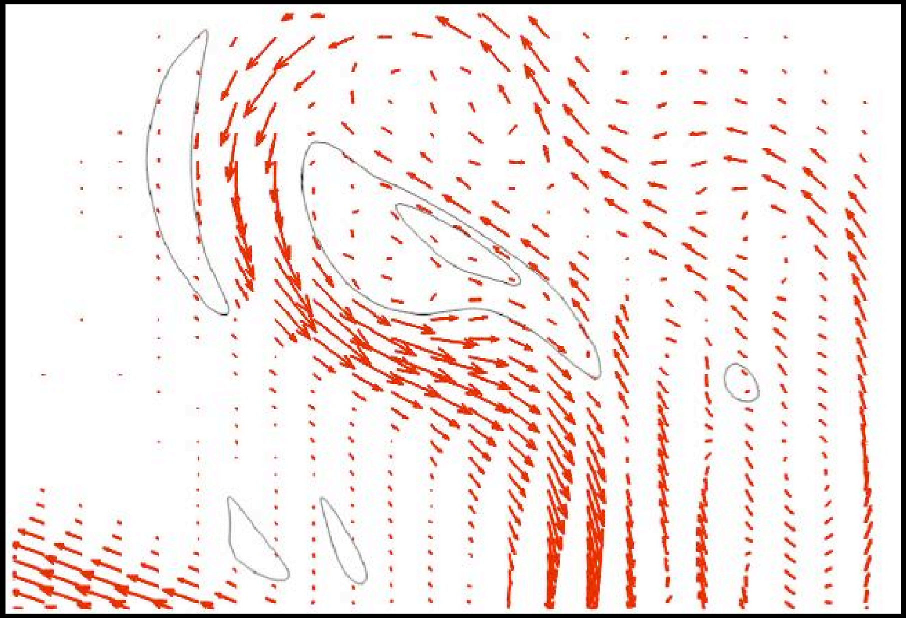
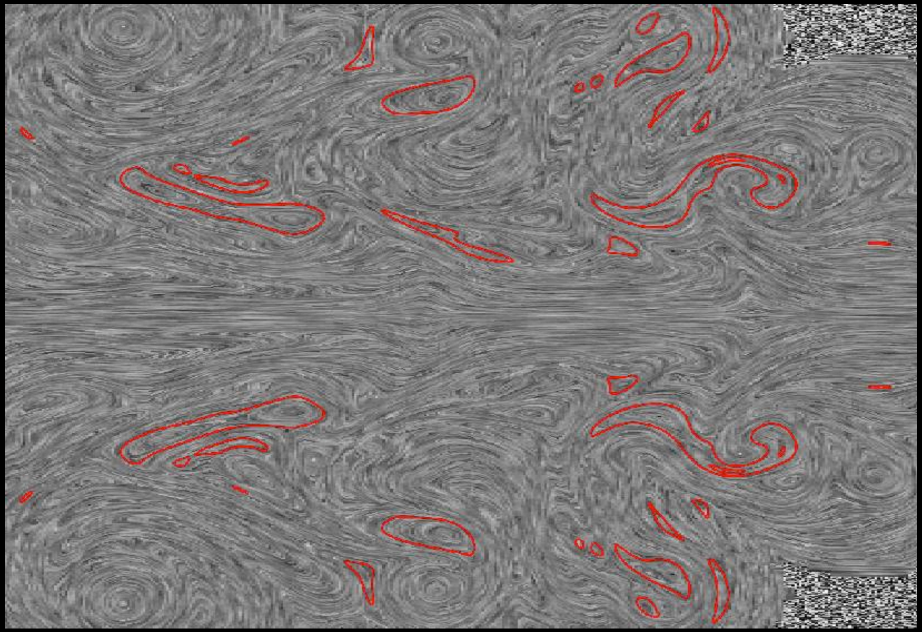
- Detect closed cell cycle
- Check for flow exit along boundary
- Find exact position with Poincaré map





Closed Orbit Extraction

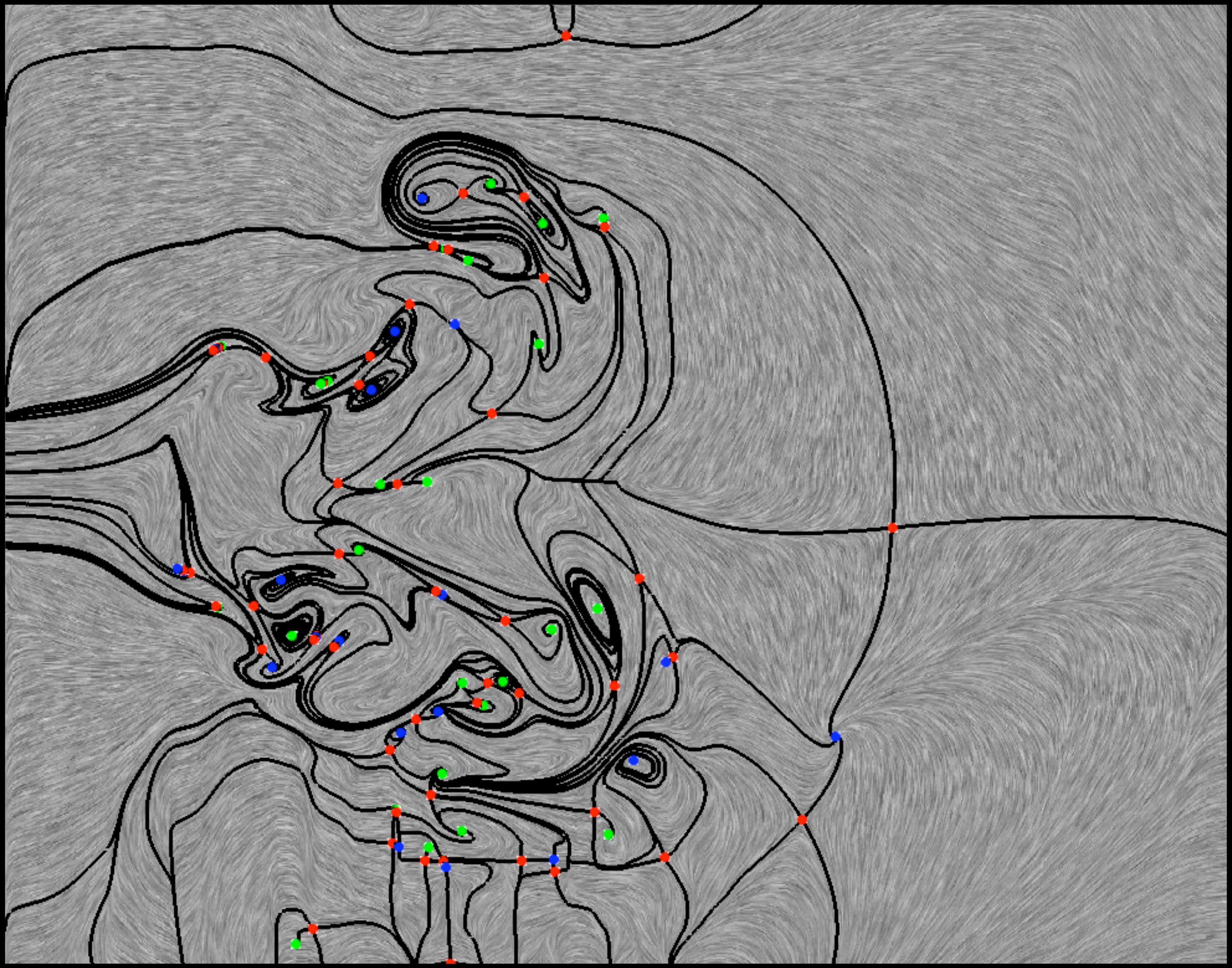
- Results

Topological Graph

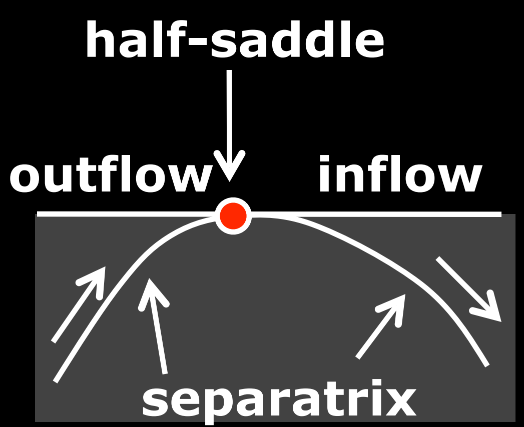
- Graph
  - *Nodes: critical points*
  - *Edges: separatrices and closed orbits*
- Remark
  - *All streamlines in a given region have same α- and ω-limit set*
- Problem
  - *Definition does not account for bounded domain*

Topological Graph

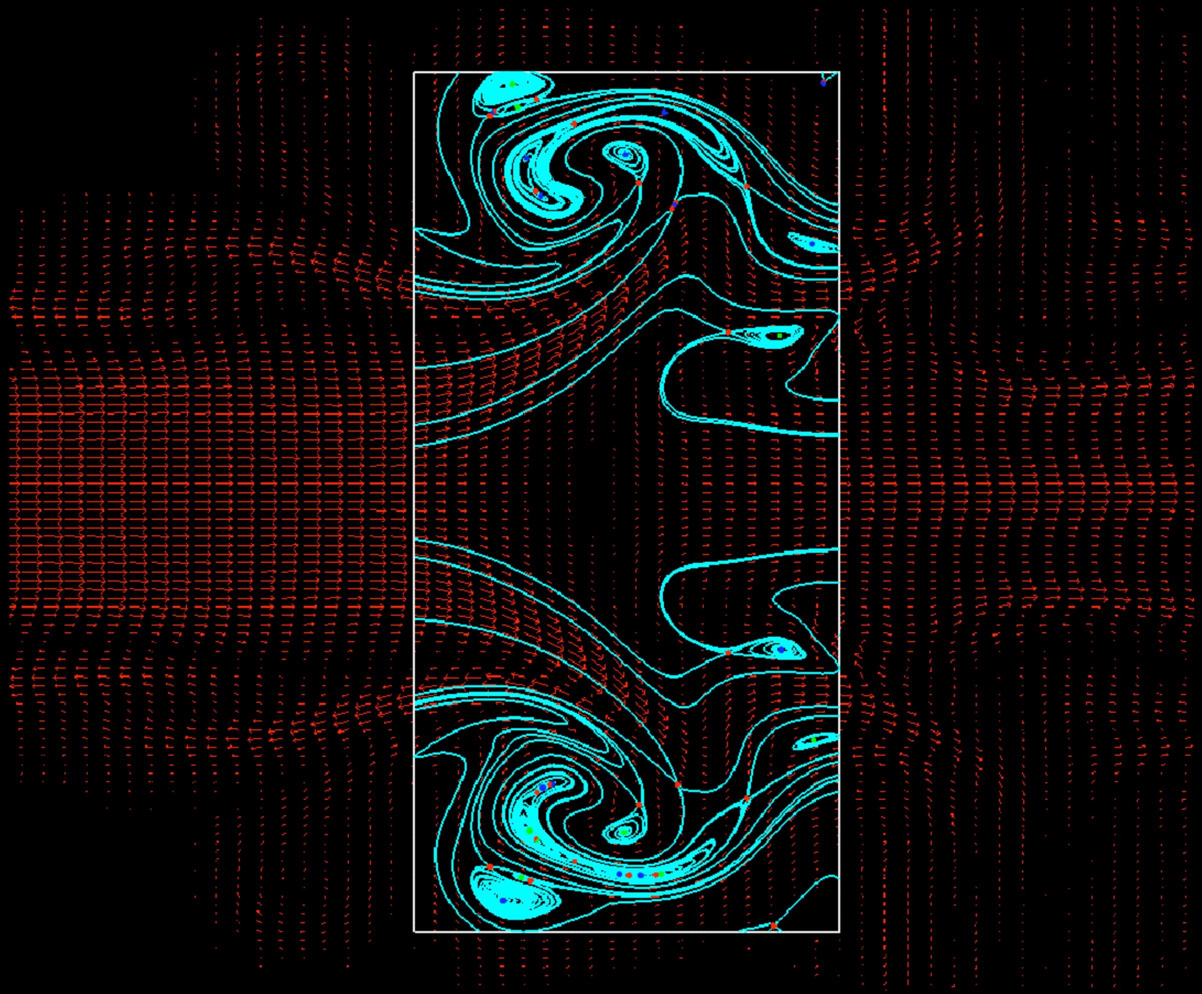


Local Topology

- Classification w.r.t. asymptotic convergence
- On bounded domain: streamlines leave domain in finite time
- Extend definition of topology
  - *Inflow boundaries ≡ sources*
  - *Outflow boundaries ≡ sinks*
  - *Bounded by half-saddles*
  - *Additional separatrices*

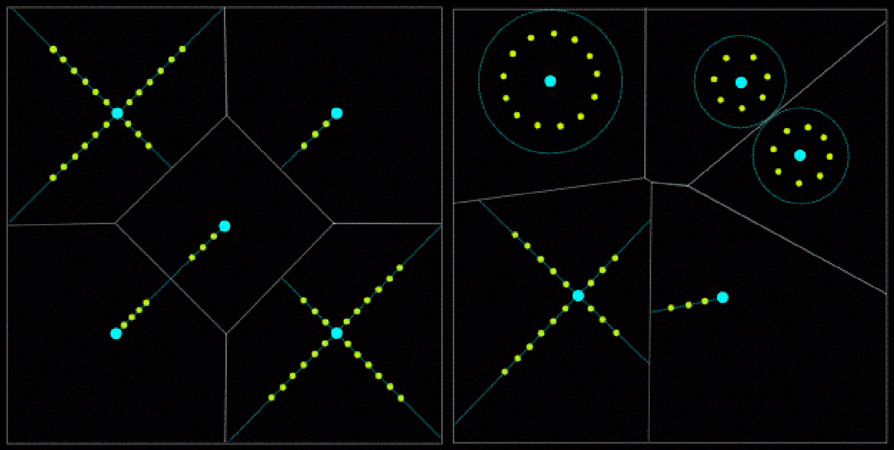


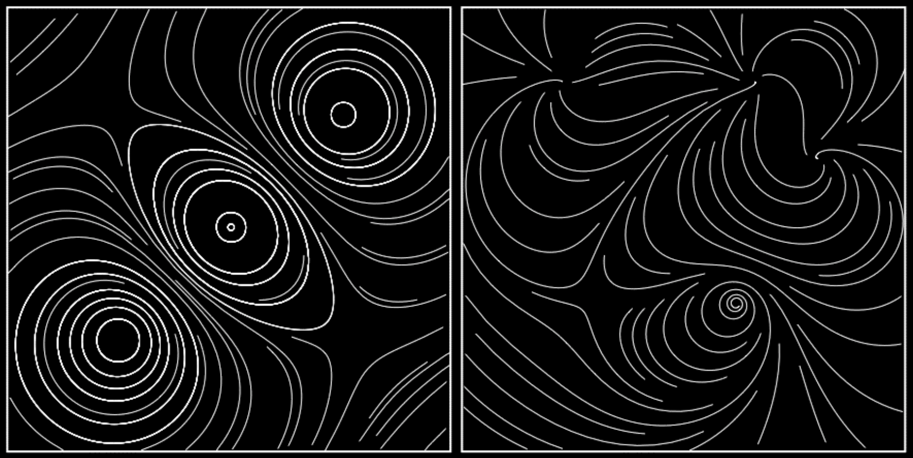
Local Topology



Applications

- Can be combined with
  - *Texture-based flow visualization*
  - *Color-coding of associated quantity*
  - *Topology-based streamline seeding*





Applications

- Can be applied to the gradient of a related scalar field (cf. Morse theory)
- Jacobian matrix is symmetric!
  - *Eigenvalues are real: rotation free*
  - *Linear critical points are saddle and nodes*
  - *Interpretation as height field*

What about transient flows?

- Parameter dependent topology:
  - *Critical points move, appear, vanish, transform*
  - *Graph connectivity changes*
- Structural stability (Peixoto): topology is stable w.r.t. small but arbitrary changes of parameter(s) if and only if

  1) Number of critical points and closed orbits is finite and all are hyperbolic

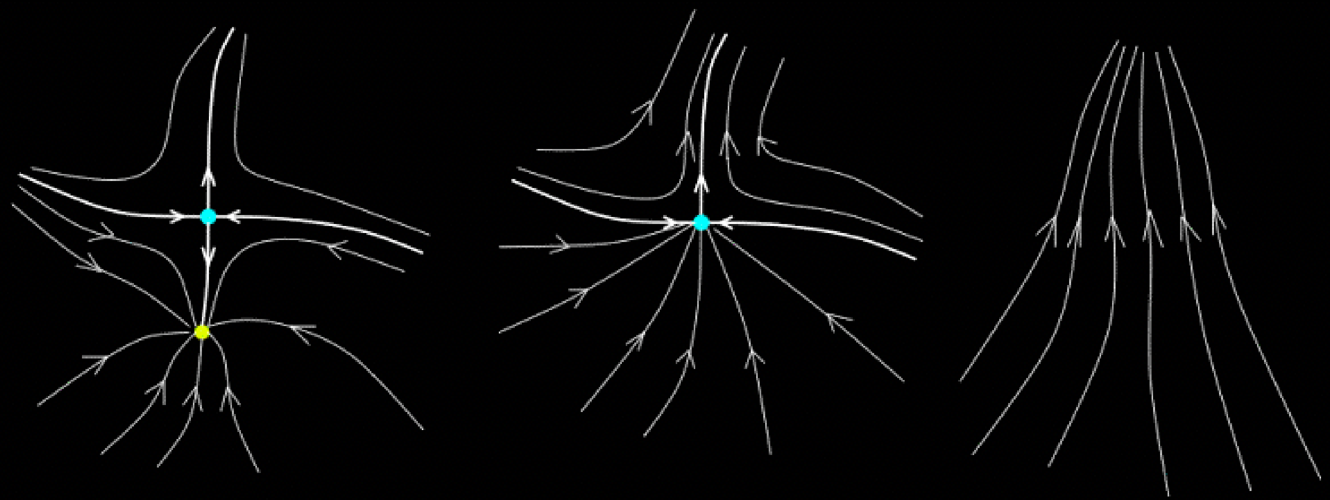
  2) No saddle-saddle connection

Bifurcations

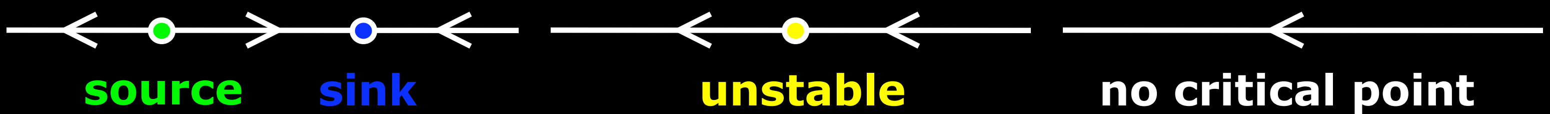
- Transition from one stable structure to another through unstable state
- Bifurcation value: parameter value inducing the transition
- Local vs. global bifurcations

Local Bifurcations

- Transformation affects local region
- Fold bifurcation: saddle + sink/source

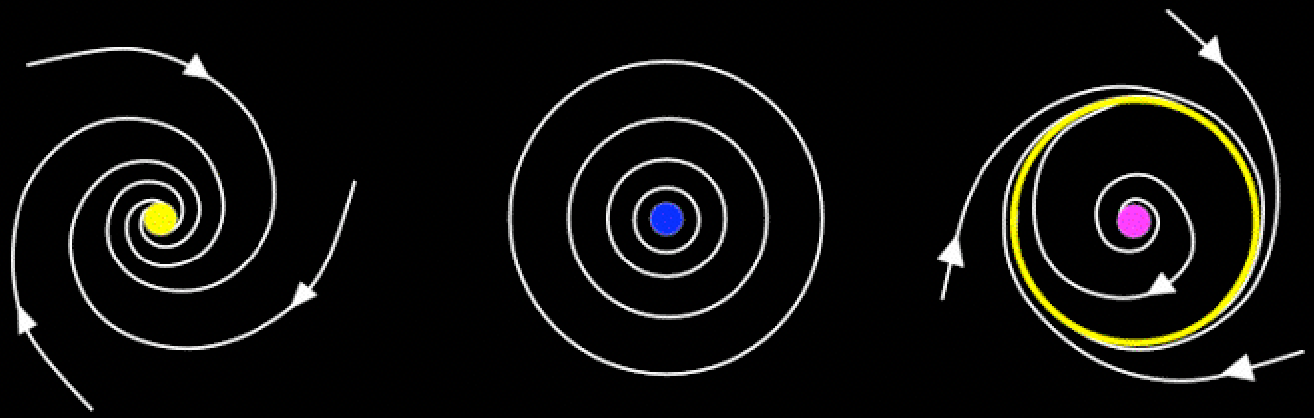


1D equivalent:



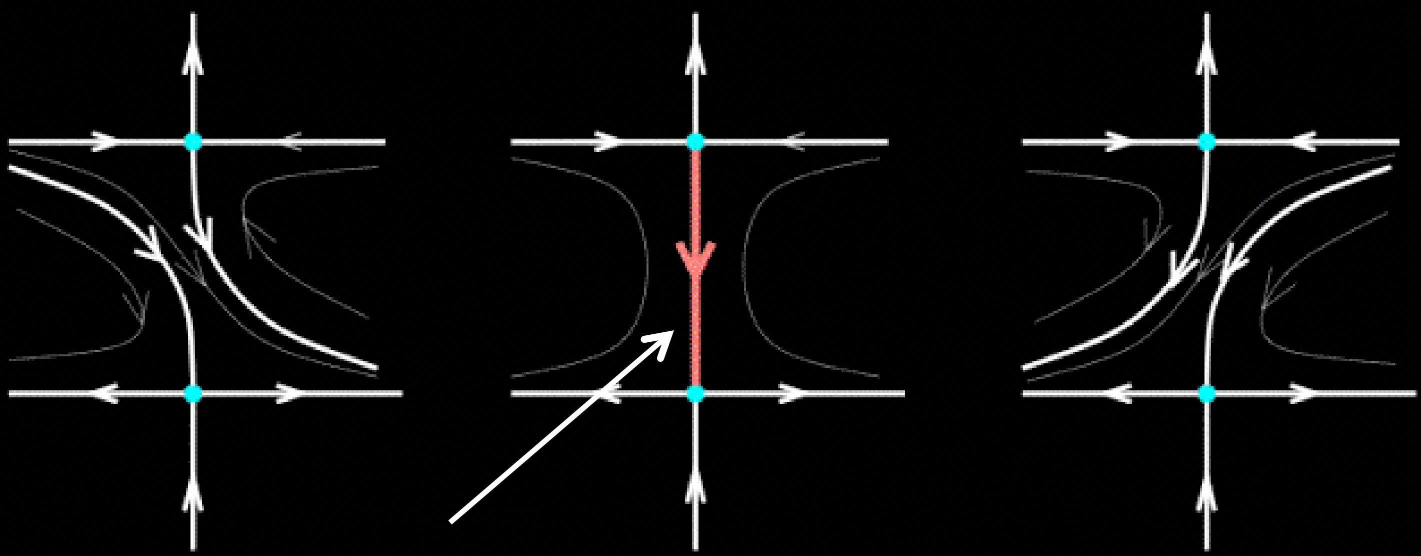
Local Bifurcations

- Transformation affects local region
- Hopf bifurcation: sink/source + closed orbit



Global Bifurcations

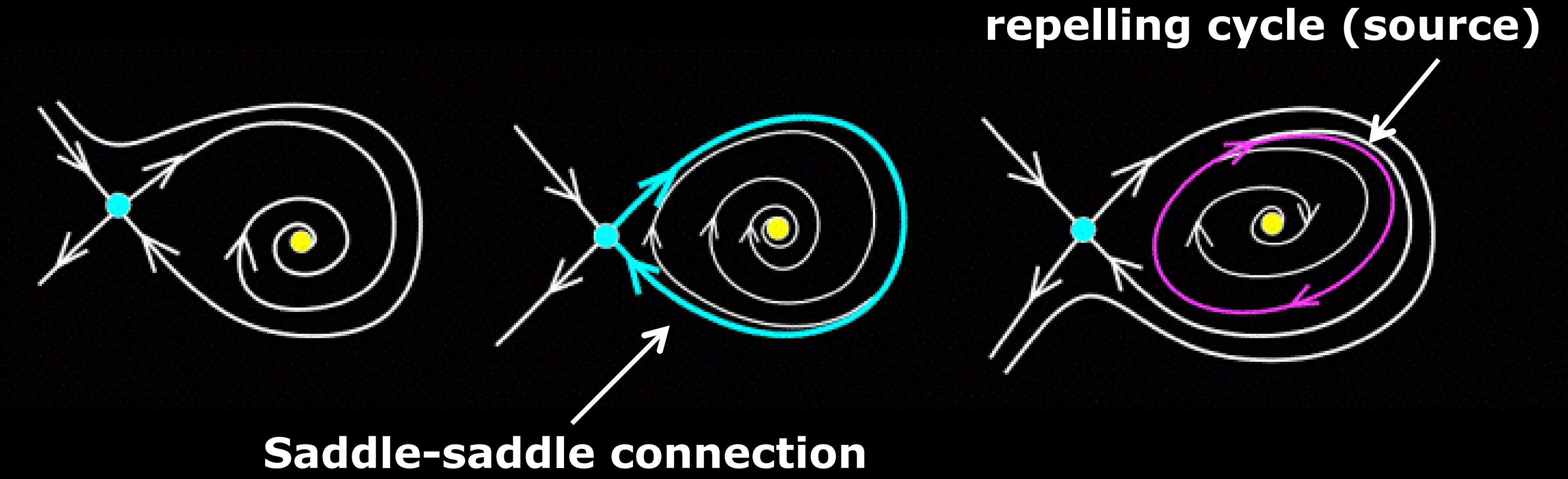
- Affects overall topological connectivity
- Basin bifurcation



Saddle-saddle connection

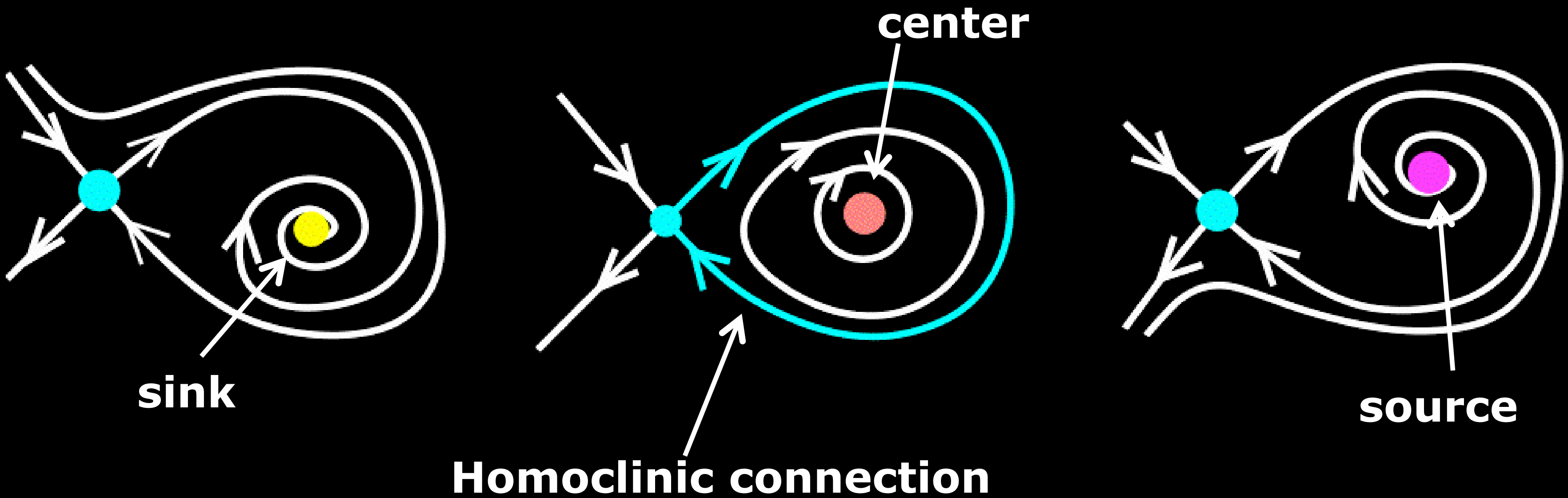
Global Bifurcations

- Modifies overall topological connectivity
- Homoclinic bifurcation



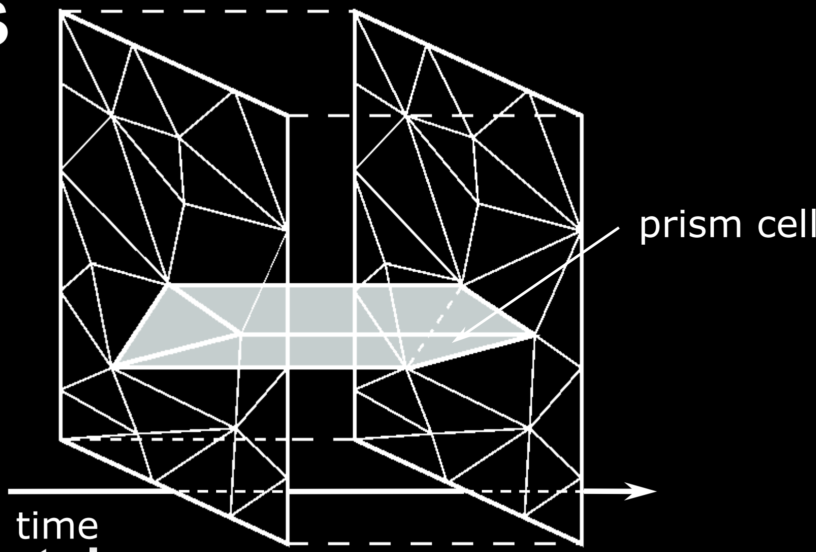
Global Bifurcations

- Modifies overall topological connectivity
- Periodic blue sky



2+1D Topology *Cell-wise Tracking*

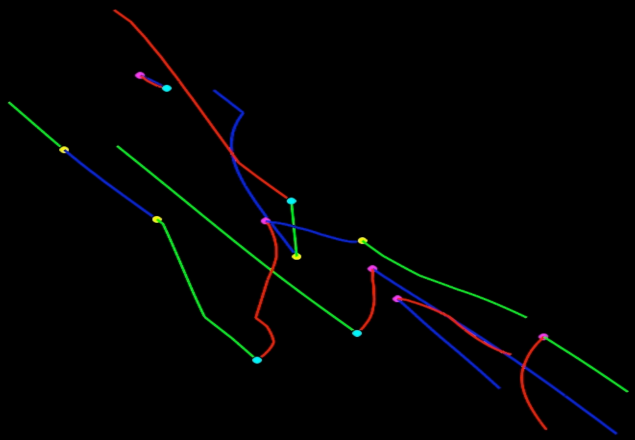
- Time-wise interpolation
- Cell-wise tracking over 2+1D grid
- Detect local bifurcations

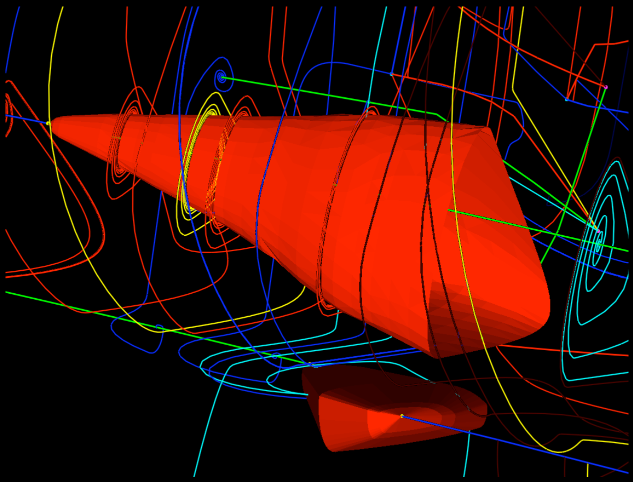


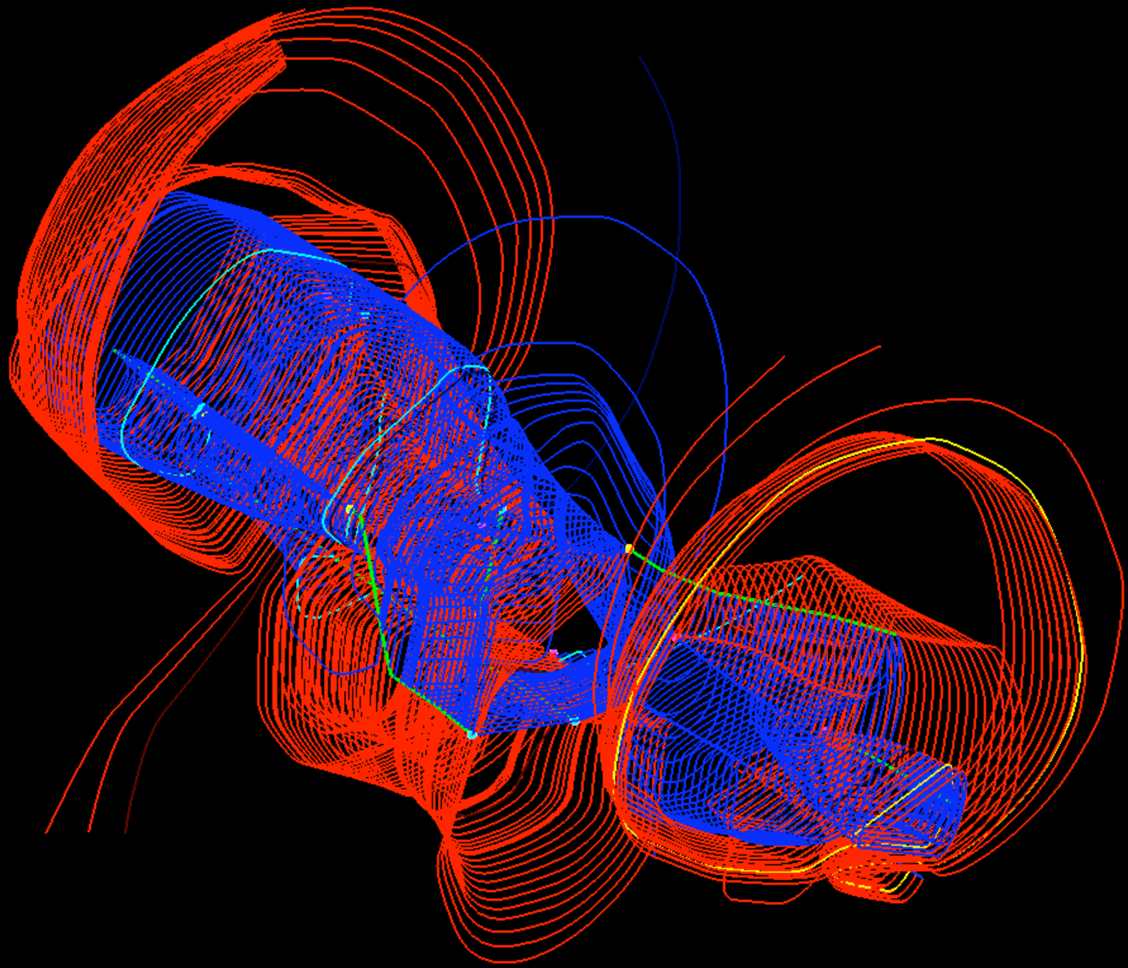
- Track associated separatrices ($\rightarrow$ surfaces)

2+1D Topology

Cell-wise Tracking

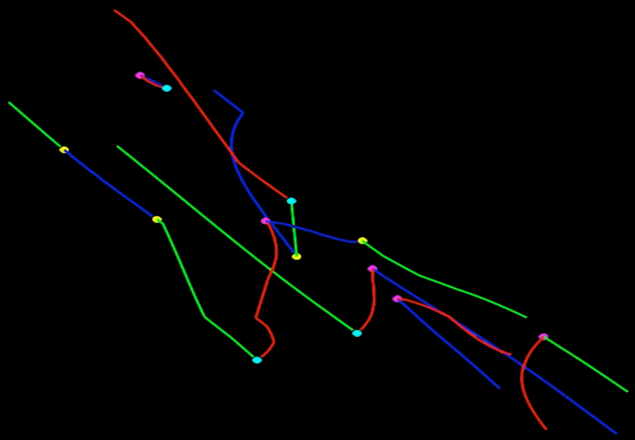


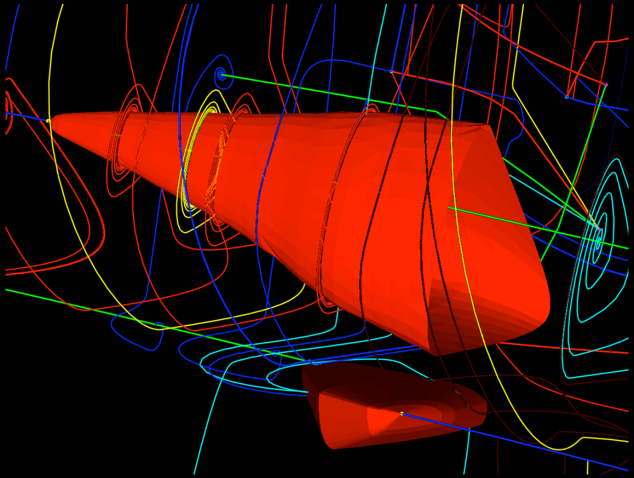


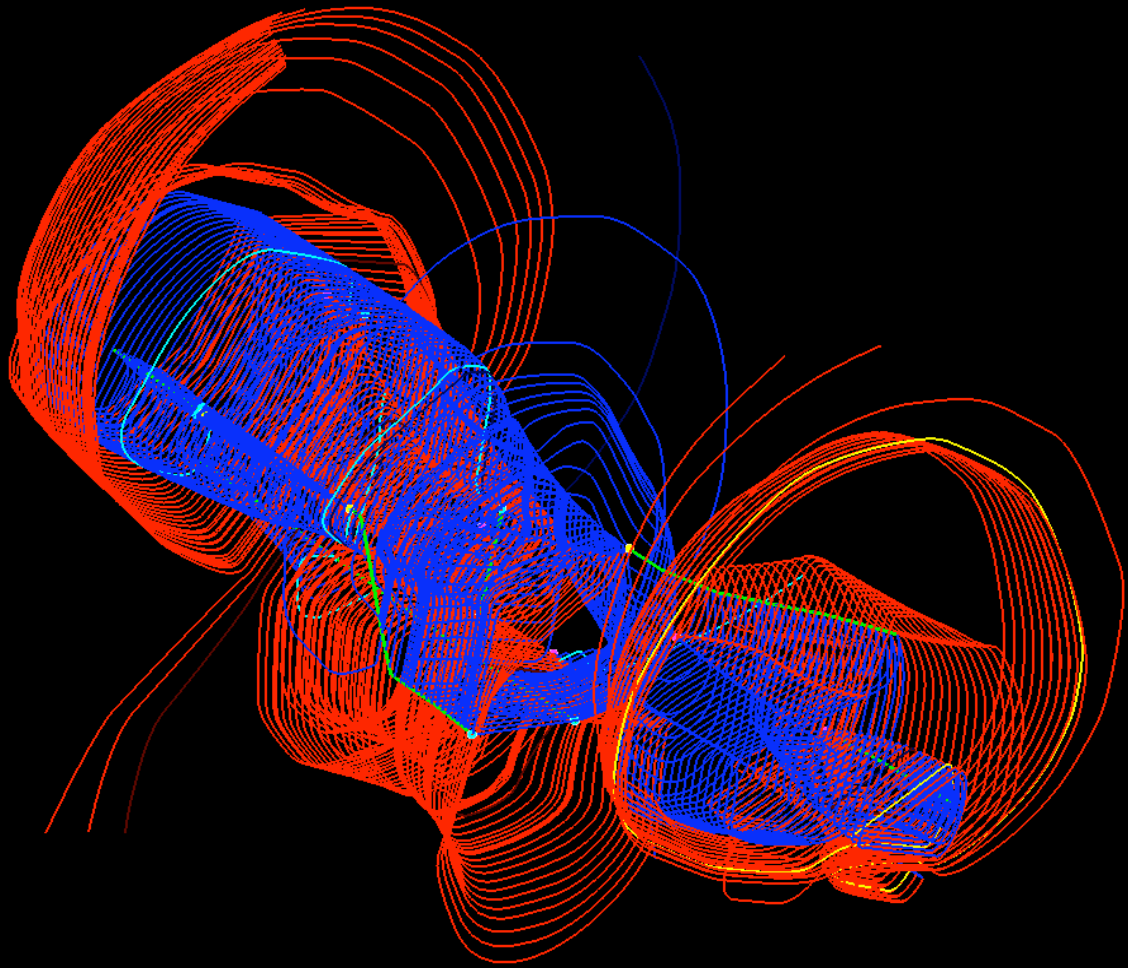


2+1D Topology

Cell-wise Tracking

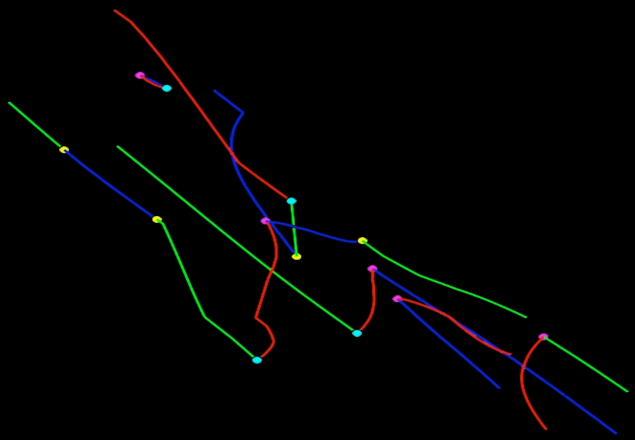


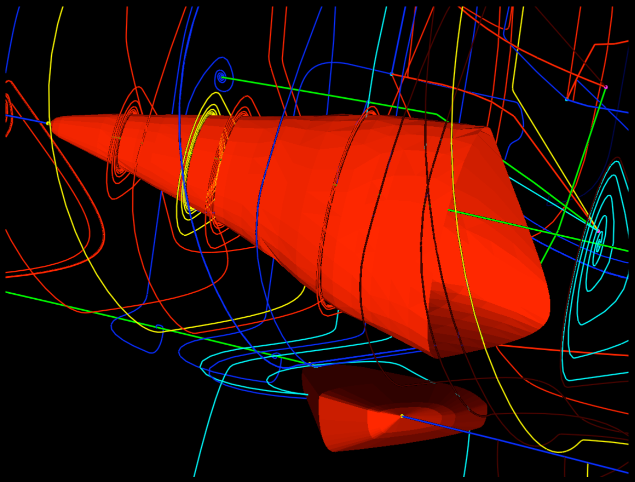


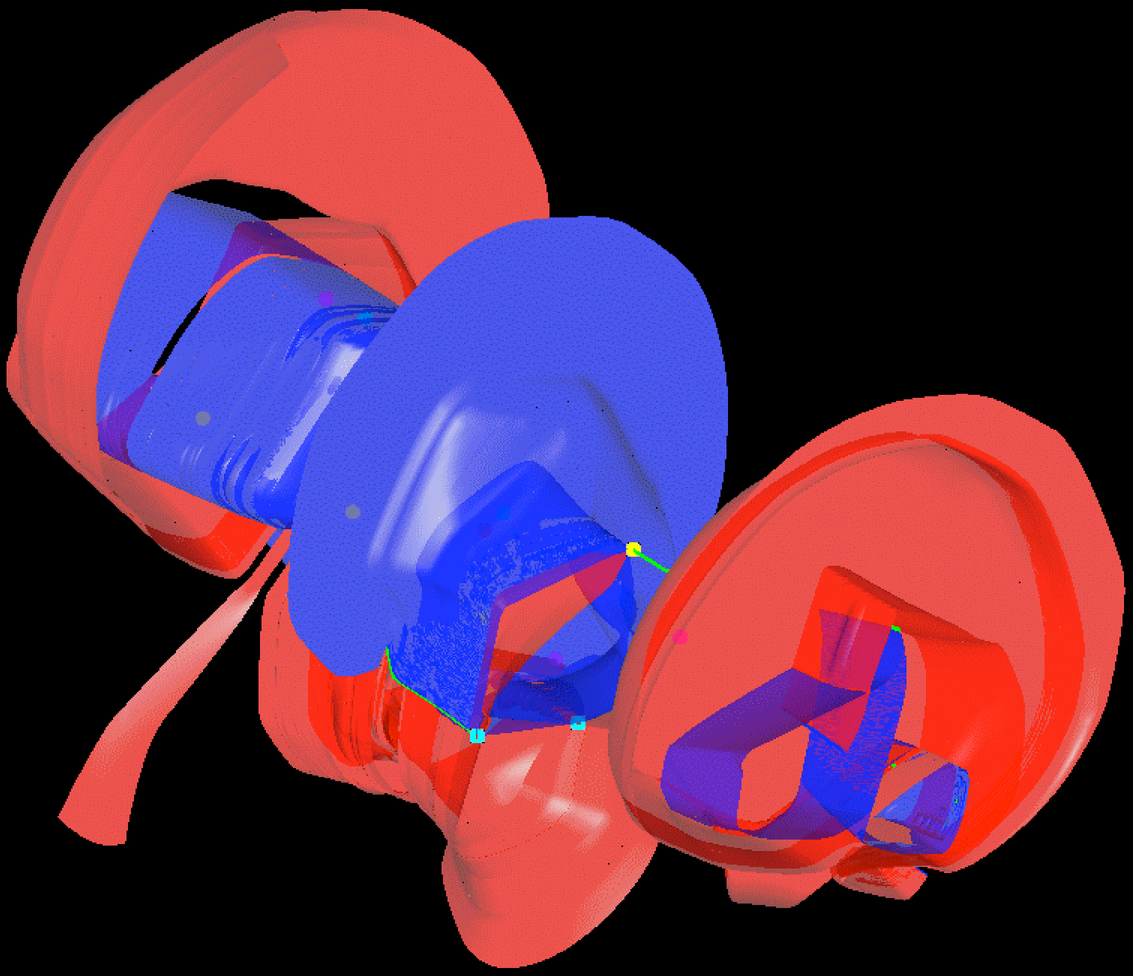


2+1D Topology

Cell-wise Tracking







2+1D Topology *Feature Flow Field*

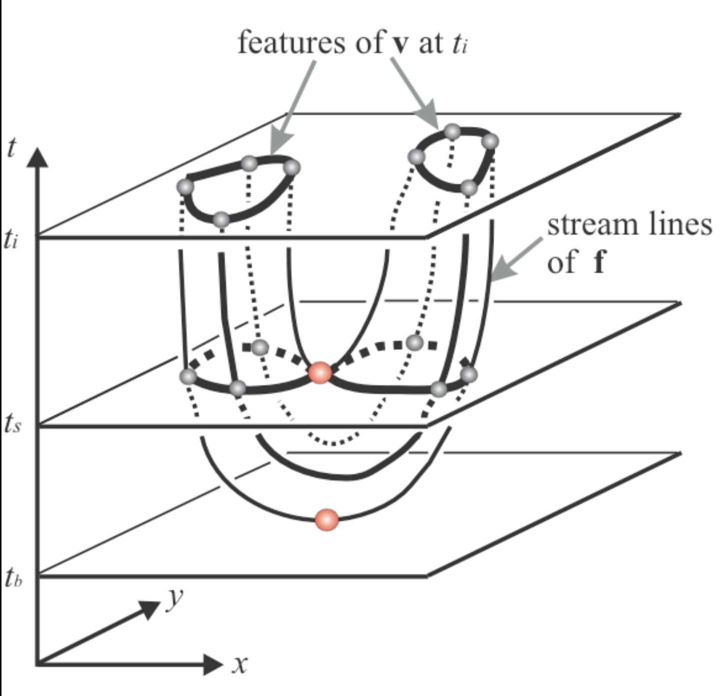
- Feature Flow Field
  - *Track path of critical points by streamline integration in vector field defined over (n+1)D space-time domain*

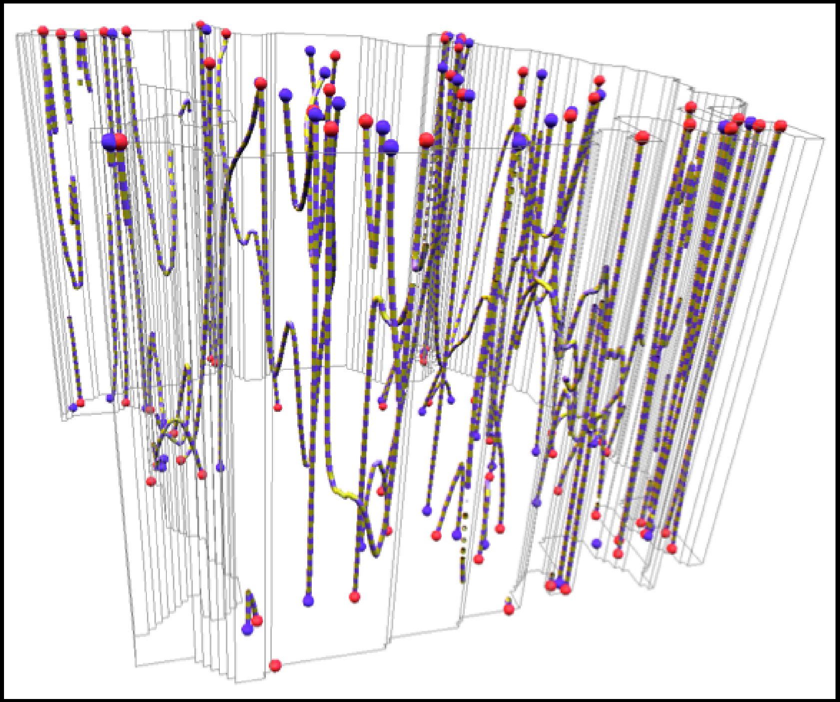
$$\vec{f}(x,y,t) = \begin{pmatrix} u(x,y,t) \\ v(x,y,t) \end{pmatrix} \quad \vec{g}(x,y,t) = \nabla u \times \nabla v$$

  - *The value of $\vec{f}$ (e.g. $\vec{0}$) is constant along streamlines of $\vec{g}$*

2+1D Topology

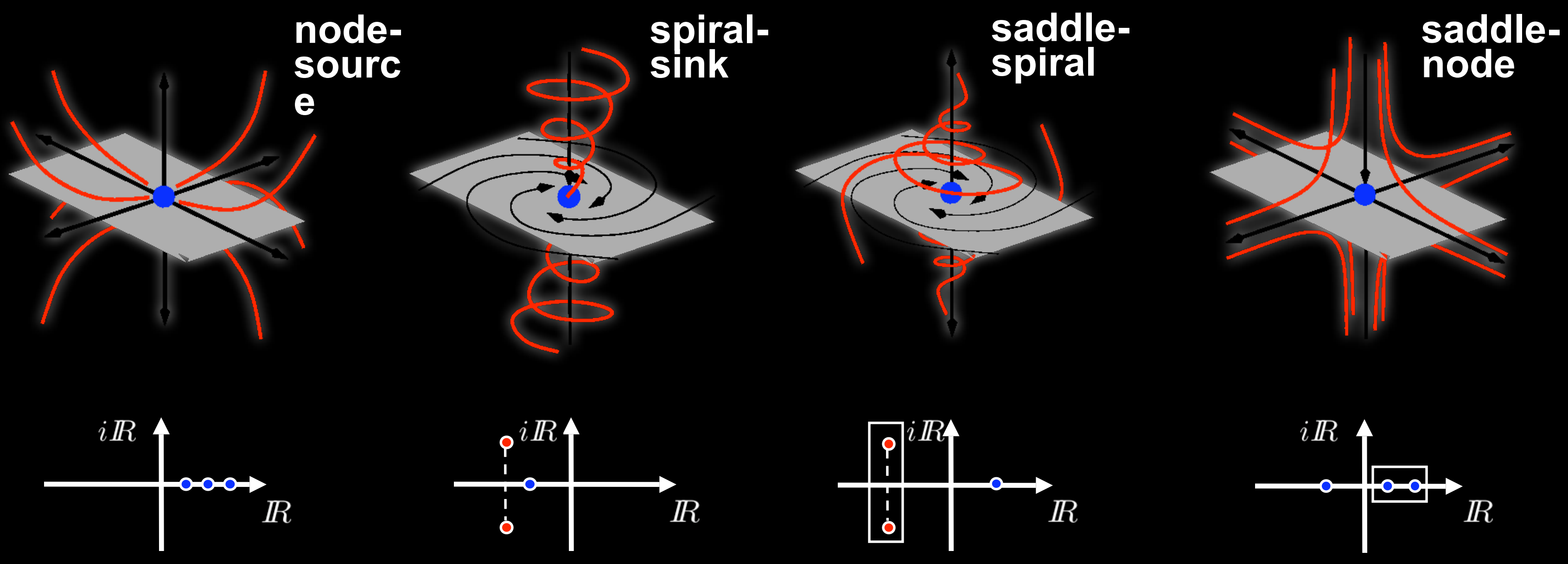
Feature Flow Field





3D Flow Topology

- Critical points

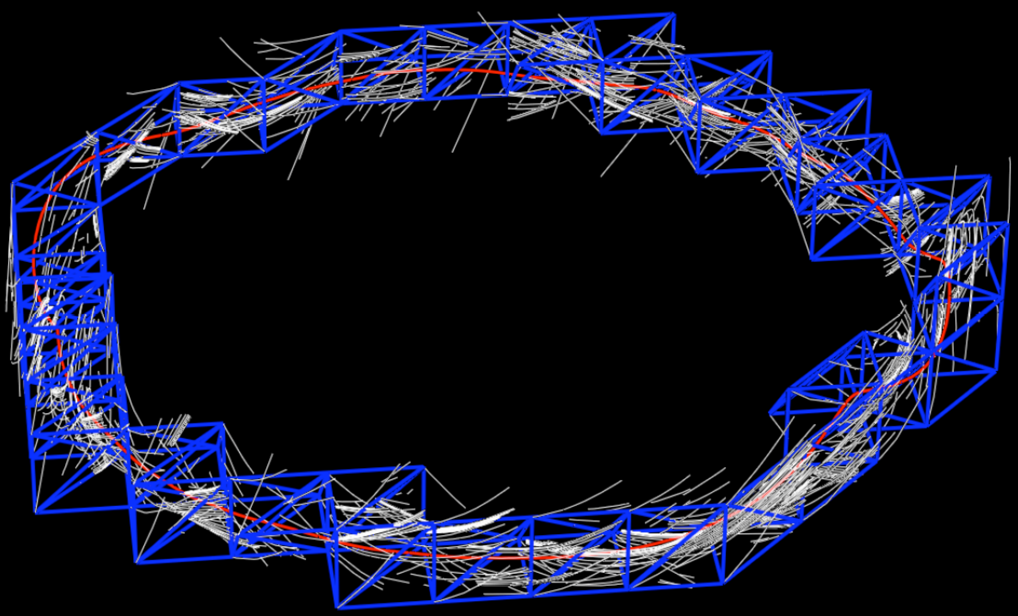


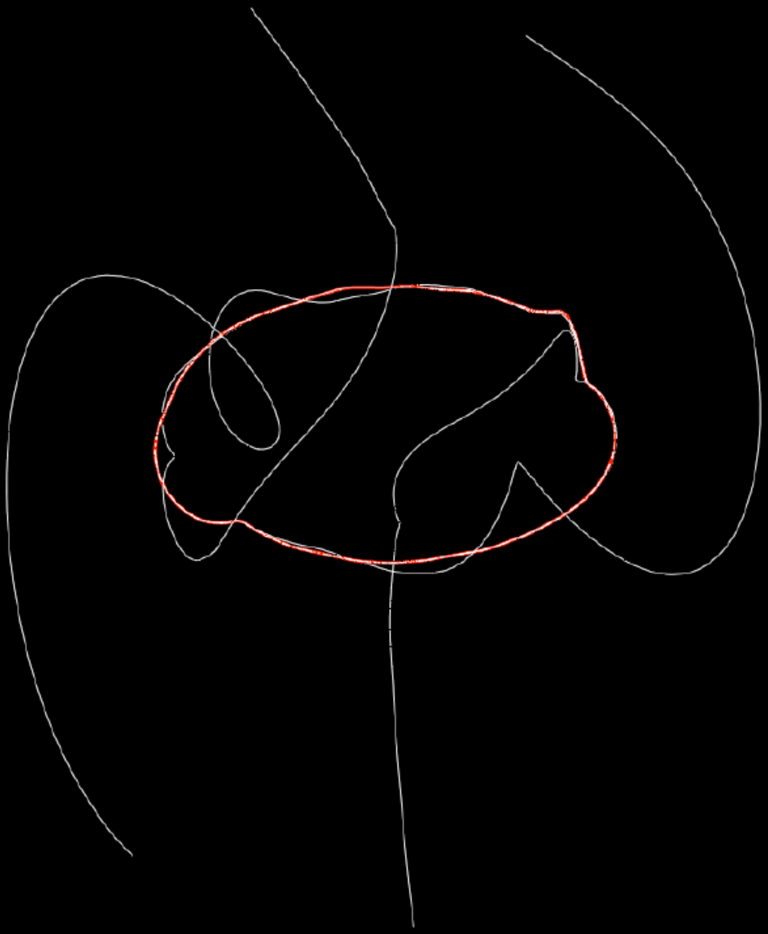
- Both line and surface separatrices exist

3D Cycles

- Similar principle as in 2D
  - *Isolate closed cell chain in which streamline integration appears captured*
  - *Start stream surface integration along boundary of cell-wise region*
  - *Use flow continuity to exclude reentry cases*

3D Cycles





3D Topology Extraction

- Cell-wise critical point extraction:
  - *Compute root of linear / trilinear expression*
  - *Compute Jacobian at found position*
  - *If type is saddle compute eigenvectors*
- Extract closed streamlines
- Integrate line-type separatrices
- Integrate surface separatrices as stream surfaces

3D Topology Extraction