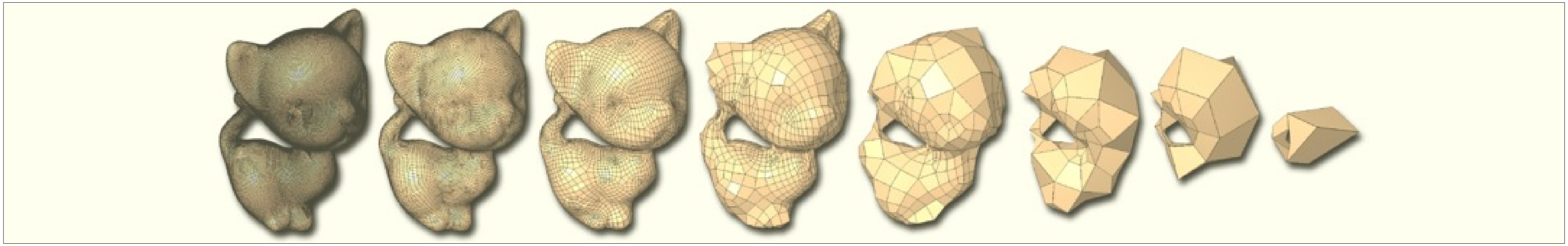
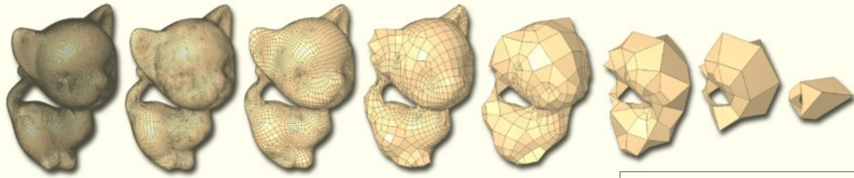


Spectral Processing

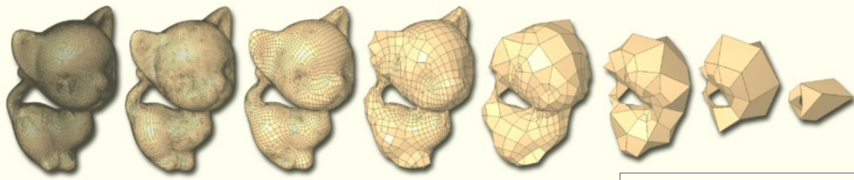
Part I

Geometry Processing
CS 6968 Spring 2009



Spectral Processing

Although relatively recent in the context of geometry processing, spectral methods have already experienced a large development in the field of spectral graph theory.



Spectral Processing

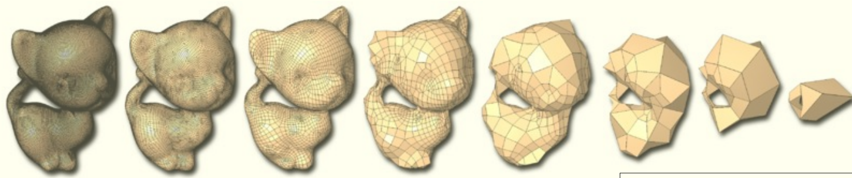
Although relatively recent in the context of geometry processing, spectral methods have already experienced a large development in the field of spectral graph theory.

Treating the vertex coordinates as a signal defined over the graph structure supplied by the underlying mesh, G. Taubin first introduced the Spectral Theory in the context of geometry processing.

Spectral Processing

Spectral methods strongly rely on handling the eigendecomposition of a matrix, called *Laplacian Matrix*, which incorporates the adjacency and/or geometrical information derived from the mesh.

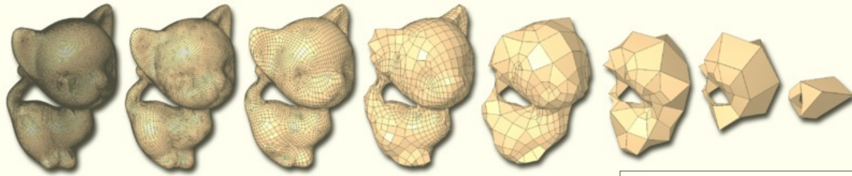
Lets stop a moment and review the definition and some properties of eigenvectors and eigenvalues.



Short Review of Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix. A vector $u \in \mathbb{R}^n$ is an *eigenvector* of A iff there is an scalar λ, called *eigenvalue* associated to u, such that:

$$Au = \lambda u$$



Spectral Processing

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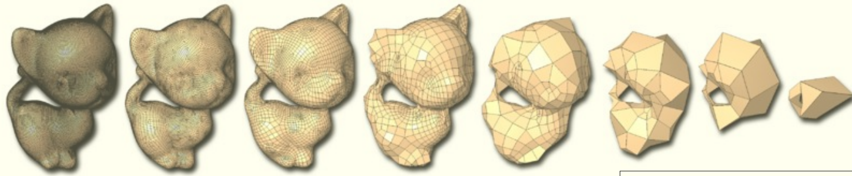
Let $U_i = \{\alpha u_i \,|\, a \in \mathbb{R}\}$, where u_i is an eigenvector of A, and

$$V = U_1 \oplus U_2 \cdots U_m$$

If we consider A a linear operator $A : V \subset \mathbb{R}^n \to V$ then the subspaces U_i are invariant under A.



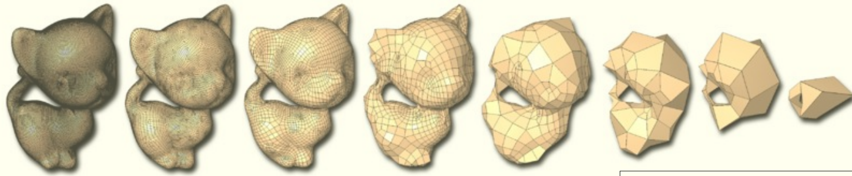
Short Review of Eigenvalues and Eigenvectors

The eigenvalues and eigenvectors associate to a matrix A can be complex numbers, even if A is a real matrix

Example: $A(x,y) = (-y,x)$

Short Review of Eigenvalues and Eigenvectors

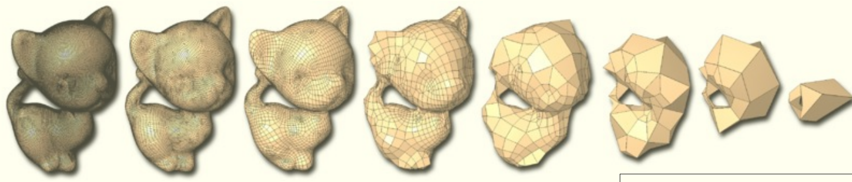
Theorem: If $\{u_1, \ldots, u_m\}$ is a set of eigenvectors associated to distinct eigenvalues $\lambda_1, \ldots, \lambda_m$ then $\{u_1, \ldots, u_m\}$ is linearly independent.



Short Review of Eigenvalues and Eigenvectors

Theorem: If $\{u_1, \ldots, u_m\}$ is a set of eigenvectors associated to distinct eigenvalues $\lambda_1, \ldots, \lambda_m$ then $\{u_1, \ldots, u_m\}$ is linearly independent.

Corollary: An $n \times n$ matrix A has at most n distinct eigenvalues.



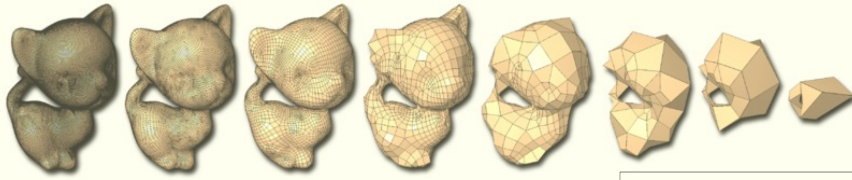
Short Review of Eigenvalues and Eigenvectors

If $\{v_1, \ldots, v_n\}$ is a basis for the space $V \subset \mathbb{R}^m$ and $T : V \subset \mathbb{R}^n \to V$ is a linear operator then the matrix representation of T with respect to $\{v_1, \ldots, v_n\}$ is given by:

$$Tv_k = a_{1k}v_1 + \cdots + a_{nk}v_n, \quad k = 1, \ldots, n$$

that is,

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$



Short Review of Eigenvalues and Eigenvectors

If the matrix representation of a linear operator T is diagonal, that is:

$$\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

then there is a base $\{v_1, \ldots, v_n\}$ such that:

$$Tv_i = \lambda_i v_i$$

Short Review of Eigenvalues and Eigenvectors

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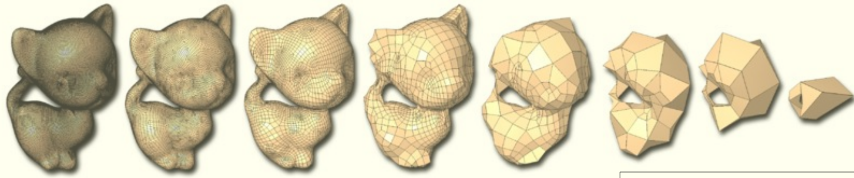
Thus a linear operator $T : V \to V$ has a diagonal matrix representation if only if V has a basis consisting of eigenvectors of T.

Short Review of Eigenvalues and Eigenvectors

If an operator $T : V \to V$ is represented by a symmetric matrix (T is self-adjoint operator) then:

1. The eigenvalues of T are real
2. The set of eigenvectors forms an orthogonal basis for V



Short Review of Eigenvalues and Eigenvectors

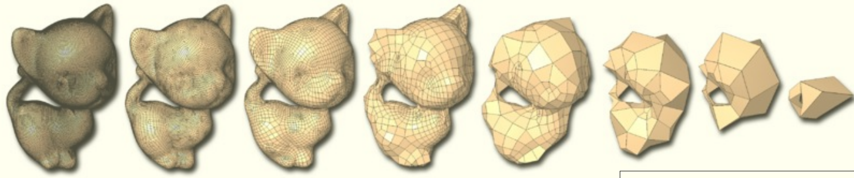
Given a basis $\{v_1, \ldots, v_m\}$ of a subspace $V \subset \mathbb{R}^n$, the projection $\hat{x}$ of a vector $x \in \mathbb{R}^n$ onto V can be computed by solving the following linear system:

$$\begin{bmatrix} < v_1, v_1 > & \cdots & < v_1, v_m > \\ \vdots & & \vdots \\ < v_m, v_1 > & \cdots & < v_m, v_m > \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} = \begin{bmatrix} < x, v_1 > \\ \vdots \\ < x, v_m > \end{bmatrix}$$

$$\hat{x} = \sum_{i=1}^{m} \alpha_i v_i$$

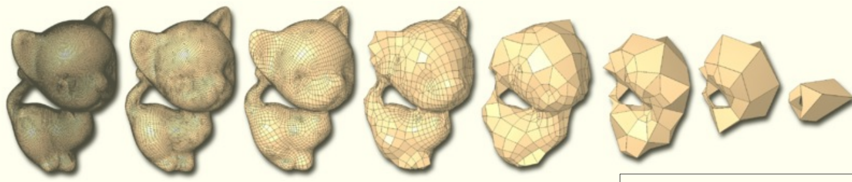
For further details see:

[C. Meyer. Matrix Analysis and Applied Linear Algebra, Siam 2000]
[S. Axler. Linear Algebra Done Right, Springer 1997]



Spectral Processing

Back to Spectral Mesh Theory !



Spectral Processing

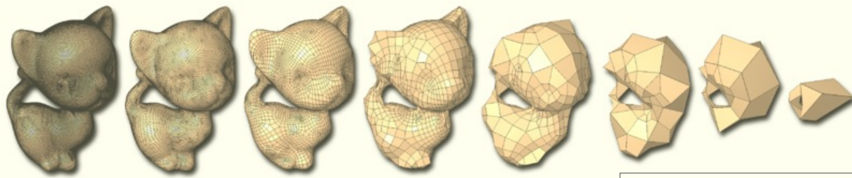
Back to Spectral Mesh Theory !

Three main steps are involved in most mesh processing methods:

1. Construction of the matrix M that incorporates the pairwise relations between mesh elements (adjacencies and geometry).

2. The eigendecomposition of M.

3. Handling of the eigendecomposition towards obtaining the desired mesh processing.

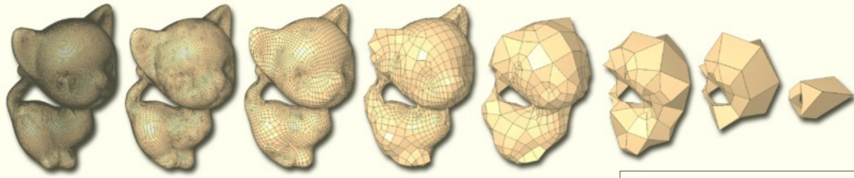


Spectral Processing

Given a triangulated surface S, an usual way to define the matrix M is as follows:

$$m_{ij} = \begin{cases} -w_{ij} & i \neq j \text{ and } j \in N(i) \\ w^*_{ii} & i = j \\ 0 & \text{otherwise} \end{cases}$$

where $w_{ij} = w_{ji}$ is a weight assigned to the edge connecting the vertices v_i ans v_j, $N(i)$ are the indexes of neighbours of vertex i, and $w^*_{ii} = \sum_{j \in N(i)} w_{ij}$



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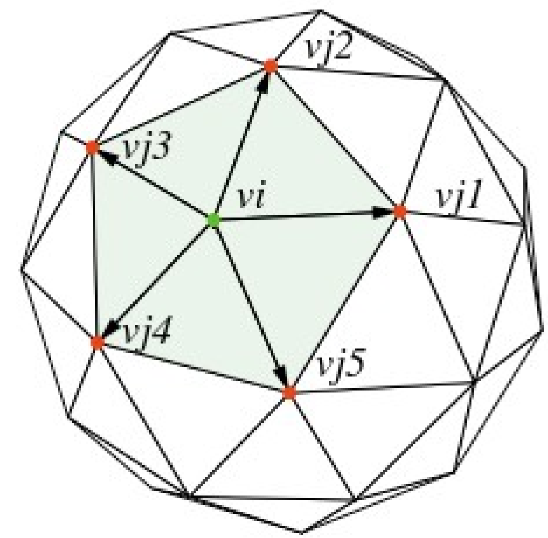
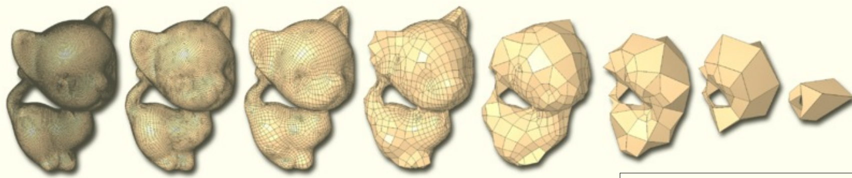


$$\begin{bmatrix} w^*_{11} & \cdots & & & & & & & & & & & & & \\ & & & & & & \vdots & & & & & & & & \\ 0 & \cdots & -w_{ij_1} & \cdots & -w_{ij_2} & \cdots & w^*_{ii} & \cdots & -w_{1j_3} & \cdots & -w_{1j_4} & \cdots & -w_{1j_5} & 0 \\ & & & & & & \vdots & & & & & & & & \\ \cdots & & & & & & & & & & & & \cdots & w^*_{nn} \end{bmatrix}$$

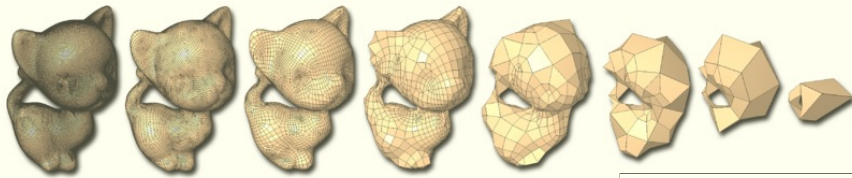


Spectral Processing

The Laplacian operator is defined as follows:

$$L = B^{-1}M$$

where B is a diagonal matrix. Although L is not symmetric in general, it is similar to the symmetric matrix $O = B^{-1/2}MB^{-1/2}$, thus L has real eigenvalues (the same as O), being $u_i = B^{-1/2}v_i$ its eigenvectors (v_i being the eigenvectors of O).



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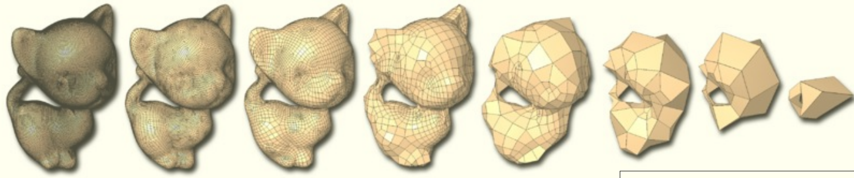
Let $f : S \rightarrow \mathbb{R}$ be a function that assigns a real scalar to each vertex of the surface S (notice that f can be thought as a vector in $\mathbb{R}^m$, where m is the number of vertices in S). Then, apply the Laplacian operator in f is equivalent to:

$$(Lf)_i = b_i^{-1} \sum_{j \in viz(i)} w_{ij}(f_i - f_j)$$

where b_i is the i^{th} diagonal element of B and f_k is the value of f in the vertex v_k.



Spectral Processing

Examples: [Taubin 1995]: Signal Processing for to mesh fairing
[Tutte 1963]: graph drawing

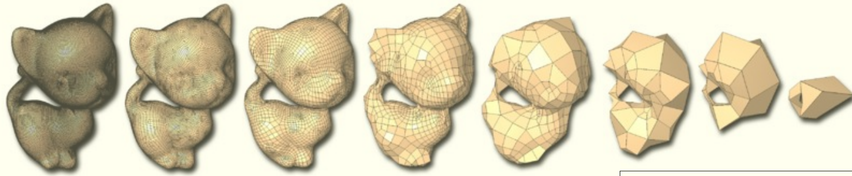
(combinatorial Laplacian)

$$b_i = |N(i)| \quad \text{(number of elements in the neighborhood of } v_i\text{)}$$
$$w_{ij} = 1$$

IF S is a polygon the Laplacian operator defined from $b_i = |N(i)|$ and $w_{ij} = 1$ is:

$$(Lf)_i = \frac{1}{2}(f_{i-1} - f_i) + \frac{1}{2}(f_{i+1} - f_i)$$

$$L = \frac{1}{2}\begin{bmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ \vdots & & & & \\ -1 & & & -1 & 2 \end{bmatrix}$$



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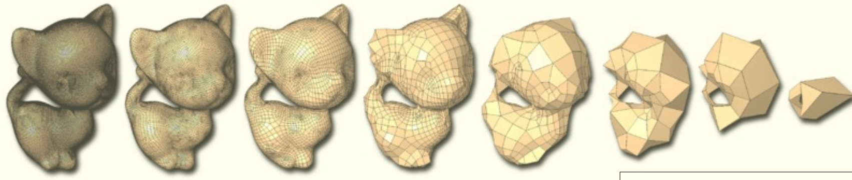
Eigenvalues are:

$$\lambda_j = 1 - \cos(2\pi \lfloor j/2 \rfloor / n)$$
$$0 \leq \lambda_1 \leq \ldots \lambda_n \leq 2$$

Eigenvectors are:

$$(u_i)_h = \begin{cases} \sqrt{1/n} & i = 1 \\ \sqrt{2/n}\sin(2\pi h \lfloor j/2 \rfloor / n) & i \text{ is even} \\ \sqrt{2/n}\cos(2\pi h \lfloor j/2 \rfloor / n) & i \text{ is odd} \end{cases}$$



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Equivalent to the Fourier Basis

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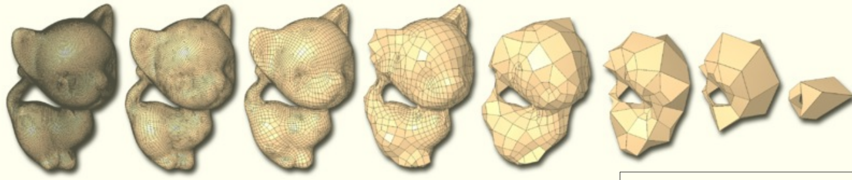
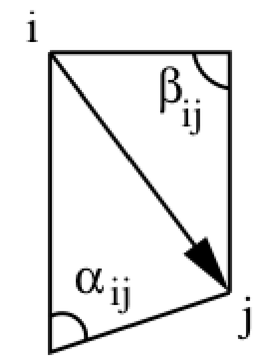


Spectral Processing

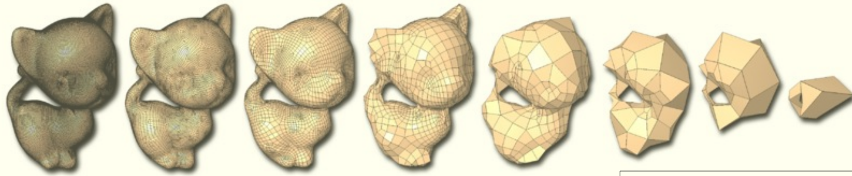
Examples: [Pinkall and Polthier 1993]: Geometrical information

$$(Lf)_i = \sum_{j \in N(i)} \frac{1}{2}(\cot(\alpha_{ij}) + \cot(\beta_{ij}))(f_i - f_j)$$

$$b_i = 1\,; \quad w_{ij} = \frac{1}{2}(\cot(\alpha_{ij}) + \cot(\beta_{ij}))$$



What do the weights as defined above means?



Spectral Processing

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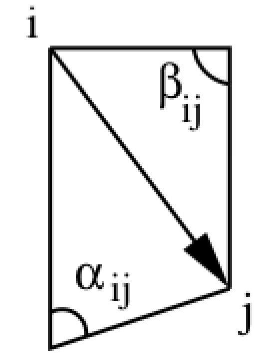


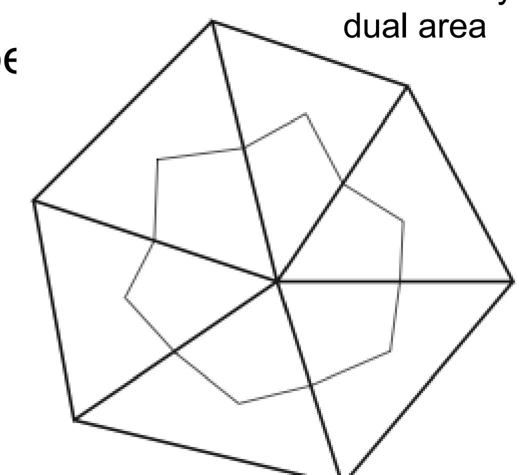
What do the weights as defined above means?

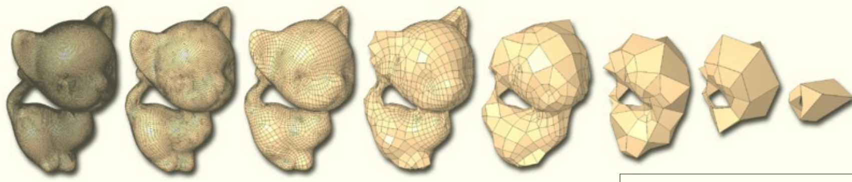
It aims at approximating the Laplace-Beltrami operator which can be derived from the mean curvature normal operator:

$$K(x) = 2\kappa_H(x)\vec{n}(x)$$

$$K(x_i) = \frac{1}{2|x_i^*|} \sum_{j \in N(x_i)} (\cot(\alpha_{ij}) + \cot(\beta_{ij}))(x_i - x_j)$$

vertex barycentric dual area

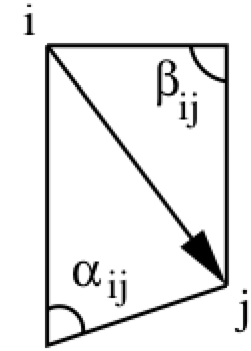




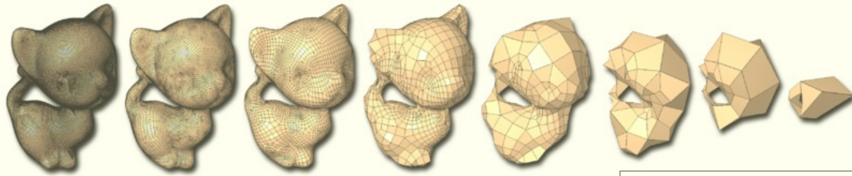
Spectral Processing

Examples: [Vallet and Levy 2008]: Manifold Harmonics

$$(Lf)_i = \sum_{j \in N(i)} \frac{(\cot(\alpha_{ij}) + \cot(\beta_{ij}))}{|v_i^*|}(f_i - f_j)$$



where $|v_i^*|$ is the barycentric dual area.



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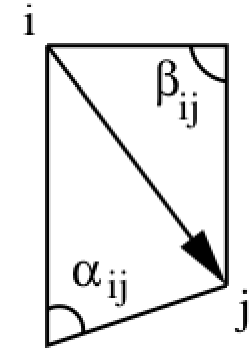


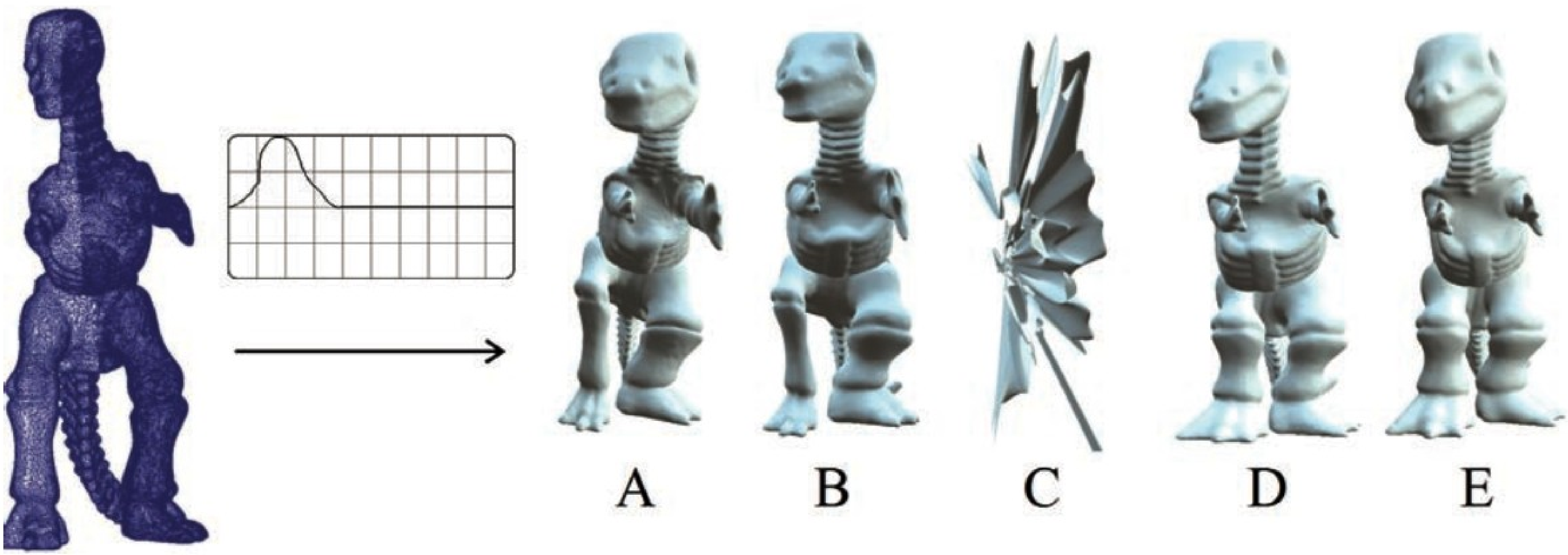
where $|v_i^*|$ is the barycentric dual area.

As $w_{ij} \neq w_{ji}$ this operator is not symmetric, furthermore, it can not be written as $L = B^{-1}M$ where M is a symmetric matrix.

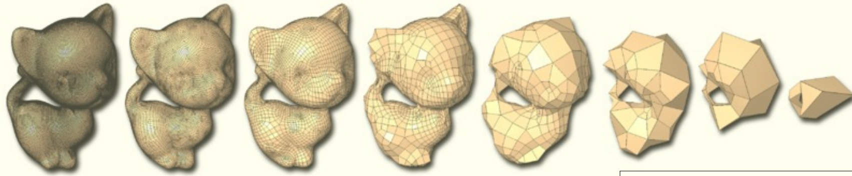
$$(Lf)_i = \sum_{j \in N(i)} \frac{(\cot(\alpha_{ij}) + \cot(\beta_{ij}))}{\sqrt{|v_i^*||v_j^*|}}(f_i - f_j)$$

$$b_i = 1\,; \quad w_{ij} = \frac{(\cot(\alpha_{ij}) + \cot(\beta_{ij}))}{\sqrt{|v_i^*||v_j^*|}}$$

Comparison: [Vallet and Levy 2008]



A: combinatorial
B: cotangents
C: weighted cotangents
D: symmetrized cotangents
E: symmetric weights

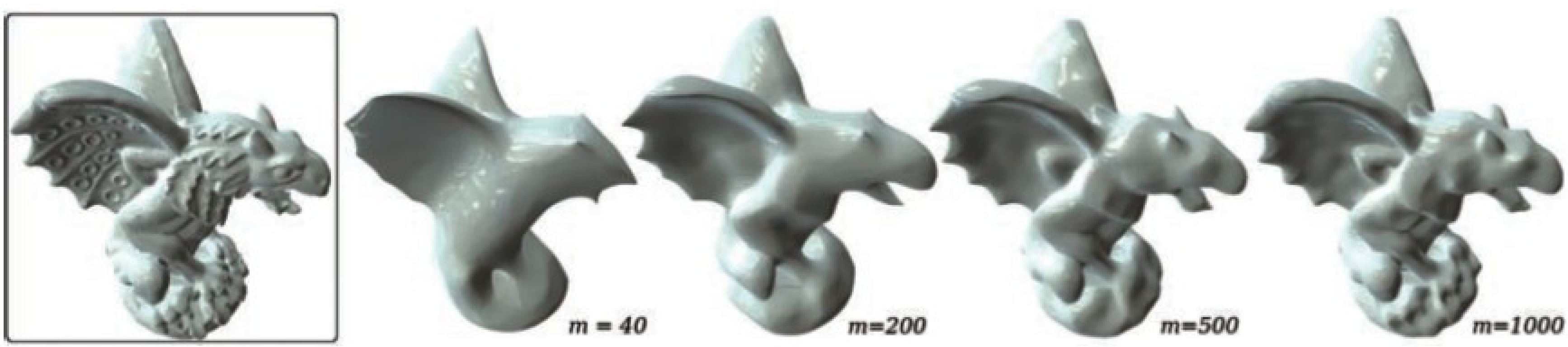


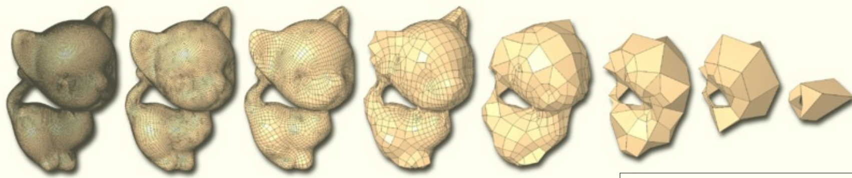
Given the orthonormal basis v_i, we can compute the coefficients:

$$\alpha_i = < x, v_i >$$

such that x is projected on the eigenspace by:

$$x \sim \hat{x} = \sum \alpha_i v_i$$

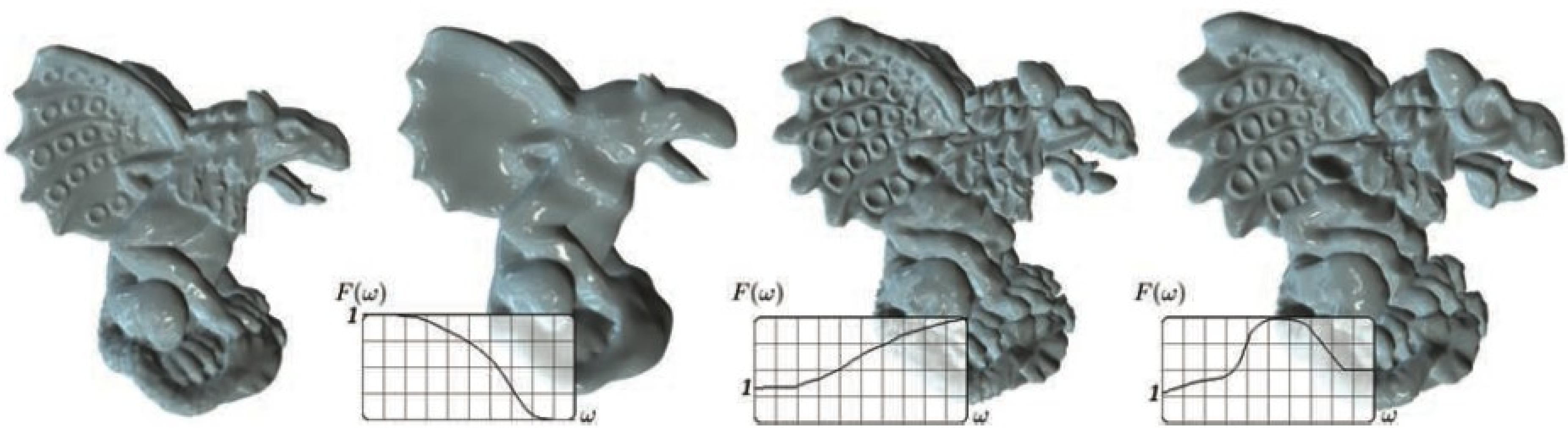


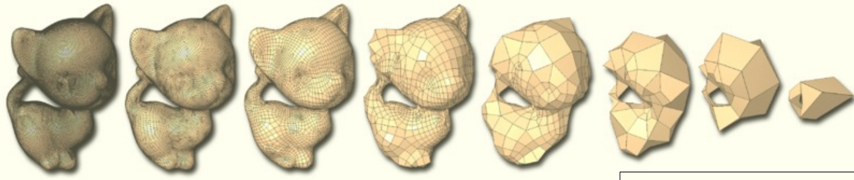


Spectral Processing

In the context of Manifold Harmonics, Vallet and Levy propose a filtering scheme based on “transfer functions” acting on the coefficients:

$$\hat{x}^F = \sum F(\omega)\alpha_i v_i = \sum F(\sqrt{\lambda_i})\alpha_i v_i$$





Spectral Processing

There are other operators with similar structure to Laplacian operators:

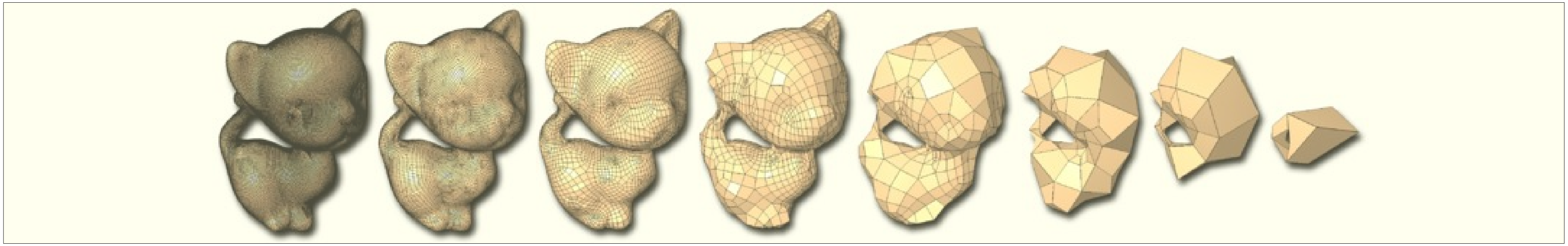
- Schrödinger operator

$$H_{ij} = \begin{cases} \text{negative real number} & i \neq j \text{ and } j \in viz(i) \\ \text{any real number} & i = j \\ 0 & \text{otherwise} \end{cases}$$

- Affinity Matrix

w_{ij} represents a numerical relation between i and j

$$w_{ij} = e^{-dist(i,j)/2\sigma^2}$$

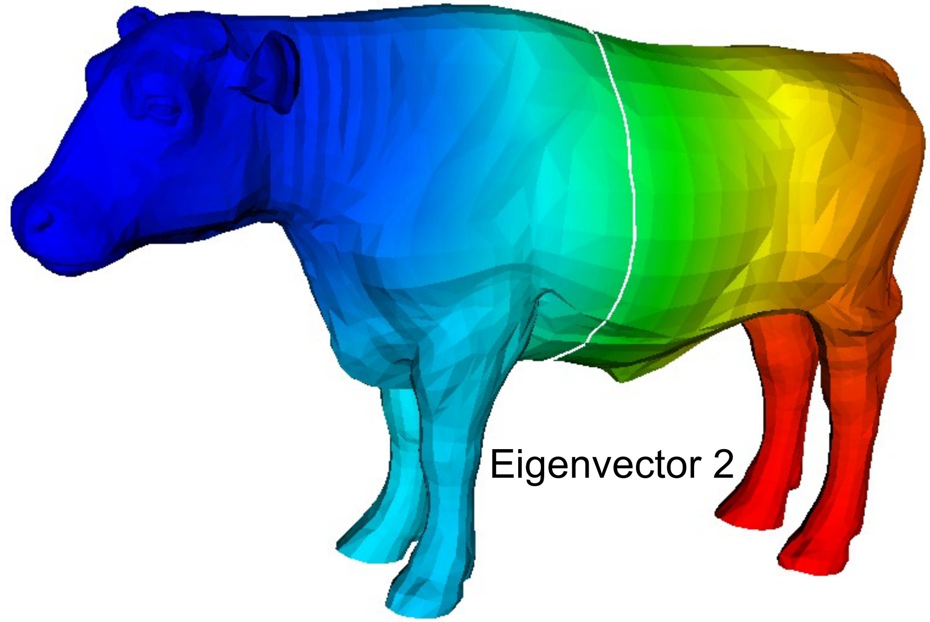


Spectral Processing

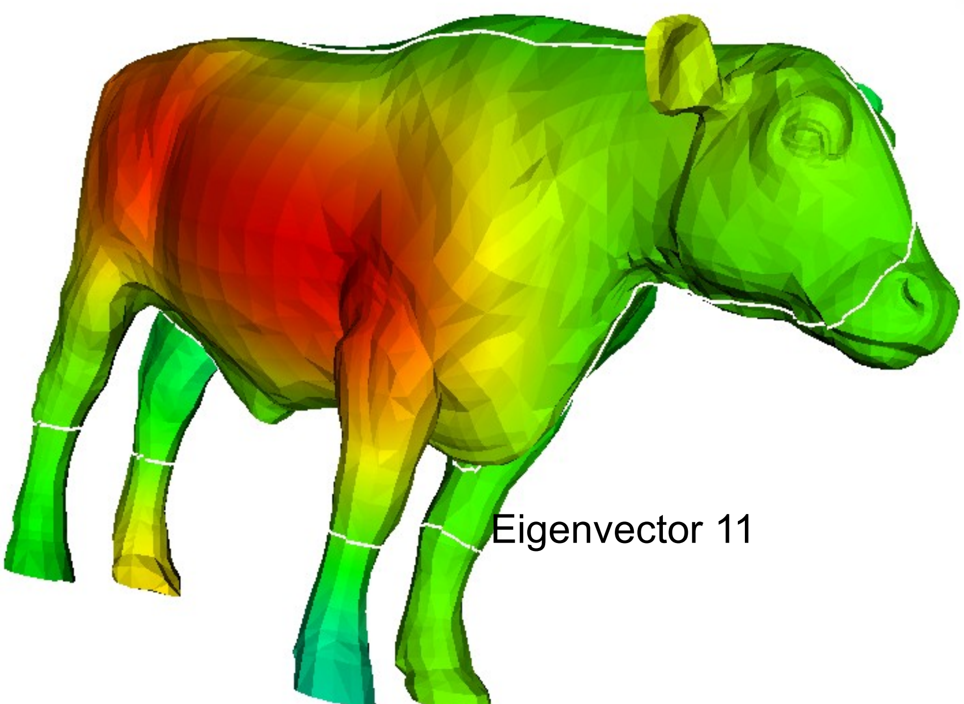
Part II

Geometry Processing

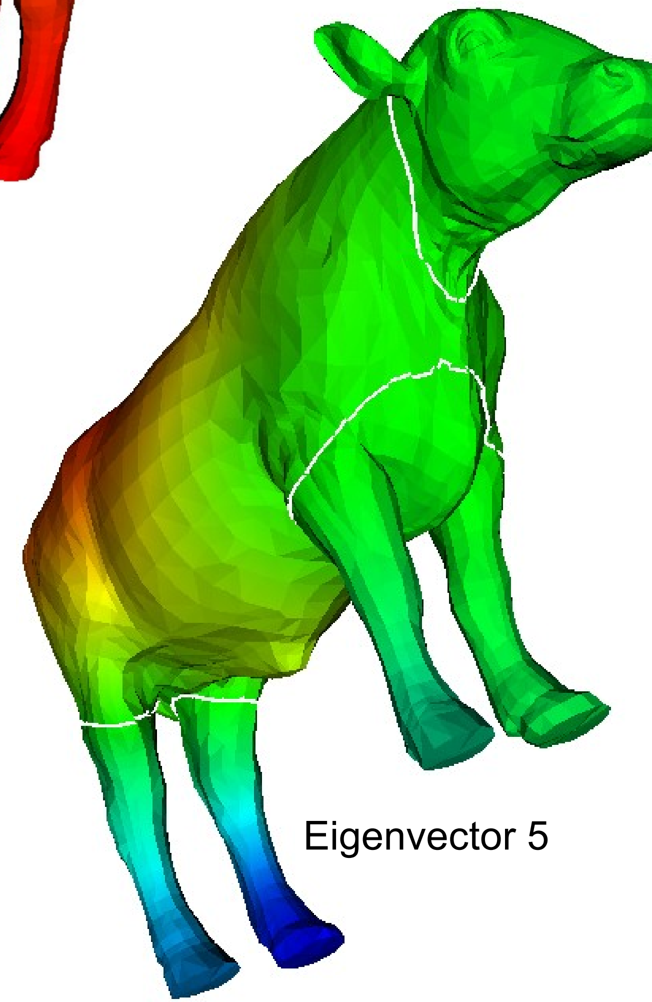
CS 6968 Spring 2009



Eigenvector 2



Eigenvector 11

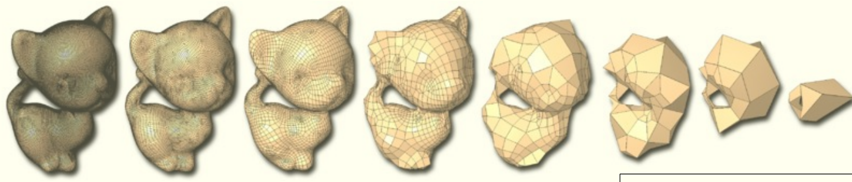


Eigenvector 5

Nodal Domain: The *nodal set* of an eigenfunction is composed of points at which the eigenfunction is zero. The regions bounded by the nodal set are called *nodal domains*.

PS. The eigenfunction is built by interpolating the values of the eigenvector.

Courant's Nodal Theorem: Let the eigenvectors of the Laplace operator be labeled in increasing order in accordance with the corresponding eigenvalues. Then, the *k-th* eigenfunction can have at most k nodal domains, that is, the *k-th* eigenfunction can separate the surface into at most k connected components.



Spectral Processing

In the case of combinatorial Laplacian:

$$\frac{f^T L f}{f^T f} = \frac{1/2 \sum_{i,j} w_{ij} (f_i - f_j)^2}{||f||^2}$$

L is positive semi-definite, so
the smallest eigenvalue is zero with
a constant associated eigenvector.

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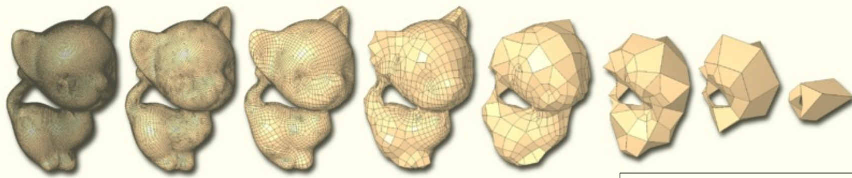
Similar results can also be obtained for other Laplacians !



Spectral Processing

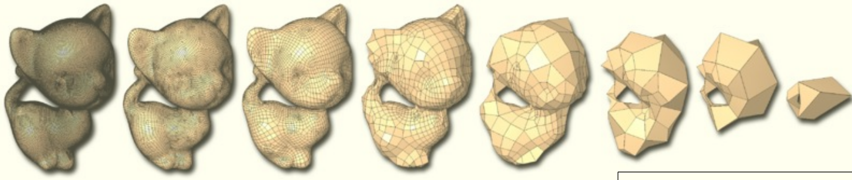
In the case of combinatorial Laplacian:

The second eigenvector (called Fiedler vector) is characterized by:

$$v_2(L) = \arg\min_u \sum_{i,j} w_{ij}(u_i - u_j)^2$$

$$\text{Subject to:} \sum u_i = 0 \text{ and } \sum u_i^2 = 1$$

The solution of this problem is related with one-dimensional embedding of the graph and can be used for finding vertices ordering.

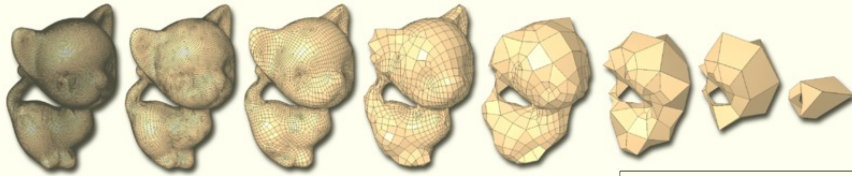


Spectral Processing

Computational Aspects:

Computing the eigendecomposition of matrix efficiently has been a challenge problem for many years. Much effort has been devoted to devise robust techniques that speed-up the eigendecomposition, as efficiency is a concern for large meshes.

In the context of geometry processing, eigendecomposition techniques aim at accelerating the computation by exploring the sparsity of the matrices together with the shift property of the eigenvalues.



Computational Aspects:

The numerical analysis literature describes efficient techniques to compute the largest eigenvalue and associated eigenvector of a given matrix, as for example, the iterative power method.

The following properties allow the computation of other elements of the spectrum:

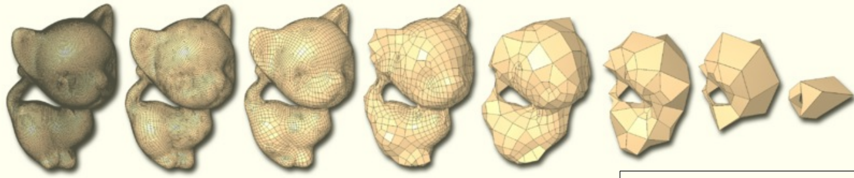
Shift Property:

If λ is an eigenvalue of L with associated eigenvector v then v is also an eigenvector of $L - \sigma I$ with $\lambda - \sigma$ its associated eigenvalue.

Inversion Property:

If λ is an eigenvalue of L with associated eigenvector v then $1/\lambda$ is an eigenvalue of L^{-1} and v is its associated eigenvector.

$$B = (L - \sigma I)^{-1}$$



Spectral Processing

Computational Aspects:

Available Libs:

1. ARPACK: Based on Arnoldi's Method

2. Sparse Factorization: TAUCS