1 Extension to the Design of 2D Tensor Fields

The present framework can be easily extended to the design of 2D time-varying second-order symmetric tensor fields. Tensor fields are currently a popular research subject in both graphics and visualization communities [1, 7, 5, 6, 2, 4]. Tensor fields have rather different characterizations from vector fields. In a tensor field, there is no definition of pathline since it is oriented-less. Instead, we study the behavior of hyperstreamlines which leads to the tensor field topology in the static case. Tensor field topology consists of the degenerate points and their connectivity through separatrices (Note that closed orbits can exist but are barely discussed in the community). Similar to our treatment to the time-varying vector fields, we can create a time-varying tensor field by modeling its instantaneous topology over time. Therefore, the discussion of the design of orientation fields is still extensible to tensor fields. That is, we can design the degenerate points and their paths, hyperstreamlines, and bifurcations of a time-varying tensor field in the similar fashion of time-varying vector fields. Therefore, the former approaches for generating a time-varying vector fields can be reformulated for the creation of a time-varying tensor field. We consider a tensor field \( T \), a \( 2 \times 2 \) symmetric and traceless matrix, which is of the form

\[
R \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}
\]

where \( R \geq 0 \) and \( \theta \in [0, 2\pi) \). A tensor field \( T \) is a continuous function that associates every point \( p = (x, y) \in \mathbb{R}^2 \) with a tensor \( T(p) \). \( p \) is said to be a degenerate point if \( T(p) = 0 \), otherwise, it is regular. More theoretical details can be found in [3].

The 2D time-varying tensor fields can also be designed using the proposed three approaches, i.e. the element-based design, the key frame design, and the field deformation. For the element-based design, the time-varying design elements adapted from the static case can be defined similarly to those for the time-varying vector fields. For instance, a degenerate point element is defined as \( D(T, P(t), M(t)) \), where \( T \) is the tensor that defines the selected type of the degenerate point, \( P(t) \) represents the path of the singular element over time, and \( M(t) \) is the affine transformation matrix (i.e. scaling and rotating) that is exerted on the element along \( P(t) \). Its basis field at time \( t \) is

\[
T_i(x; t) = e^{-d||x-p_i(t)||^2} M_i^T(t) T_i M_i(t) \begin{pmatrix} x-p_i(t) \\ y-p_i(t) \end{pmatrix}
\]

Similarly, the regular element is defined as \( R(\theta(t), P(t)) \) where \( P(t) \) is a prescribed pathline and \( \theta(t) \) is the tensor orientation at \( P(t) \) in space at a time \( t \). As such, the basis field generated from a regular element at time \( t \) is

\[
T_i(x; t) = e^{-d||x-p_i(t)||^2} M_i(t) \begin{pmatrix} \cos 2\theta(t) & -\sin 2\theta(t) \\ \sin 2\theta(t) & \cos 2\theta(t) \end{pmatrix} \begin{pmatrix} x-p_i(t) \\ y-p_i(t) \end{pmatrix}
\]

The total tensor field is then the sum of these basis fields.

\[
T(x; \lambda) = \sum_i T_i(x; \lambda)
\]

determined by their local tensor, including wedge, trisector, node, center, and saddle [7].

Note that the bifurcation theory for time-dependent tensor fields has not been well established. Thus, the bifurcation element for a time-varying tensor field is not easy to define at this moment.

Spatial-temporal constrained optimization:

In key frame design, the user can specify the instantaneous tensor fields at the desired times. A tensor-valued spatial-temporal constrained optimization can be used to compute the rest of the tensor field.

\[
\omega T(v; \lambda_j) = \sum_{k \in N(\theta)} \omega_{k} T(v_k; \lambda_j) + \omega_{j-j-1} T(v; \lambda_j) + \omega_{j-j+1} T(v; \lambda_{j+1})
\]

where \( T(v; \lambda_j) \) represents the average tensor value at position \( (v; \lambda_j) \) in the space time domain.

Figure 1 shows a time-varying tensor field generated using key frame design with the spatiotemporal constrained optimization. This demonstrates the generality of the present frame work to the higher-order field design (e.g. N-Rosy field [5]). However, due to rather complicated behaviors in a tensor field than a vector field, especially when including time dimension, further comprehensive study is needed.

![Image](image_url)

**Fig. 1.** A time-varying tensor field generated using the key frame design and the spatiotemporal constrained optimization. The first and last fields are the key frames.

References


