CS 6170: Computational Topology, Spring 2019 Lecture 06

Topological Data Analysis for Data Scientists

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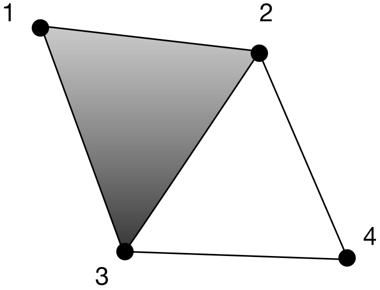
Homology and Computation

Book Chapter B.IV

Chain complexes

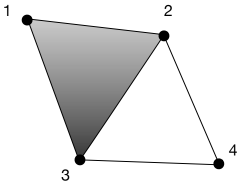
K: a simplicial complex. p: dimension.

- Modulo 2 coefficients (or $\mathbb{Z}_2$ coefficients): $a_i = 0$ or 1.
- A *p-chain* is a formal sum of p-simplices in K, $c = \sum a_i \sigma_i$, where $a_i = 0$ or 1.



$K = \{1, 2, 3, 4, 12, 24, 13, 23, 34, 123\}$ with some abuse of notations.

Chain complexes



- 0-chain, e.g., $c_0 = 1 + 2 + 3$
- 1-chain, e.g., $c_1 = 12 + 23 + 34 + 24$
- 2-chain, e.g., $c_2 = 124$
- Chain additions are done component-wise:
  - $c_0 = 1 + 2 + 3$, $c_0' = 1 + 3 + 4$
  - $c_0 = c_0' = 2 + 4$
  - $c = \sum a_i \sigma_i$, $c' = \sum b_i \sigma_i$, then $c + c' = \sum (a_i + b_i) \sigma_i$

Review: Groups and Abelian Groups

A *group* is a set G with an operation $\bullet$ such that,

- Closure: $a, b \in G \implies a \bullet b \in G$.
- Associativity: $a, b, c \in G \implies (a \bullet b) \bullet c = a \bullet (b \bullet c)$.
- Identity element: $\exists$ an element $e \in G$ s.t. $\forall a \in G$, $e \bullet a = a \bullet e = a$.
- Inverse element: $\forall a \in G$, $\exists b \in G$ s.t. $a \bullet b = b \bullet a = e$

An *abelian group* is a group where the operation also satisfies

- Commutativity: $\forall a, b \in G$, $a \bullet b = b \bullet a$.

Example: Check to see that $(\mathbb{Z}, +)$ and $(\mathbb{Z}_2, +)$ are both abelian groups.

Chain groups

The p-chains together with the additional operation form *a group of p-chains*, denoted as $(\mathsf{C}_p, +)$, or $\mathsf{C}_p = \mathsf{C}_p(K)$.

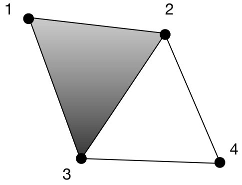
- C_0: 0-chain group (elements are sums of vertices)
- C_1: 1-chain group (elements are sums of edges)
- C_2: 2-chain group (elements are sums of triangles)

Now check that C_p with $\mathbb{Z}_2$ coefficients is indeed an abelian group.

- $c, c' \in \mathsf{C}_p \implies c' + c \in \mathsf{C}_p$
- $(c + c') + c'' = c + (c' + c'')$
- $0 + c = c + 0 = 0$
- $c + c = 0$
- $c + c' = c' + c$

Boundaries

- The *boundary* of a p-simplex is the sum of its $(p-1)$-dimensional faces.
- $\sigma = [u_0, \cdots, u_p]$
- $\partial\sigma = \sum_{j=0}^{p}[u_i, \cdots, \hat{u}_j, \cdots, u_p]$, where $\hat{u}_j$ means omitting u_j.



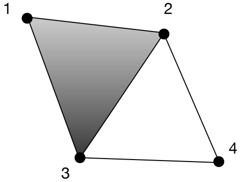
Example: $c = 12 + 23 + 34$.
What is ∂c? What is $\partial^2(c)$?

Fundamental lemma of homology

Lemma (Fundamental lemma of homology)

$$\partial_p \partial_{p+1} d = 0$$

For every integer p and every $(p+1)$-chain d.



Check $\partial^2(123) = 0$.

Homomorphism (in our context)

- A *homomorphism* is a map between groups that commutes with the group operation, that is, $f : A \to B$ for groups $(A, \bullet)$ and $(B, \circ)$, we have $f(A \bullet B) = f(A) \circ f(B)$.
- The boundary map $\partial_p : \mathsf{C}_p \to \mathsf{C}_{p-1}$ is a homomorphism.
- A *chain complex* is the sequence of chain groups connected by boundary homomorphisms,

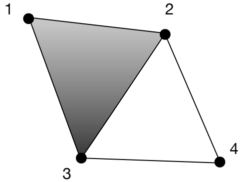
$$\cdots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \cdots$$

Cycles

- A *p-cycle* is a p-chain with empty boundary, $\partial c = 0$.
- A *group of p-cycles* is $\mathsf{Z}_p = \mathsf{Z}_p(K)$, which is a subgroup of C_p.

$$\mathsf{Z}_p = \ker\ \partial_p$$

- The boundary of every vertex is 0: $\mathsf{C}_{-1} = 0$; $\mathsf{Z}_0 = \ker \partial_0 = \mathsf{C}_0$.
- For $p > 0$, usually $\mathsf{Z}_p \neq \mathsf{C}_p$.

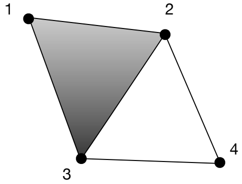


Check: $c = 24 + 23 + 34$ is a 1-cycle.

Boundaries

- A *p-boundary* is a p-chain that is the boundary of a $(p+1)$-chain, $c = \partial d$ with $d \in \mathsf{C}_{p+1}$.
- E.g., a 1-boundary is a 1-chain that is the boundary of a 2-chain.
- $c = 12 + 23 + 13$ "bounds something".
- $c = 23 + 34 + 24$ "bounds nothing".
- A group of p-boundaries form the *group of p-boundaries*, $\mathsf{B}_p = \mathsf{B}_p(K)$,

$$B_p = \mathrm{im}\partial_{p+1}$$

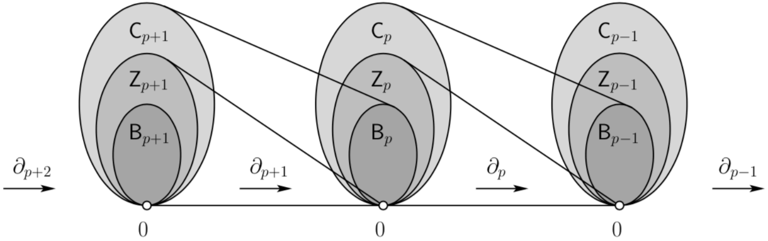


$c = 12 + 23 + 13$ is a 1-boundary.

Homology groups

- The *p-th homology group* is the p-th cycle group modulo the p-th boundary group.

$$\mathsf{H}_p = \mathsf{Z}_p / \mathsf{B}_p.$$



- $\mathsf{C}_p, \mathsf{Z}_p, \mathsf{B}_p$ are all abelian groups.
- Observe their subgroup relations.
- "Cycles that do not bound".

Edelsbrunner and Harer (2010), page 81

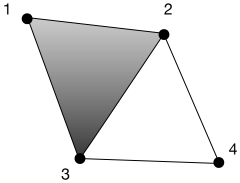
Homology groups

- The *p-th homology group* is the p-th cycle group modulo the p-th boundary group.

$$\mathsf{H}_p = \mathsf{Z}_p / \mathsf{B}_p$$

- The *p-th Betti number* is the rank of H_p

$$\beta_p = \text{rank } \mathsf{H}_p$$

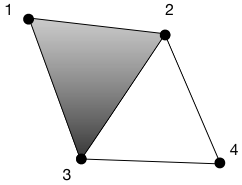


- $c = 23 + 34 + 24$, $c \in \mathsf{H}_1$
- $c' = 12 + 24 + 34 + 13$, $c' \in \mathsf{H}_1$
- Compute H_1? Naively, looking at all cycles that do not bound.

Rank of a group

- If S is a subset of a group G, then $\langle S \rangle$ is the subgroup of all elements of G that can be expressed as the finite product of elements in S and their inverses. The elements in $\langle S \rangle$ are *generators*.
- If S is finite, then a group $G = \langle S \rangle$ is called finitely generated.
- The *rank* of a group G, $\mathrm{rank}(G)$, is the smallest cardinality of a generating set for G, that is

$$\mathrm{rank}(G) = \min\{|X| : X \subseteq G, \langle X \rangle = G\}.$$



$$\mathrm{rank}(\mathsf{H}_1) = 1.$$

Prepare for Project 1: Learn to Use Ripser

https://github.com/Ripser/ripser

References I

Edelsbrunner, H. and Harer, J. (2010). *Computational Topology: An Introduction*. American Mathematical Society, Providence, RI, USA.