Dr. Bei Wang

School of Computing
Scientific Computing and Imaging Institute (SCI)
University of Utah
www.sci.utah.edu/~beiwang
beiwang@sci.utah.edu

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Topological Spaces

Book Chapter A.I.
Definition (Topological space)

A topological space is an ordered pair \((X, U)\), where \(X\) is a set, \(U\) is a collection of subsets of \(X\), such that:

- The empty set and \(X\) itself belong to \(U\).
- Any arbitrary (finite or infinite) union of members of \(U\) still belongs to \(U\).
- The intersection of any finite number of members of \(U\) still belongs to \(U\).

The elements of \(U\) are called open sets and the collection \(U\) is called a topology on \(X\).

- The pair \((X, U)\) is called a topological space.
- We usually assume that \(U\) is understood, and refer to \(X\) as a topological space.
Topological spaces

Topologies on a three-point set.

https://commons.wikimedia.org/wiki/File:Topological_space_examples.svg
An open set is an abstract concept generalizing the idea of an open interval in the real line.

A set is open if it doesn’t contain any of its boundary points.

Any collection of sets can be called open, as long as the union of an arbitrary number of open sets is open, the intersection of a finite number of open sets is open, and the space itself is open.

A subset $U \subset \mathbb{R}^n$ is called open if given any point $x \in U$, there exists a real number $\epsilon > 0$ such that for all points $y \in \mathbb{R}^n$ such that $\text{dist}(x, y) < \epsilon$, then $y \in U$.

A closed set if a set whose complement is an open set.
Examples of topological spaces \((X, U)\):

- \(X = \{1, 2, 3\}, U = \{\emptyset, \{1, 2, 3\}\}\). \(U\) is a (trivial) topology on \(X\).
- \(X = \{1, 2, 3\}, U\) is the power set of \(X\).
- \(X\) is the real line \(\mathbb{R}^1\). \(U\) is the set of all open intervals on the real line \(\mathbb{R}^1\).
- We refer to \(X\) as a topological space assuming \(U\) is understood.
A continuous function is a function for which sufficiently small changes in the input result in arbitrarily small changes in the output [wikipedia].

A function \( f : X \rightarrow Y \) is *continuous* if the pre-image of every open set is open.

Example of a discontinuous function, \( f : \mathbb{R} \rightarrow \mathbb{R} \),

\[
f(x) = \begin{cases} 
0, & \text{if } x \leq 0; \\
1, & \text{otherwise.} 
\end{cases}
\]

For any \( 0 < \epsilon < 1 \), \((-\epsilon, +\epsilon)\) is open, \( f^{-1}((-\epsilon, +\epsilon))\) is not.
Connectivity of a topological space

Definition (Path connectivity)
A topological space is *path-connected* if every pair of points is connected by a path.

- A function between topological spaces \( f : X \rightarrow Y \) is *continuous* if the preimage of every open set is open.
- A *path* is a continuous function from the unit interval \( \gamma : [0, 1] \rightarrow X \).
A separation of a topological space $\mathbb{X}$ is a partition $\mathbb{X} = U \cup W$ into two non-empty, open subsets that are disjoint. A topological space is connected if it has no separation.

- Connectedness is strictly weaker than path-connectedness.
- A topological space that is connected but is not path-connected: topologist’s sine curve: $T = \{(x, \sin(\frac{1}{x})) : x \in (0, 1]\} \cup \{(0, 0)\}$
Union-Find

Book Chapter A.I.
Given a set of elements which are partitioned into subsets, you have to keep track of the connectivity of each element in a particular subset or connectivity of subsets with each other.

To do this operation efficiently, you can use union-find data structure. Galler and Fischer (1964); Galil and Italiano (1991)

Algorithmic question of deciding connectedness

- Union-find: a disjoint-set data structure.
- A data structure that tracks a set of elements partitioned into a number of disjoint (non-overlapping) subsets.
- Represent each set as a tree of elements (thinking grapes).
- Maintain a collection of sets under three operations (book notations):
  - \textbf{MakeSet}(x): Create a set containing a single element \( x \).
  - \textbf{Find}(x): Return the root of the tree containing \( x \).
  - \textbf{Union}(x,y): Make the root of tree containing \( x \) to also be the root of tree containing \( y \).
There are 5 people A, B, C, D and E. A is a friend of B, B is a friend of C, D is a friend of E. As we can see:

- A, B and C are connected to each other.
- D and E are connected to each other.

Use union-find to check whether one friend is connected to another in a direct or indirect way or not.

We can also determine the two different disconnected subsets: \{A, B, C\} and \{D, E\}.

You have to perform two types of operations here:

- **Union (A, B):** connect two elements A and B.
- **FindPath(C, D):** find if there is any path connecting two elements C and D.
