Topological Data Analysis for Vector Fields

The Robustness

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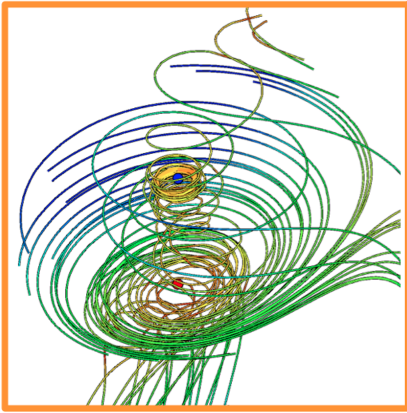
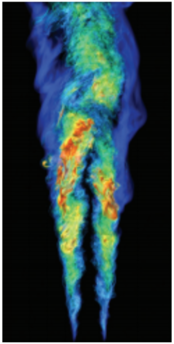
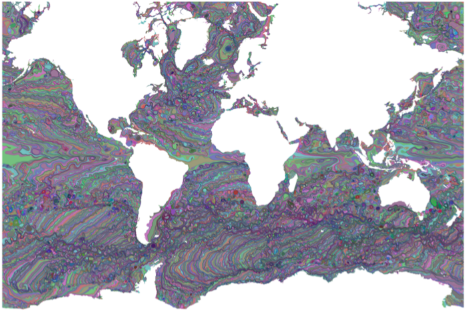
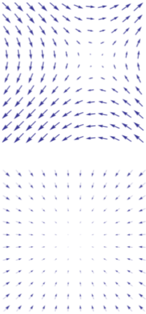


Robust Feature Extraction and Visualization of Vector Fields

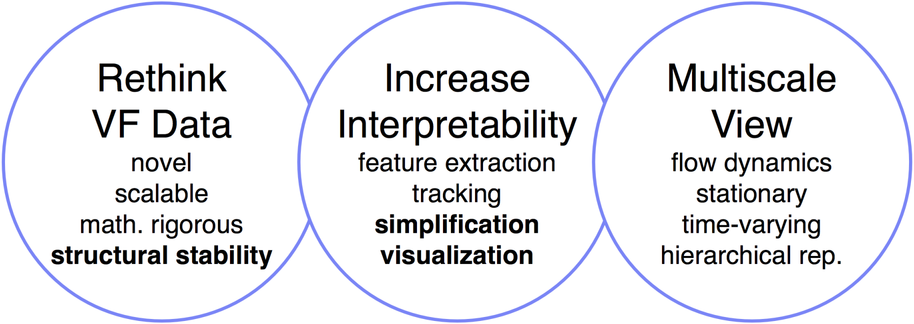
Understanding VF is indispensable for many applications

- Turbulence combustion, global oceanic eddies simulations, etc.
- A d-dim VF: a function that assigns to each point a d-dim vector
- $f : S \subset \mathbb{R}^d \to \mathbb{R}^d$, $d = 2$ or 3
- Critical point x: $f(x) = 0$

[Yu, Wang, Grout, Chen, Ma 2010] [Maltrud, Bryan, Peacock, 2010] [Levine, Jadhav, Bhatia, Pascucci, Bremer, 2012]

VF analysis and visualization

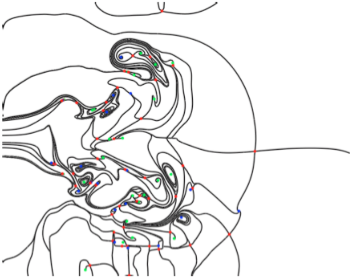


Simplifying 2D VF: independent of topological skeleton

First 3D VF simplification based on critical point cancellation

VF simplification

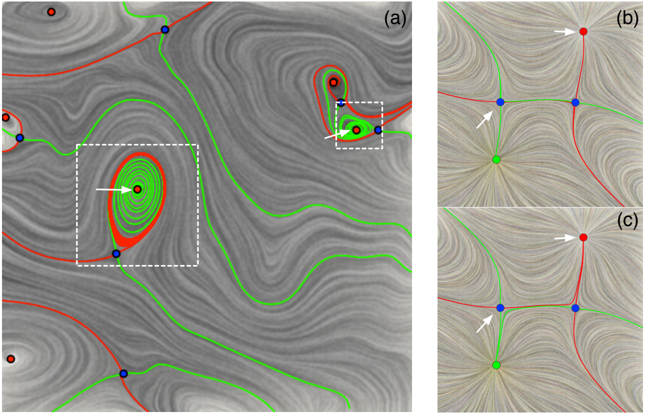
- Prior work: canceling nearby critical points based on topological skeleton: critical points connected by separatrices that divide domain into regions of uniform flow behavior
- Preserve important scientific properties of the data
- Obtain compact representation for interpretation
- Derive multi-scale view of the flow dynamics



Swirling jet simulation [Tricoche, Scheuermann, Hagen 2001]

Challenges with prior work

Topological skeleton can be unstable due to numerical instability



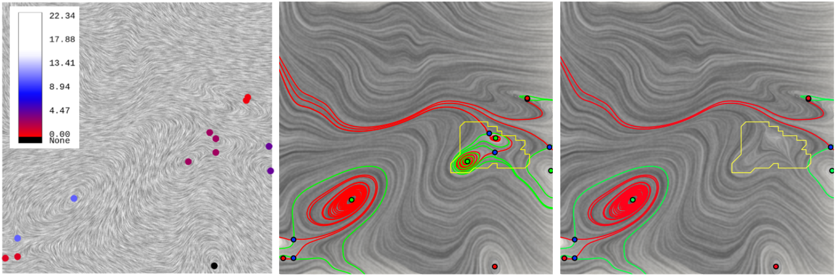
(a) Highly rotational flow, near Hopf bifurcations:
diff separatrices intersect/switch.

(b-c) Separatrices are unstable w.r.t perturbations.

Sink, saddle-sink, saddle, source, saddle-source

Contributions: Robustness-based simplification

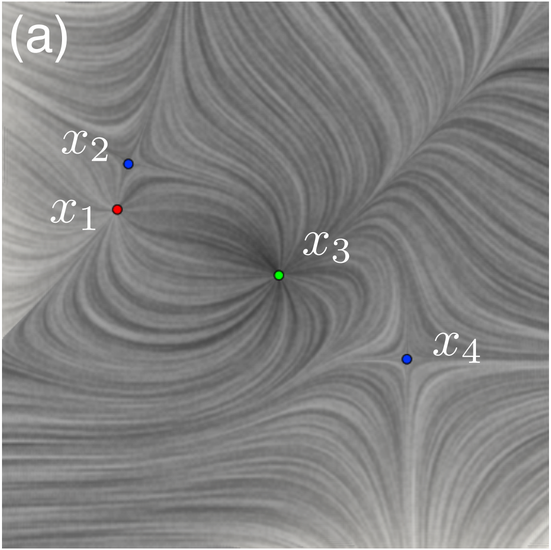
- Canceling critical points based on stability measured by robustness
- Complementary view, independent of topological skeleton
- Efficient computation for large data, avoid numerical integration
- Handle complex boundary configurations
- Analysis generalizes to higher dimensions



Robustness-based simplification in a nutshell

- In the space of all VFs, find the one closest to the original VF with a particular set of critical points removed, bases on the L_∞ norm
- Results are optimal: no other simplification with a smaller perturbation

Some teaser results: synthetic A



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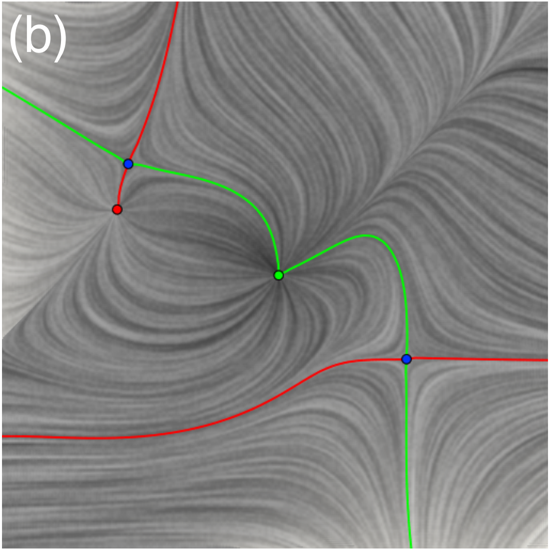

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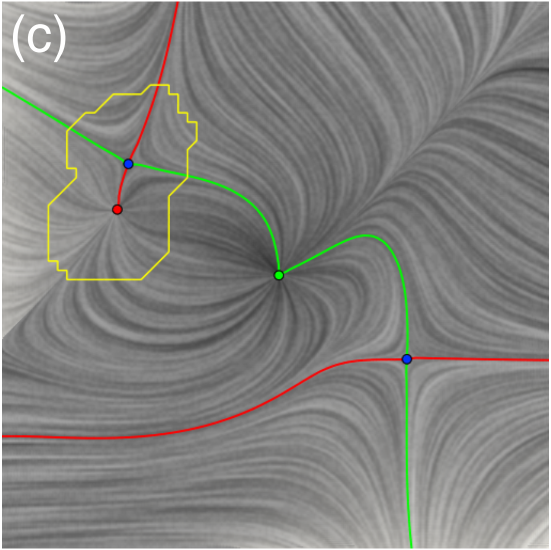

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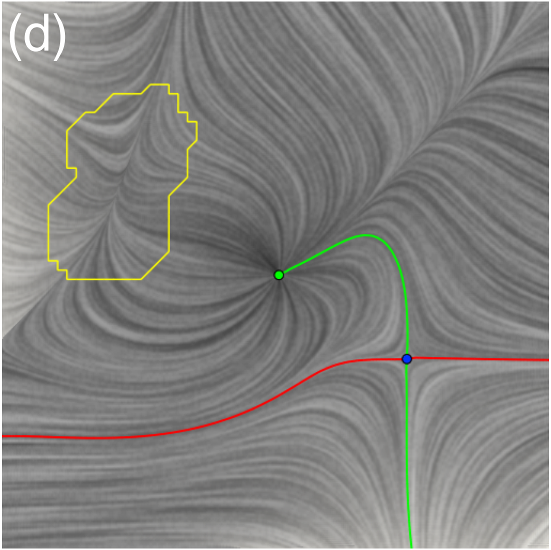

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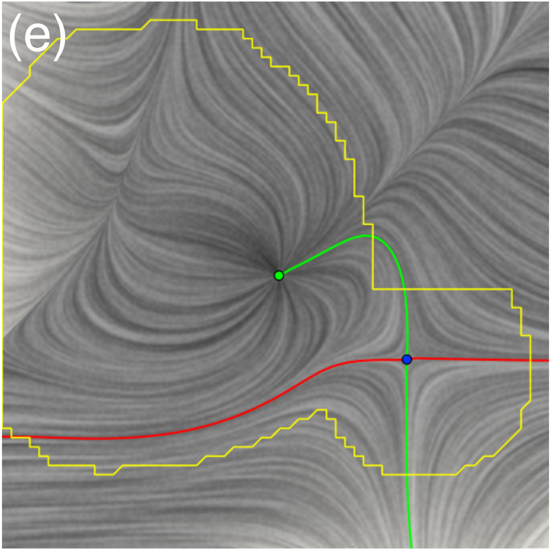

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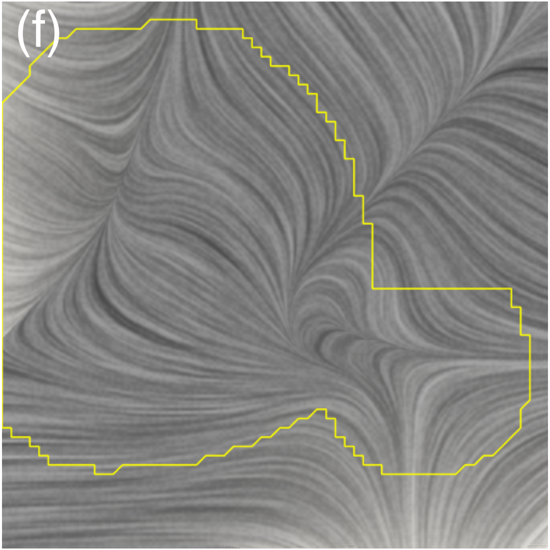
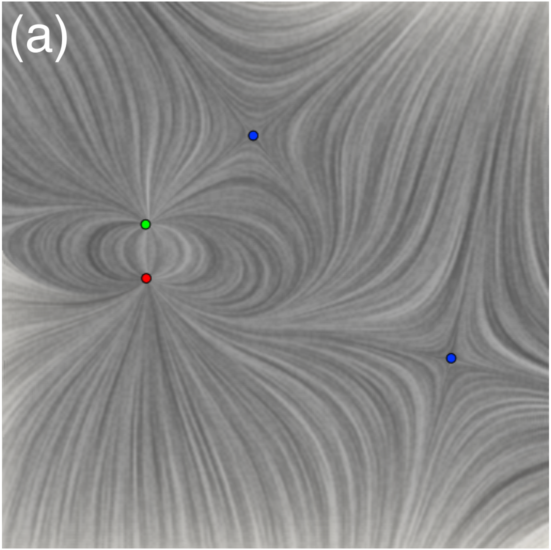


Some teaser results: synthetic B



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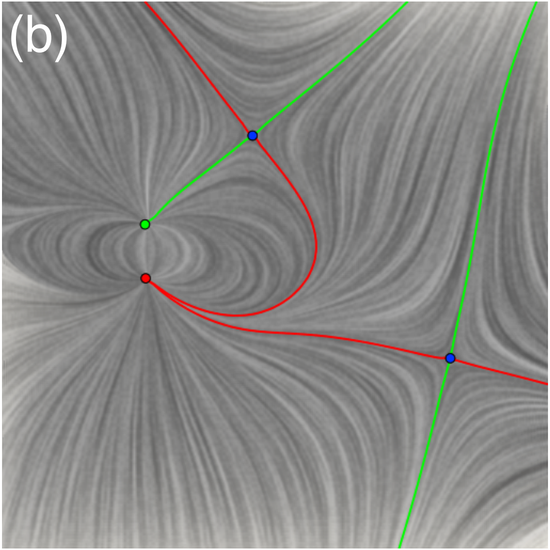

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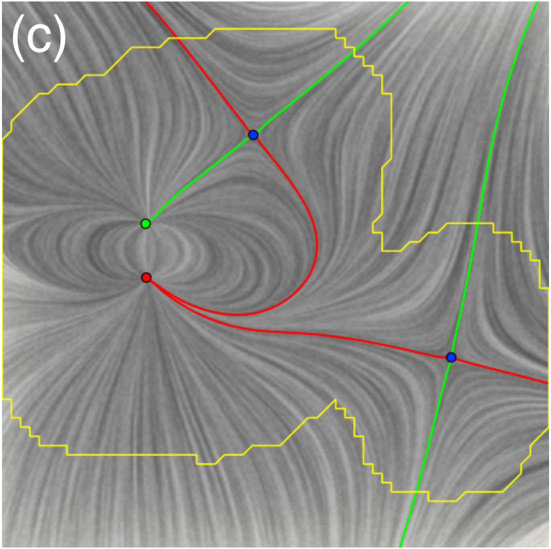

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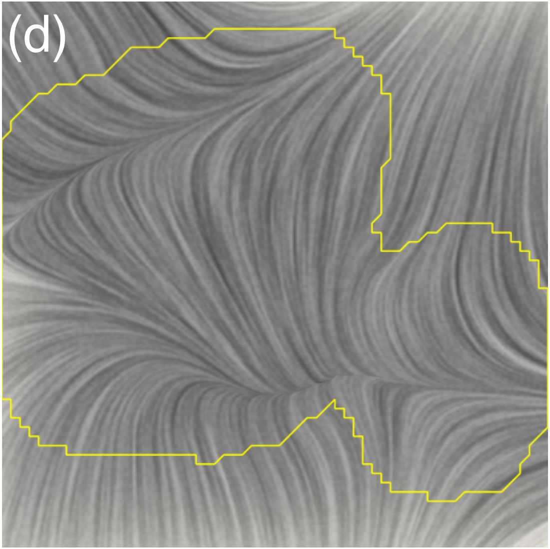
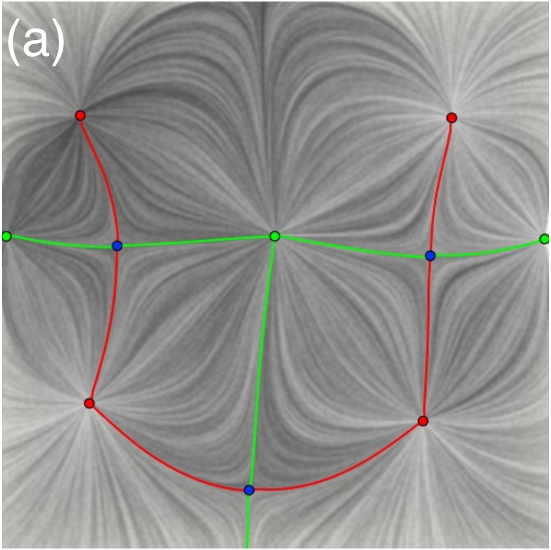


Some teaser results: synthetic C



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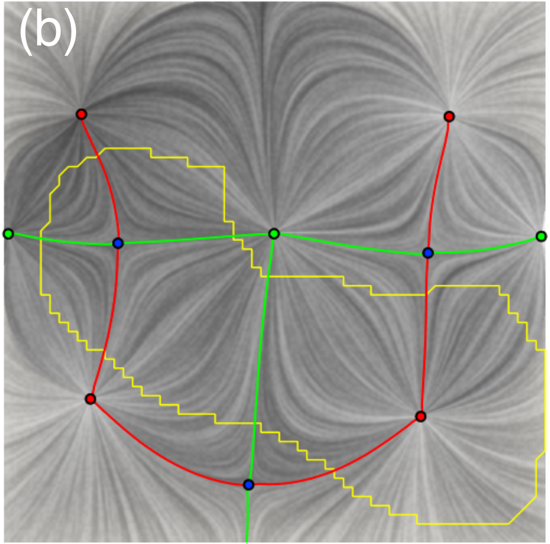

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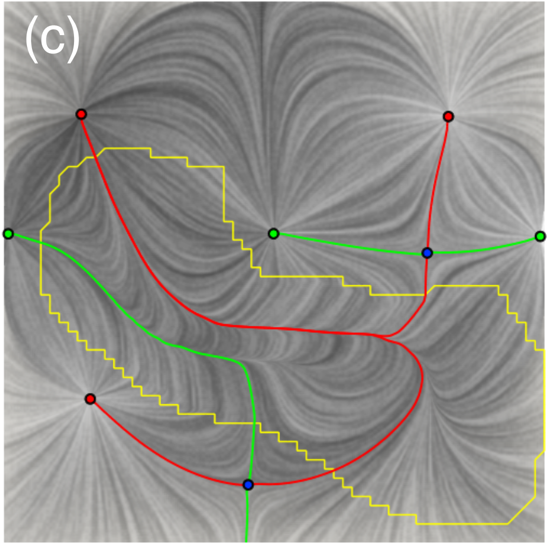
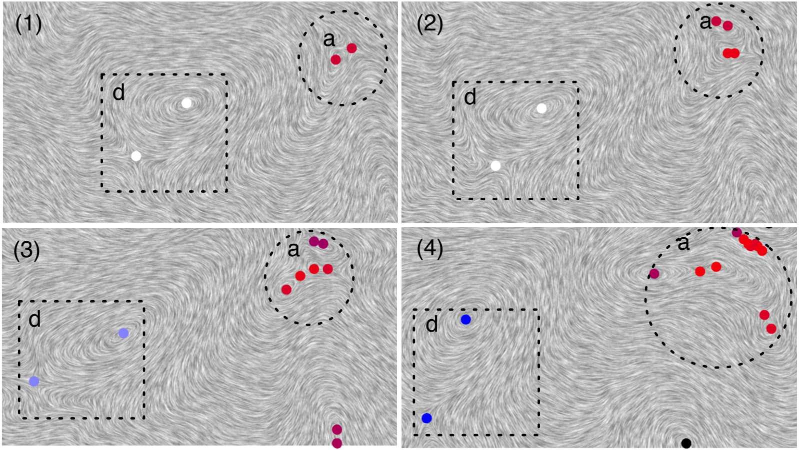


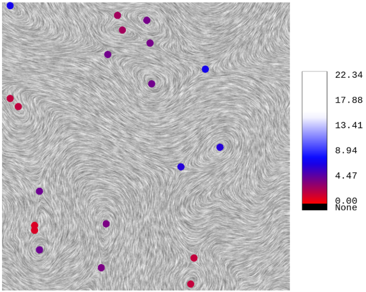
Visualizing Robustness of Critical Points



Critical points clustered by robustness for time-varying ocean eddie simulation
[**Wang**, Rosen, Skraba, Bhatia and Pascucci (EuroVis) 2013]

Robustness of critical points

- Robustness: quantify the stability of critical points
- Intuitively, the robustness of a critical point is the minimum amount of perturbation necessary to cancel it within a local neighborhood
- Well group theory
- [Edelsbrunner, Morozov and Patel 2010, 2011], [Chazal, Patel and Skraba 2012].
- Robustness computation: based on degree theory and merge tree

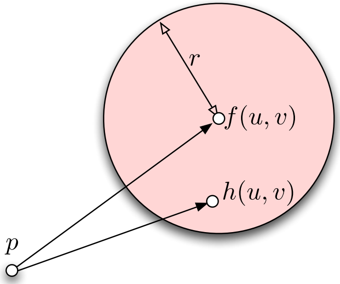


r-perturbation: L_∞-norm of the VF

Let $f, h : \mathbb{R}^2 \to \mathbb{R}^2$ be two continuous 2D vector fields. Define the distance between the two mapping as

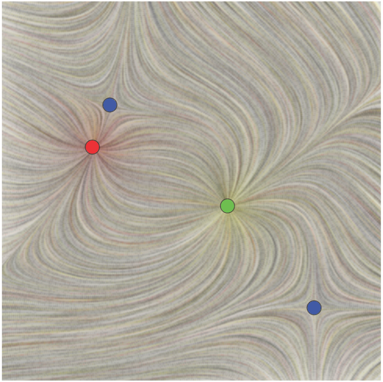
$$\mathrm{d}(f,h) = \sup_{x \in \mathbb{R}^2} ||f(x) - h(x)||_2.$$

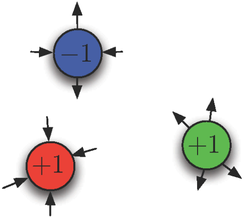
We say h is an r-perturbation of f, if $\mathrm{d}(f,h) \leq r$.



Degrees

- In 2D, $\deg(x)$ of a critical point x equals its Poincaré index.
- Source $+1$, sink $+1$, saddle -1.
- A connected component C, $\deg(C) = \sum_i \deg(x_i)$.
- Corollary of Poincaré-Hopf thm: if C in $\mathbb{R}^2$ has degree zero, then it is possible to replace the VF inside C with a VF free of critical points





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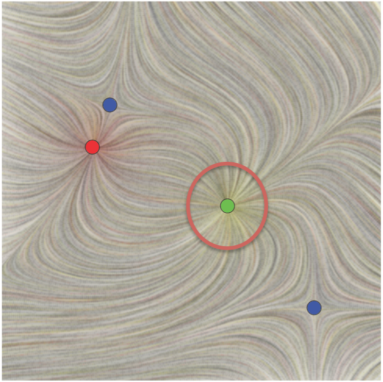

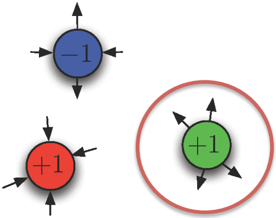

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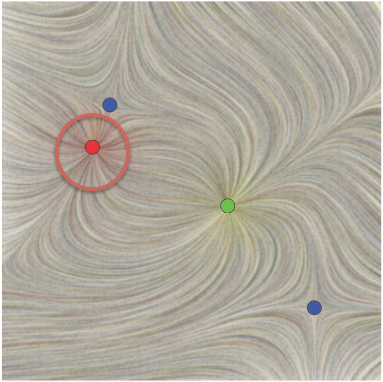

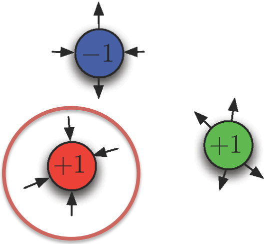

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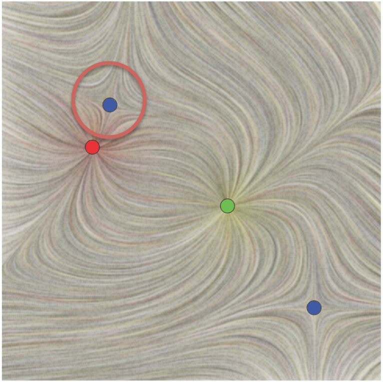

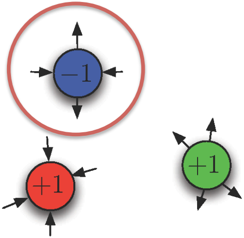

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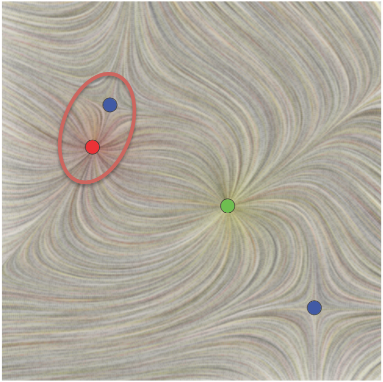
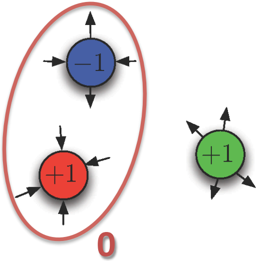




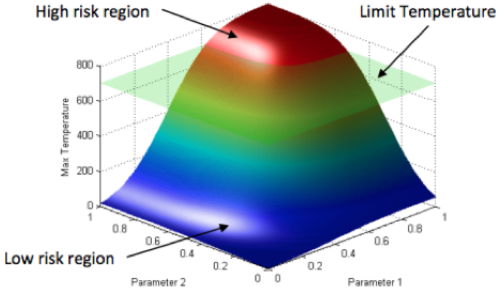
Sublevel set

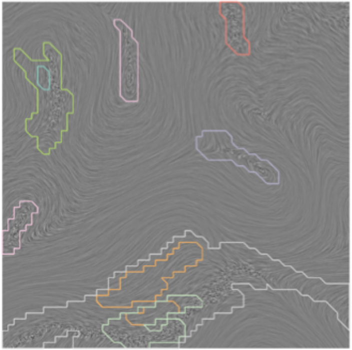
Given $f : \mathbb{R}^2 \to \mathbb{R}^2$, define its norm (speed of flow) $f_0 : \mathbb{R}^2 \to \mathbb{R}$ as

$$f_0(x) = ||f(x)||_2$$

For some $r \geq 0$, define the sublevel set of f_0 as

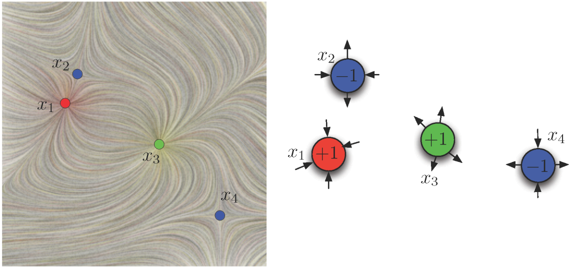
$$\mathbb{F}_r = f_0^{-1}[0, r].$$





Merge tree of f_0

Track components of $\mathbb{F}_r$ as they appear and merge, as r increases from 0



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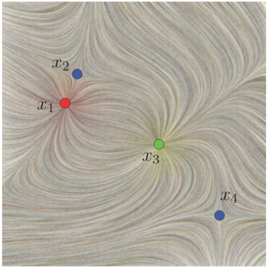

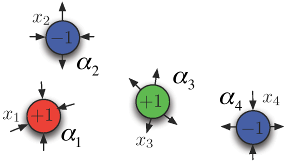

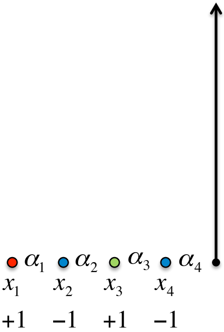

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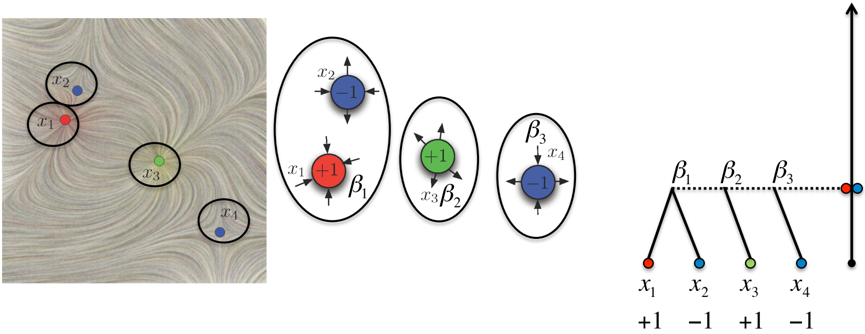

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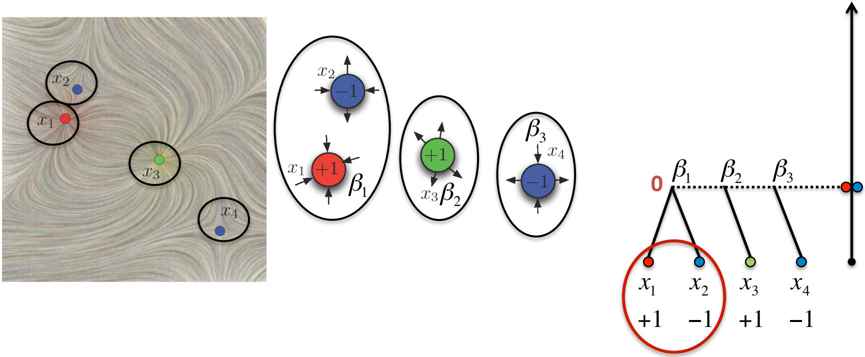

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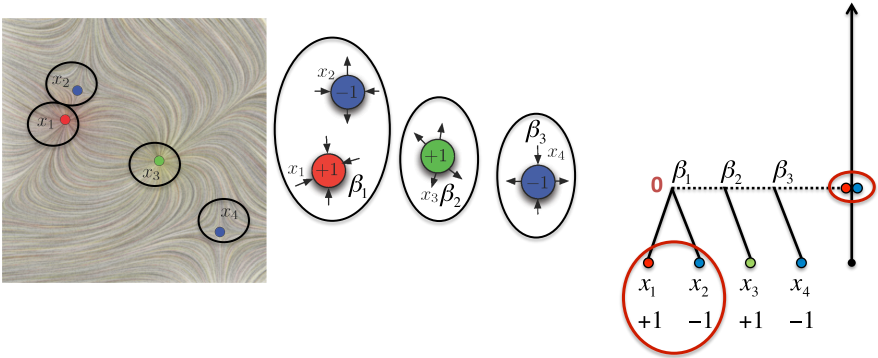

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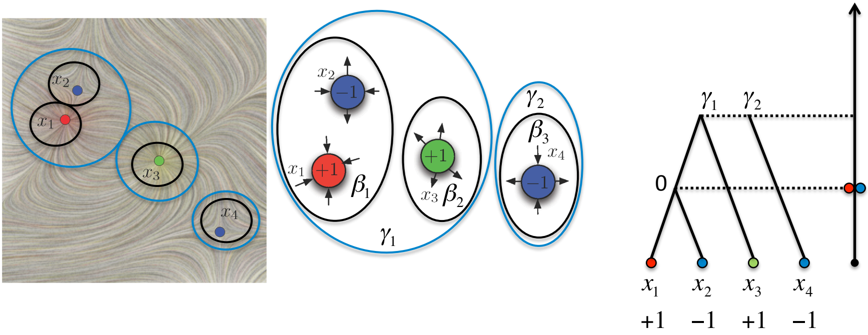

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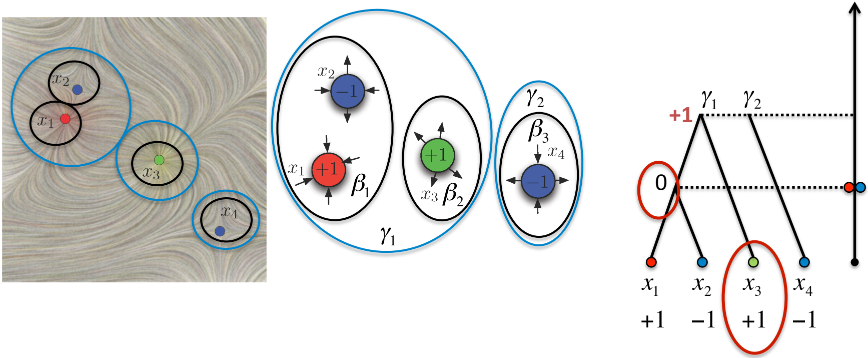

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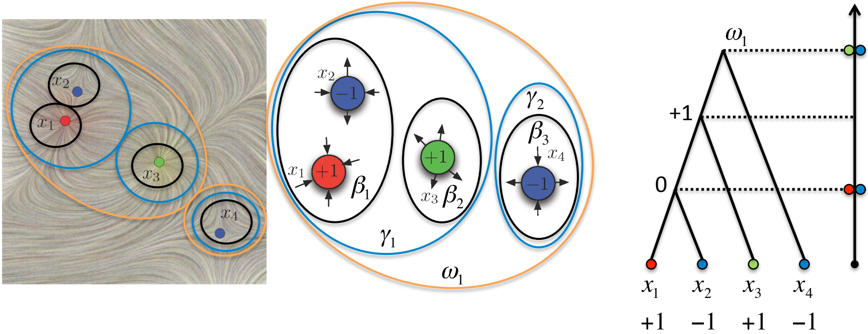

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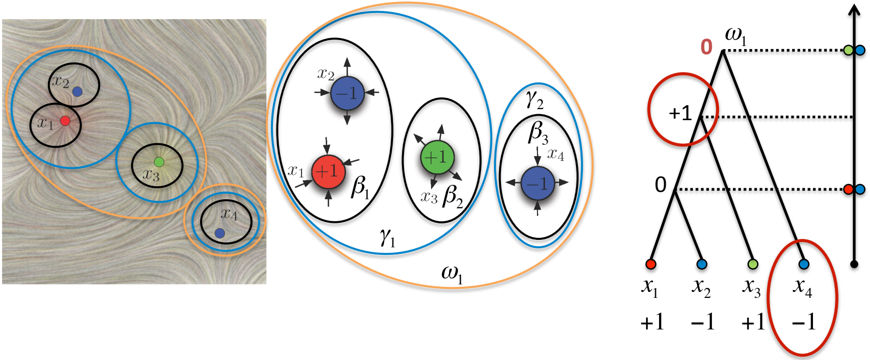

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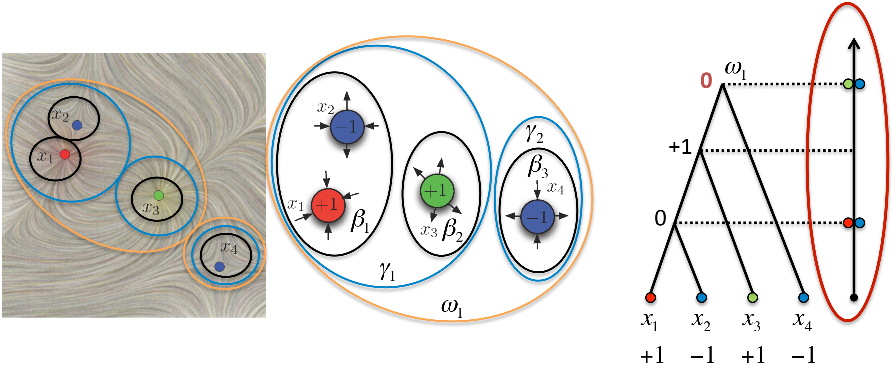
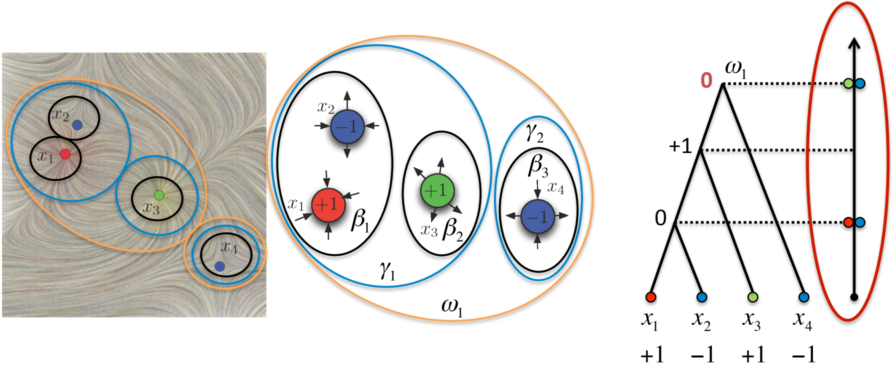


Merge tree of f_0 and robustness

The robustness of a critical point is the height of its lowest degree zero ancestor in the merge tree. [Chazal, Patel, Skraba 2012]
Interpretation: robustness is the min amount of perturbation necessary to cancel a critical point.



Robustness: $\mathrm{rb}(x_1) = \mathrm{rb}(x_2)$, $\mathrm{rb}(x_3) = \mathrm{rb}(x_4)$.

Well group

[Edelsbrunner, Morozov and Patel 2010, 2011], [Chazal, Patel and Skraba 2012]
Suppose h is an r-perturbation of f.
$\mathbb{H}_0 = h^{-1}(0)$ is the set of critical points of h. We have inclusion:

$$i : \mathbb{H}_0 \to \mathbb{F}_r$$

i induces linear map:

$$j_h : \mathsf{H}(\mathbb{H}_0) \to \mathsf{H}(\mathbb{F}_r)$$

The well group, $\mathsf{U}(r)$, is the subgroup of $\mathsf{H}(\mathbb{F}_r)$, whose elements belong to the image of each j_h, for all r-perturbation h of f:

$$\mathsf{U}(r) = \bigcap_h \operatorname{im} j_h$$

Intuitively, an element in $\mathsf{U}(r)$ is considered a stable element in $\mathsf{H}(\mathbb{F}_r)$ if it does not disappear with respect to any r-perturbation.

Robustness Properties

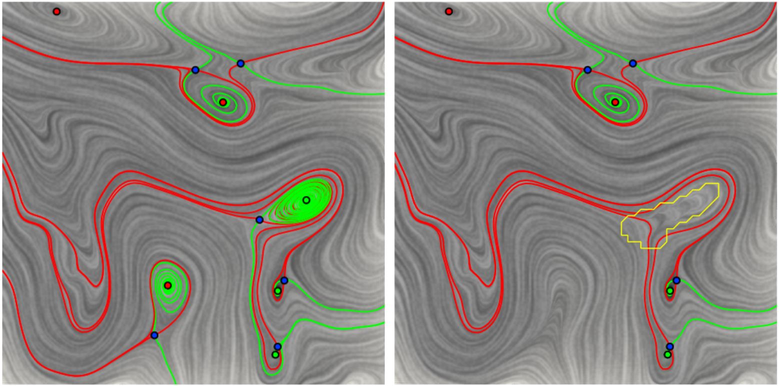
Robustness quantifies the stability of a critical point w.r.t. perturbations of the VFs.

If a critical point x has a robustness r:

- Need $(r + \delta)$-perturbation to cancel x, for arbitrarily small $\delta > 0$
- Any $(r - \delta)$-perturbation is not enough to cancel x.

Visualizing robustness: Video, combustion simulation

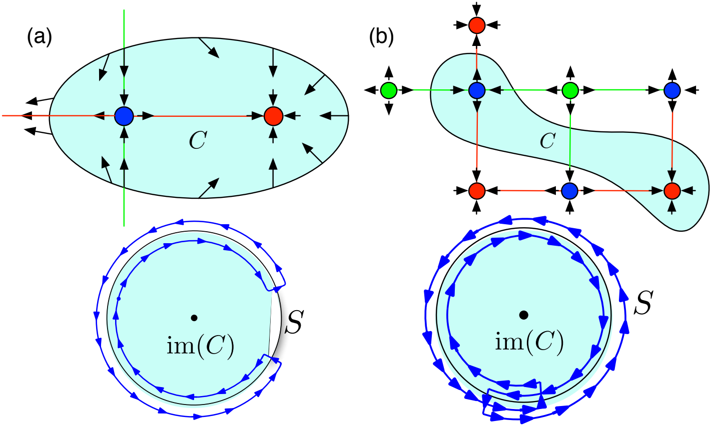
2D VF Simplification Based on Robustness



[Skraba, **Wang**, Chen and Rosen (PacificVis Best Paper) 2014]
[Skraba, **Wang**, Chen and Rosen (TVCG) 2015]

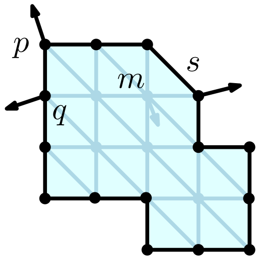
Image space $\mathrm{im}(C)$ of a zero-degree component $C \subseteq \mathbb{F}_r$

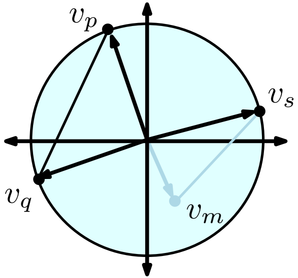
- Map each vector in C to its vector coordinates
- Critical points map to the origin of $\mathrm{im}(C)$
- $\mathrm{im}(C)$ is part of a disk of radius r, whose boundary S could be uncovered/covered.



PL Image space

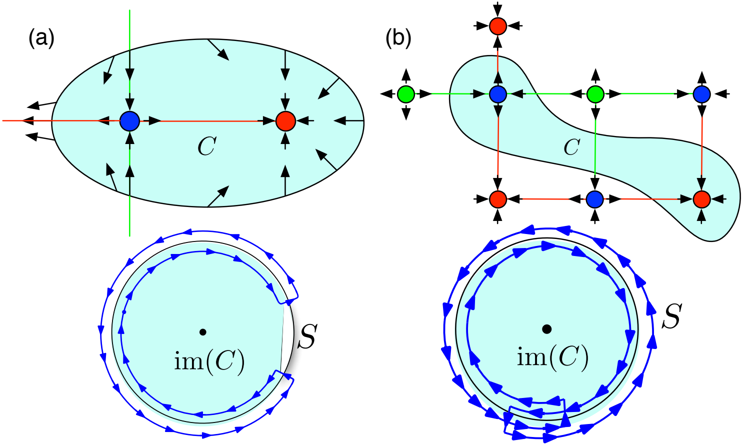
$f : K \to \mathbb{R}^2$, K is a triangulation of C
Linear interpolation: edges and triangles in K map to thoese in $\mathrm{im}\,(C)$.





Simplification: Key ideas

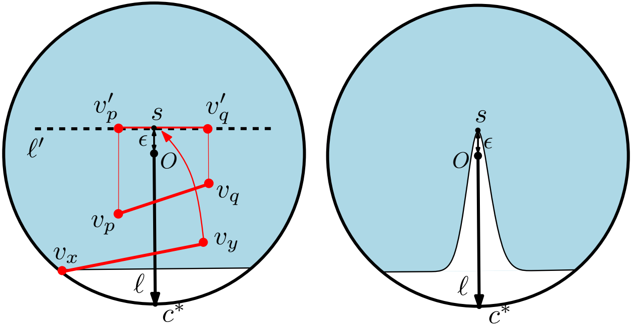
- A region contains critical points if its image space contains the origin
- Simplification: deform the VF to create a void surrounding the origin
- Simple boundary: boundary of $\mathrm{im}(C)$ is uncovered
- Complex boundary: boundary of $\mathrm{im}(C)$ is covered



Cut: Create a void surrounding the origin

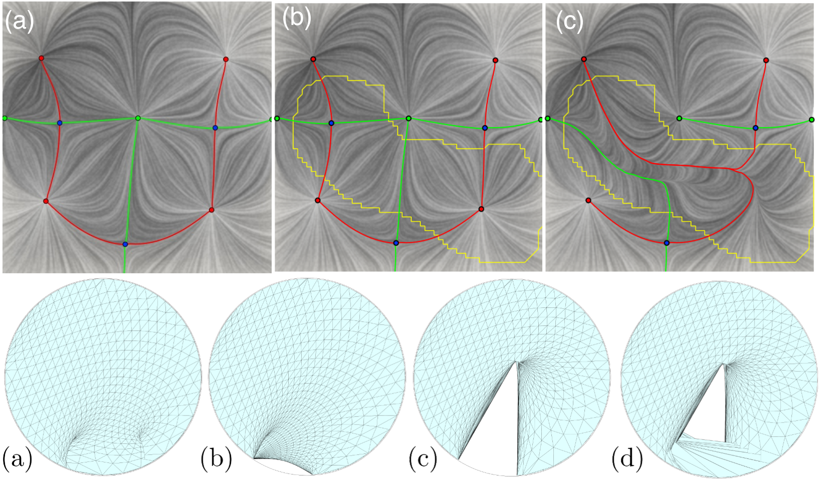
Deform $\mathrm{im}(C)$ to create a void surrounding the origin.

c^*: cut point



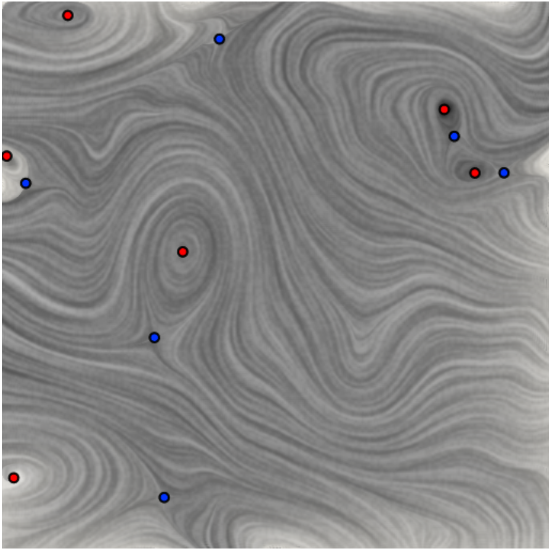
By construction: amount of perturbation $< r + \epsilon$

Example revisited: Synthetic C complex boundary

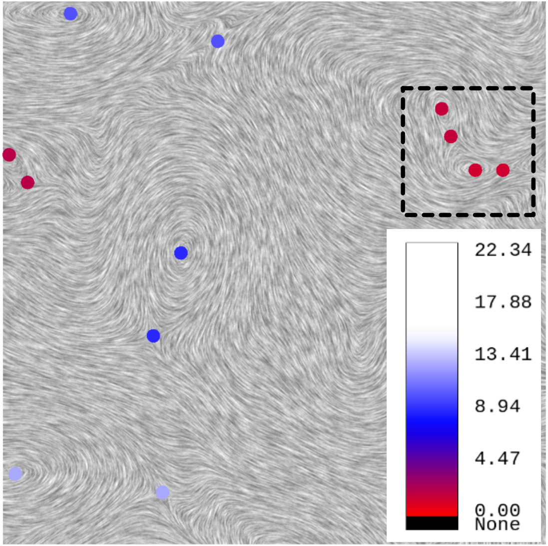


(a) original, (b) after **Unwrap**, (c) after **Cut** and (d) final output after **Restore**

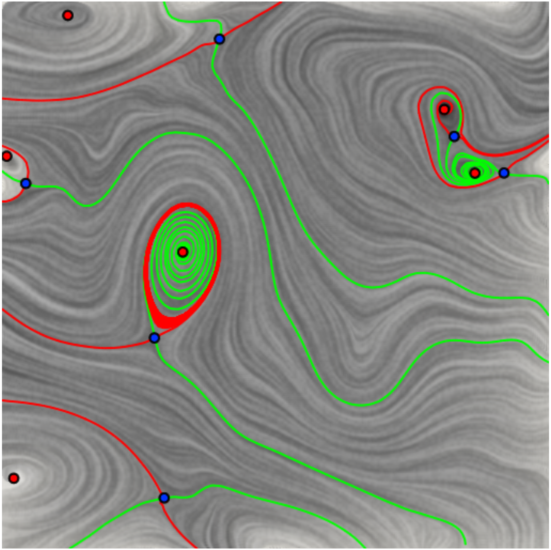
Ocean eddie simulation



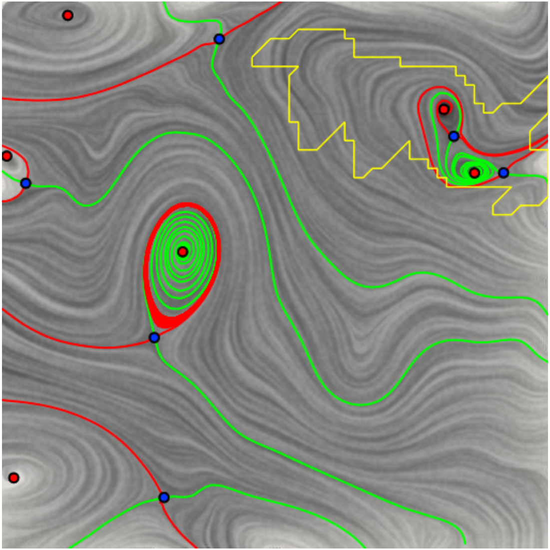
Ocean eddie simulation



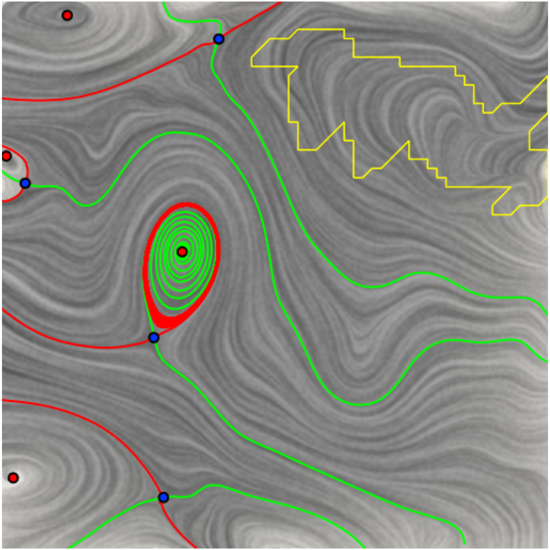
Ocean eddie simulation



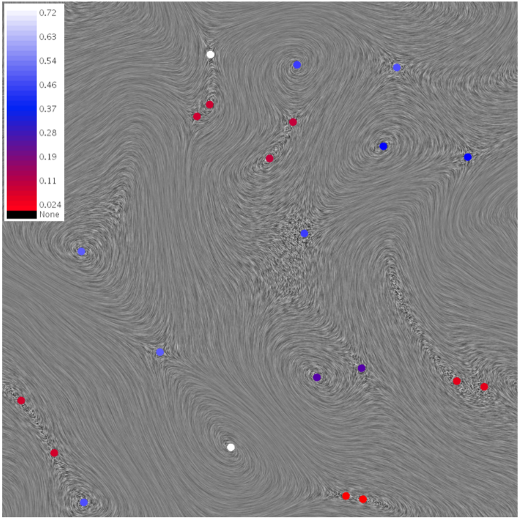
Ocean eddie simulation



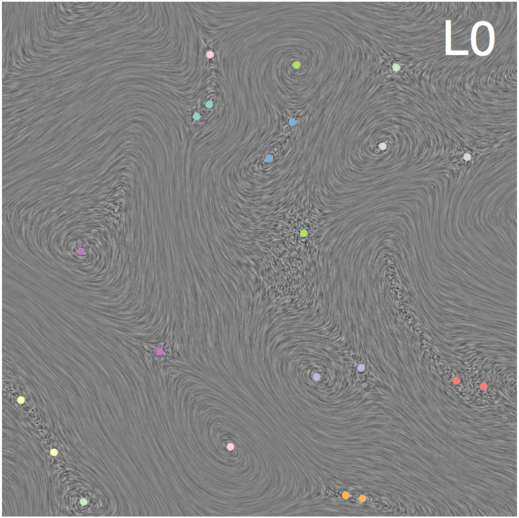
Ocean eddie simulation



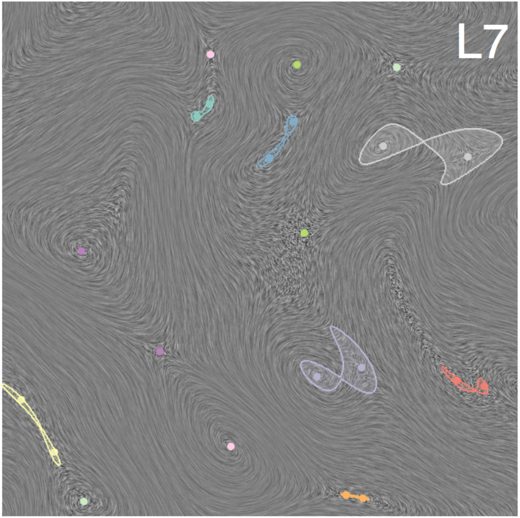
Combustion simulation: Hierarchical simplification



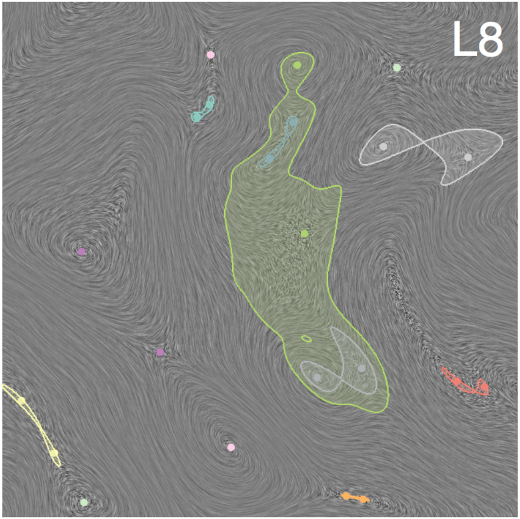
Combustion simulation: Hierarchical simplification



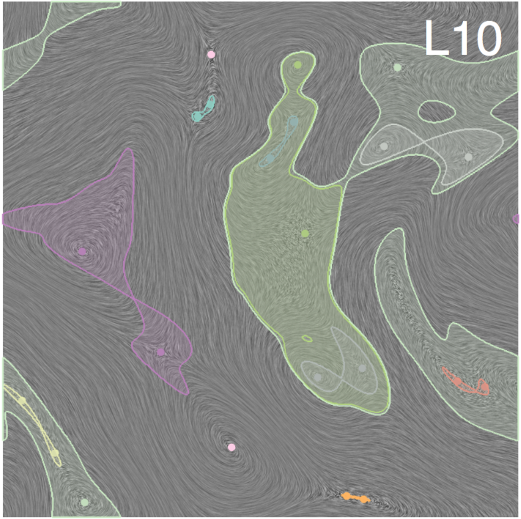
Combustion simulation: Hierarchical simplification



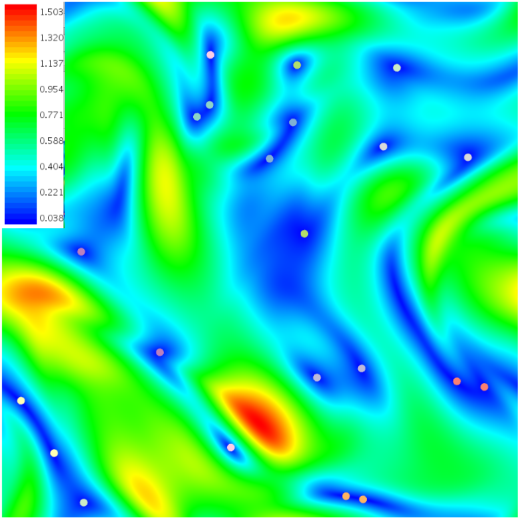
Combustion simulation: Hierarchical simplification



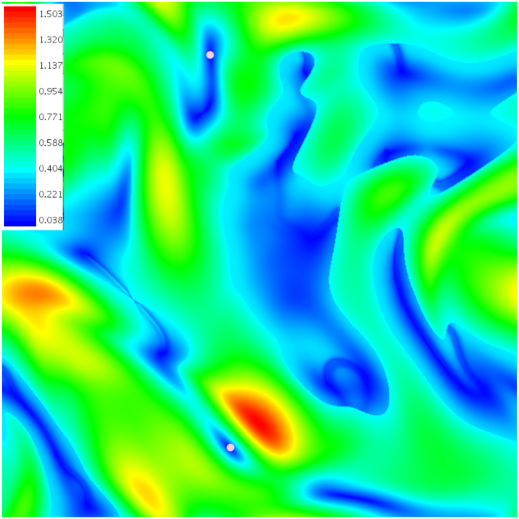
Combustion simulation: Hierarchical simplification



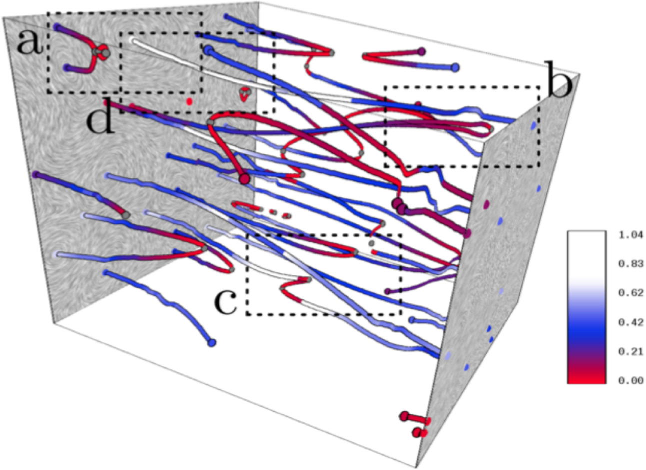
Combustion simulation: Hierarchical simplification



Combustion simulation: Hierarchical simplification



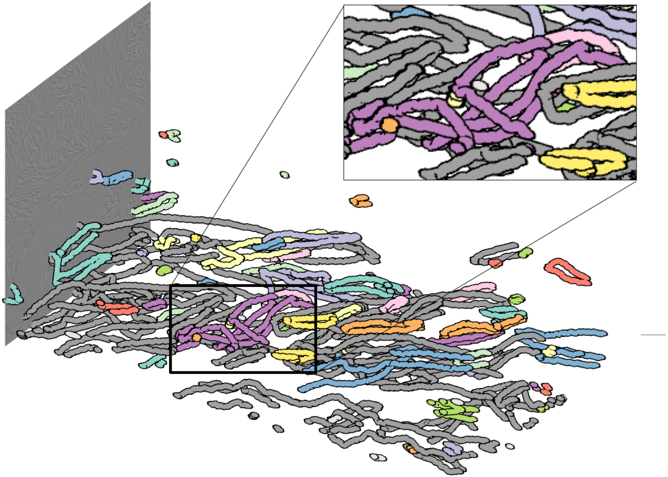
Feature Tracking for 2D Time-Varying VF



Stable critical points could provably be tracked more easily and more accurately in the time-varying setting.

[Skraba, **Wang** (TopoInVis/Book Chapter), 2014][a]

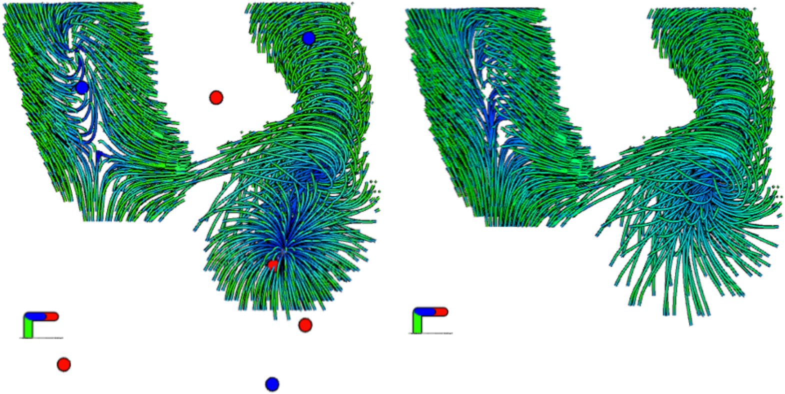
Simplifying 2D Time-Varying VF



[Skraba, **Wang**, Chen, Rosen (TVCG), 2015]

2D Time-varying VF simplification: Video

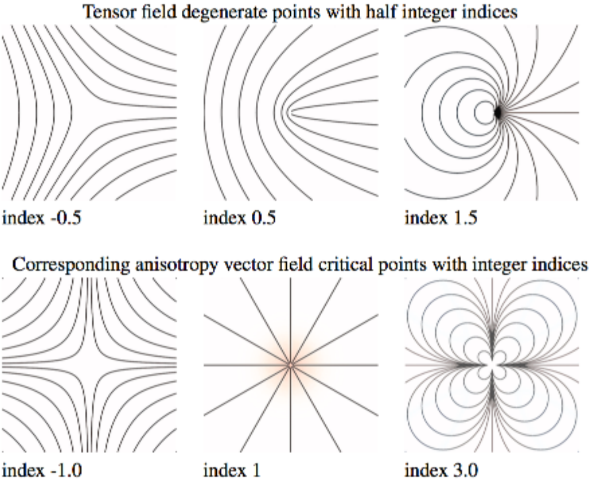
Simplifying 3D VF



[Skraba, Rosen, **Wang**, Chen, Bhatia, Pascucci (PacificVis/TVCG), 2016]

3D VF simplification

Next? Tensor field simplification, stress tensor, DTI...



[**Wang**, Hotz, 2017]