Topological Data Analysis for Vector Fields
The Robustness

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Robust Feature Extraction and Visualization of Vector Fields
Understanding VF is indispensable for many applications

- Turbulence combustion, global oceanic eddies simulations, etc.
- A $d$-dim VF: a function that assigns to each point a $d$-dim vector
- $f : S \subset \mathbb{R}^d \to \mathbb{R}^d, \ d = 2 \text{ or } 3$
- Critical point $x$: $f(x) = 0$

Simplifying 2D VF: independent of topological skeleton
First 3D VF simplification based on critical point cancellation
VF simplification

- Prior work: canceling nearby critical points based on topological skeleton: critical points connected by separatrices that divide domain into regions of uniform flow behavior
- Preserve important scientific properties of the data
- Obtain compact representation for interpretation
- Derive multi-scale view of the flow dynamics

Swirling jet simulation [Tricoche, Scheuermann, Hagen 2001]
Challenges with prior work

Topological skeleton can be unstable due to numerical instability

(a) Highly rotational flow, near Hopf bifurcations: diff separatrices intersect/switch.

(b-c) Separatrices are unstable w.r.t perturbations. Sink, saddle-sink, saddle, source, saddle-source
Contributions: Robustness-based simplification

- Canceling critical points based on stability measured by robustness
- Complementary view, independent of topological skeleton
- Efficient computation for large data, avoid numerical integration
- Handle complex boundary configurations
- Analysis generalizes to higher dimensions
In the space of all VFs, find the one closest to the original VF with a particular set of critical points removed, based on the $L_\infty$ norm.

Results are optimal: no other simplification with a smaller perturbation.
Some teaser results: synthetic A

(a)

\[ x_2 \]
\[ x_1 \]
\[ x_3 \]
\[ x_4 \]
Some teaser results: synthetic A
Some teaser results: synthetic A
Some teaser results: synthetic A
Some teaser results: synthetic A
Some teaser results: synthetic A
Some teaser results: synthetic B
Some teaser results: synthetic B
Some teaser results: synthetic B
Some teaser results: synthetic B
Some teaser results: synthetic C
Some teaser results: synthetic C
Some teaser results: synthetic C
Visualizing Robustness of Critical Points

Critical points clustered by robustness for time-varying ocean eddie simulation

[Wang, Rosen, Skraba, Bhatia and Pascucci (EuroVis) 2013]
Robustness of critical points

- Robustness: quantify the stability of critical points
- Intuitively, the robustness of a critical point is the minimum amount of perturbation necessary to cancel it within a local neighborhood
- Well group theory
- [Edelsbrunner, Morozov and Patel 2010, 2011], [Chazal, Patel and Skraba 2012].
- Robustness computation: based on degree theory and merge tree
Let $f, h : \mathbb{R}^2 \to \mathbb{R}^2$ be two continuous 2D vector fields. Define the distance between the two mappings as

$$d(f, h) = \sup_{x \in \mathbb{R}^2} ||f(x) - h(x)||_2.$$ 

We say $h$ is an $r$-perturbation of $f$, if $d(f, h) \leq r$. 

**Diagram**: Illustration of $f(u, v)$ and $h(u, v)$ with $p$.
In 2D, \( \text{deg}(x) \) of a critical point \( x \) equals its Poincaré index.

*Source* +1, *sink* +1, *saddle* −1.

A connected component \( C \), \( \text{deg}(C) = \sum_i \text{deg}(x_i) \).

Corollary of Poincaré-Hopf thm: if \( C \) in \( \mathbb{R}^2 \) has degree zero, then it is possible to replace the VF inside \( C \) with a VF free of critical points.
In 2D, \( \deg(x) \) of a critical point \( x \) equals its Poincaré index.

Source +1, sink +1, saddle −1.

A connected component \( C \), \( \deg(C') = \sum_i \deg(x_i) \).

Corollary of Poincaré-Hopf thm: if \( C \) in \( \mathbb{R}^2 \) has degree zero, then it is possible to replace the VF inside \( C \) with a VF free of critical points.
In 2D, \( \text{deg}(x) \) of a critical point \( x \) equals its Poincaré index.

- **Source** +1, **sink** +1, **saddle** −1.

- A connected component \( C \), \( \text{deg}(C) = \sum_i \text{deg}(x_i) \).

- Corollary of Poincaré-Hopf thm: if \( C \) in \( \mathbb{R}^2 \) has degree zero, then it is possible to replace the VF inside \( C \) with a VF free of critical points.
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- **Source** $+1$, **sink** $+1$, **saddle** $-1$.

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Corollary of Poincaré-Hopf thm: if $C$ in $\mathbb{R}^2$ has degree zero, then it is possible to replace the VF inside $C$ with a VF free of critical points.
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**Source** $+1$, **sink** $+1$, **saddle** $-1$.

A connected component $C$, $\deg(C) = \sum_i \deg(x_i)$.

Corollary of Poincaré-Hopf thm: if $C$ in $\mathbb{R}^2$ has degree zero, then it is possible to replace the VF inside $C$ with a VF free of critical points.
Sublevel set

Given $f : \mathbb{R}^2 \to \mathbb{R}^2$, define its norm (speed of flow) $f_0 : \mathbb{R}^2 \to \mathbb{R}$ as

$$f_0(x) = \|f(x)\|_2$$

For some $r \geq 0$, define the sublevel set of $f_0$ as

$$\mathbb{F}_r = f_0^{-1}[0, r].$$
Track components of $\mathbb{F}_r$ as they appear and merge, as $r$ increases from 0.
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Merge tree of $f_0$

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Track components of $F_r$ as they appear and merge, as $r$ increases from 0.
The robustness of a critical point is the height of its lowest degree zero ancestor in the merge tree. [Chazal, Patel, Skraba 2012]

Interpretation: robustness is the min amount of perturbation necessary to cancel a critical point.

Robustness: $rb(x_1) = rb(x_2), \ rb(x_3) = rb(x_4)$. 
Suppose $h$ is an $r$-perturbation of $f$. 

$\mathbb{H}_0 = h^{-1}(0)$ is the set of critical points of $h$. We have inclusion:

$$i : \mathbb{H}_0 \rightarrow \mathbb{F}_r$$

$i$ induces linear map:

$$j_h : H(\mathbb{H}_0) \rightarrow H(\mathbb{F}_r)$$

The well group, $U(r)$, is the subgroup of $H(\mathbb{F}_r)$, whose elements belong to the image of each $j_h$, for all $r$-perturbation $h$ of $f$:

$$U(r) = \bigcap_h \text{im} j_h$$

Intuitively, an element in $U(r)$ is considered a stable element in $H(\mathbb{F}_r)$ if it does not disappear with respect to any $r$-perturbation.
Robustness quantifies the stability of a critical point w.r.t. perturbations of the VFs.

If a critical point $x$ has a robustness $r$:

- Need $(r + \delta)$-perturbation to cancel $x$, for arbitrarily small $\delta > 0$
- Any $(r - \delta)$-perturbation is not enough to cancel $x$. 
Visualizing robustness: Video, combustion simulation
2D VF Simplification Based on Robustness

[Skraba, Wang, Chen and Rosen (PacificVis Best Paper) 2014]
[Skraba, Wang, Chen and Rosen (TVCG) 2015]
Map each vector in $C$ to its vector coordinates

Critical points map to the origin of $\text{im}(C)$

$\text{im}(C)$ is part of a disk of radius $r$, whose boundary $S$ could be uncovered/covered.
$f : K \rightarrow \mathbb{R}^2$, $K$ is a triangulation of $C$
Linear interpolation: edges and triangles in $K$ map to those in $\text{im} (C)$. 
Simplification: Key ideas

- A region contains critical points if its image space contains the origin
- Simplification: deform the VF to create a void surrounding the origin
- Simple boundary: boundary of $\text{im} (C')$ is uncovered
- Complex boundary: boundary of $\text{im} (C')$ is covered
Cut: Create a void surrounding the origin

Deform $\text{im}(C')$ to create a void surrounding the origin.

$c^*$: cut point

By construction: amount of perturbation $< r + \epsilon$
Example revisited: Synthetic C complex boundary

(a) original, (b) after Unwrap, (c) after Cut and (d) final output after Restore
Ocean eddie simulation
Ocean eddie simulation
Ocean eddie simulation
Ocean eddie simulation
Ocean eddie simulation
Combustion simulation: Hierarchical simplification
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Stable critical points could provably be tracked more easily and more accurately in the time-varying setting. [Skraba, **Wang** (TopoInVis/Book Chapter), 2014]
Simplifying 2D Time-Varying VF

[Skraba, Wang, Chen, Rosen (TVCG), 2015]
2D Time-varying VF simplification: Video
Simplifying 3D VF

[Skraba, Rosen, Wang, Chen, Bhatia, Pascucci (PacificVis/TVCG), 2016]
3D VF simplification
Next? Tensor field simplification, stress tensor, DTI...

[Tensor field degenerate points with half integer indices]

- Index -0.5
- Index 0.5
- Index 1.5

[Corresponding anisotropy vector field critical points with integer indices]

- Index -1.0
- Index 1
- Index 3.0

[Wang, Hotz, 2017]