22.1 Simplicial Map

Let $K$, $L$ be two finite simplicial complexes over the vertex set $V_K$ and $V_L$.

A set map $\phi : V_K \rightarrow V_L$ is a simplicial map if $\phi(\sigma) \in L \ \forall \ \sigma \in K$.

22.1.1 Definition

If we have two covers of $X$

$\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$

$\mathcal{V} = \{V_\beta\}_{\beta \in B}$

A map of covers from $\mathcal{U}$ to $\mathcal{V}$ is a set map $\Gamma : A \rightarrow B$

so that $U_\alpha \subseteq V_{\Gamma(\alpha)} \ \forall \alpha \in A$.

Given such a map of covers, there is an individual simplicial map

$\Gamma^* : N(\mathcal{U}) \rightarrow N(\mathcal{V})$ given on vertices by $\Gamma$. 
22.2 Stability

22.2.1 Persistent Diagram:

Multi set of points in the extended plane $\mathbb{R}^2 = (\mathbb{R} \cup \{\pm \infty\})^2$
contains finite number of points off the diagonal and infinite points on the diagonal

22.2.2 $L_\infty$ norm:

For two points $X = (x_1, x_2)$, $Y = (y_1, y_2)$ $L_\infty$ norm is defined as
$$\| x - y \|_\infty = \max \{|x_1 - y_1|, |x_2 - y_2|\}$$

22.2.3 Bottleneck Distance:

Let $X, Y$ be two persistent diagrams with $\eta: X \to Y$ as a bijection then bottle neck distance
$$W_\infty(X, Y) = \inf \sup \| X - \eta(X) \|_\infty$$
where the inf is over all possible bijections and
the sup is over all $x \in X$

Bottle Neck distance is a metric. It satisfies the following properties:

- $W_\infty(X, Y) = 0$ if $X = Y$
- $W_\infty(X, Y) = W_\infty(Y, X)$
- $W_\infty(X, Z) = W_\infty(X, Y) + W_\infty(Y, Z)$

22.2.4 Stability of a tame function:

Theorem 22.1. Let $X$ be a triangulable topological space and $f, g : X \to \mathbb{R}$ be two tame functions for each dimension $p$,
$$W_\infty(Dgm_p(f), Dgm_p(g)) \leq \| f - g \|_\infty$$
22.2.5 Definition: Tame function

A function $f : \mathbb{X} \to \mathbb{R}$ is tame if it has a finite number of homological critical values and the homological groups of all sub level sets have finite rank.

22.2.6 Definition: Homological Critical Value

A point $a \in \mathbb{R}$ is a homological critical value if there is no $\epsilon > 0$ for which $f^a_{\mathbb{X}}$ is an isomorphism for each $P$

\[
\begin{align*}
  f^a_{\mathbb{X}} : H_P(\mathbb{X}_a) &\to H_P(\mathbb{X}) \\
  \mathbb{X}_a &= f^{-1}(-\infty, a]
\end{align*}
\]

22.2.7 Definition: Wasserstein distance

Degree - q wasserstein distance between two persistence diagrams is given as

\[
W_q(X, Y) = \left( \inf_{\eta : X \to Y} \sum_{x \in X} \| x - \eta(x) \|_{\infty}^q \right)^{\frac{1}{q}}
\]

"Transportation problem" minimizes the cost of moving or transporting all $*$ points to corresponding "." points. Correspondence is given by bijection $\eta$. Minimize over all possible $\eta$. 

![Diagram](image_url)
22.2.8 Stability bound:

\[ W_\infty(Dgm_p(f), Dgm_p(g)) \leq \|f - g\|_\infty^{1 - \frac{k}{q}} \text{ for } q \geq k > j \]

C and K are constants. \( f, g : X \to \mathbb{R} \) are tame, Lipschitz functions on metric spaces whose triangulations grow polynomially with constant exponent \( g \).

\[ \exists \text{ constants } c, j \text{ such that } N(r) \leq \frac{c}{r^j} \text{ where } k: \text{ simplicial complex } N(r): \text{ number of simplexes with maximum diameter at most } r \]