5.1 Simplicial Complexes

5.1.1 Types of Simplicial Complexes

- Simplex
- Simplicial Complex (SC)
- Abstract Simplicial Complex

Typical Simplicial Complexes:

1. Vietoris-Rips (Rips Complex)
2. Cech Complex
3. Delauney Complex (Overlaps with computational geometry and related closely to Voronoi Diagram)
4. Alpha Complex (Used in protein docking: The company GeoMagic uses Alpha Complex)

Sparsified Simplicial Complexes (that we will study later in the course)

1. Witness Complex
2. Graph Induced Complex

Here is an illustration of an [Interactive Voronoi Diagram Generator](#)

5.1.2 Combinatorial Structure Point Cloud Data (PCD)

- Graph
  
  Describes a "pairwise" relation between data. An Abstract Graph is a pair $G = (V, E)$ consisting of a set of vertices $V$, and a set of Edges $E$, each a pair of vertices. The Graph is Simple if the edge set is a subset of the set of unordered pairs.

Figure 5.1: A Graph
• Simplicial Complex
  Describes "Higher-Order" interactions (includes the pairwise interactions from a graph) and the Laplacian is well defined. A Simplicial Complex is a finite collection of simplices $K$ such that
  
  $$\sigma \in K \text{ and } \tau \leq \sigma \implies \tau \in K$$

  Figure 5.2: A Simplicial Complex

• Hyper-Graph
  Describes an "in-between" structure e.g. a hyper-edge among three nodes

5.1.3 Definitions

1. k-Simplex
   A k-Simplex is the convex hull of $k+1$ affinely independent points.
   Suppose $U = \{u_0, \ldots, u_k\}$ is the set of $k+1$ affinely independent points, then
   
   $$\sigma = \text{Conv}[u_0, \ldots, u_k]$$

   Figure 5.3: k-Simplex

2. Face
   A Face of a Simplex $\sigma$ is the Convex Hull of non-empty subsets of $U$
   A Face is proper if the subset is not equal to the entire set.
3. **Boundary**

   The Boundary, $\tau$ of a simplex is the Union of all proper faces

   \[ \text{e.g.} \quad \text{bd } \sigma = \bigcup \{ 12 , 23 , 13 , 1, 2, 3 \} \]

4. **Abstract Simplicial Complex**

   An Abstract Simplicial Complex is a finite collection of sets $A$ such that

   \[ \alpha \in A , \beta \subseteq \alpha , \text{ then } \beta \in A \]

5. **Simplicial Complex**

   A Simplicial Complex $K$ is a finite collection of simplices such that

   (i) $\alpha \in K , \tau \leq \sigma \Rightarrow \tau \in K$

   (ii) $\sigma_1 , \sigma_2 \in K \Rightarrow \sigma_1 \cap \sigma_2 = \emptyset$ , or a face of both

6. **Underlying Space of $K$**

   $| K |$ the underlying space of $K$, is the union of Simplices in $K$ together with the topology of the ambient Euclidean Space those simplices live in.

7. **Subcomplex**

   $L \subseteq K$ , a Subcomplex of $K$ is the Simplicial Complex that is a subset of $K$. 
Figure 5.5: Sub Complex

L = \{ 1, 2 \} \Rightarrow \text{Not a Sub Complex}
L = \{ 12, 1, 2 \} \Rightarrow \text{Sub Complex}

8. j-Dimensional skeleton
A j-Dimensional skeleton of K contains Simplices of Dimension j or less
K(j) = \{ \sigma \in K \mid \dim \sigma \leq j \}
K(1) = \{ 1, 2, 3, 4, 12, 23, 13, 34, 24 \} \Rightarrow \text{One-Dimensional Skeleton}
K(0) = \{ 1, 2, 3, 4 \} = \text{Vertex(K)} \Rightarrow \text{0-Dimensional Skeleton}

9. Local Neighbourhood of a Simplex
Local Neighbourhood of a Simplex, a Star of $\tau$ is defined as

$$\text{St} \: \tau = \{ \sigma \in \mathcal{K} \mid \tau \leq \sigma \}$$

which implies "all the cofaces of $\tau$"

**Example:**

Let $\tau = \{ 6 \}$, then

$$\text{St} \: \tau = \{ 16, 26, 36, 46, 56, 126, 256, 564, 346, 136, 6 \}$$

10. **Closed Star**

$\bar{\text{St}}$ the closed star is the smallest sub complex that contains the star

$$\bar{\text{St}} \: \tau = \text{St} \: \tau \cup \{ 12, 25, 45, 34, 13, 1, 2, 3, 4, 5 \}$$

11. **Link**

A link is a collection of two or more disjoint knots

$$\text{Lk} \: \tau = \{ \cup \in \bar{\text{St}} \: \tau \mid \forall \cap \tau = \emptyset \}$$

**Example:**
Figure 5.8: Star Example

\[ \text{St (13)} = \{123, 134, 13\} \]

References

[JB00] ALEX BEUTEL, "Interactive Voronoi Diagram Generator" http://alexbeutel.com/webgl/voronoi.html,