Apr. 20

PLH : α-filtration, fixed r

$$H(X_\alpha \cap B_r, X_\alpha \cap \partial B_r) \longrightarrow H(X_{\alpha'} \cap B_r, X_{\alpha'} \cap \partial B_r)$$

$B_r := B_r(x_0)$, $x_0 \in X$ $\qquad \alpha \leq \alpha'$

Application of PLH:

Review LH: $x_0 \in X$ $\qquad H_p(X, X \setminus x_0)$

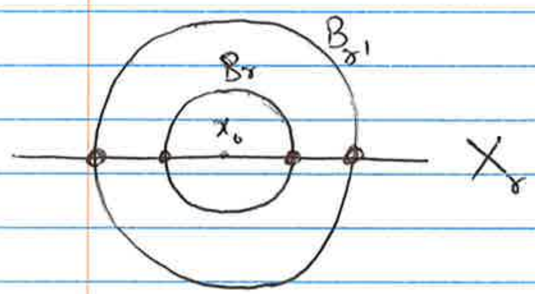
U_r is a neighborhood of x_0 then

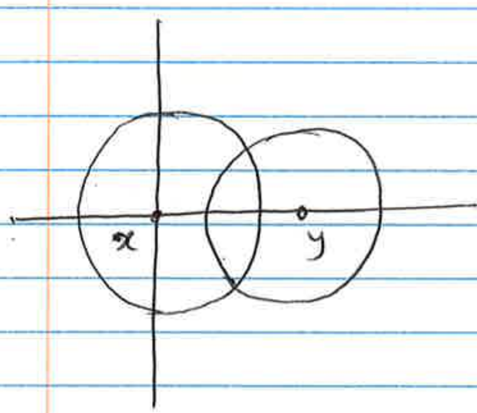
$$\lim_{r \to 0} H_p(X, X \setminus U_r)$$

$$\underline{H_p(X \cap U_r, X \cap \partial U_r)}$$

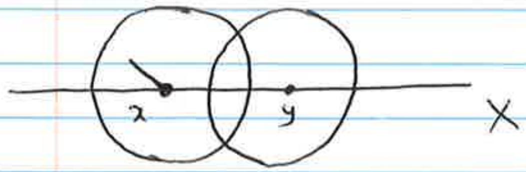
$d_X : \mathbb{R}^2 \to \mathbb{R}$, $\quad d_X(x) = \inf_{y \in X} d_X(x, y)$

then $\quad X_\alpha = d_X^{-1}[0, \alpha]$





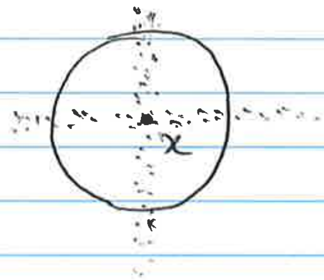
$$
\begin{array}{c}
H_1(X \cap B_r(x), X \cap \partial B_r(x)) \\
\downarrow \text{ Not isomorphism} \\
H_1(X \cap B_r(x) \cap B_r(y), X \cap \partial(B_r(x) \cap B_r(y))) \\
\uparrow \text{ isomorphism} \\
H_1(X \cap B_r(y), X \cap \partial B_r(y))
\end{array}
$$



looking at LH, we may or may not be able to account for the small branch at x (Depending on the size of branch)

We can look at LPH to differentiate between noise and true feature (small branch ~ noise)

Computation with PCD



→ Rips Complex : Only pairwise intersections

Čech Complex : requires common point of intersection betⁿ all.

Rips approx : $X \cap B_r(x)$

Rips approx : $X \cap \partial B_r(x)$

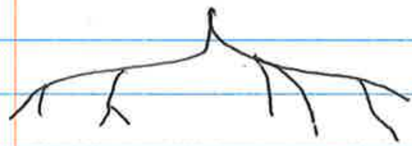
There are specific rules for computing intersections with simplicial complexes.

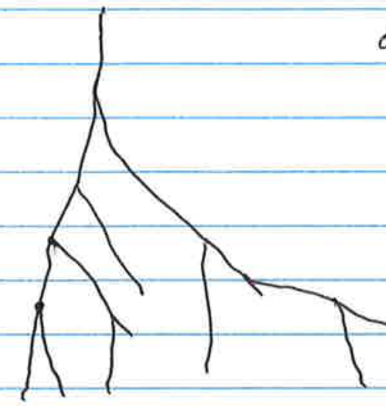
① Exact Computation : Delaunay triangulation

② Approximate LH, PLH : Can use Rips, Čech complex

Application of PLH

1. Stratification learning & clustering
2. Study of root architecture (plant roots)



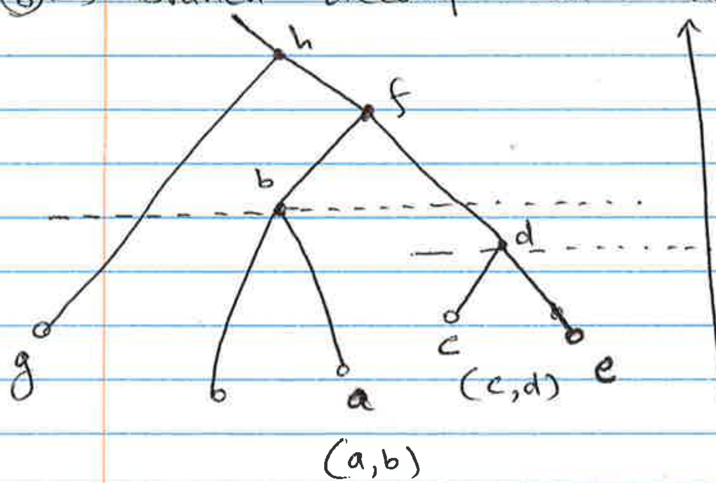


deep vs shallow roots

How to Quantify shape of roots?
How to classify different types of roots?

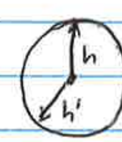
ⓐ → By length, volume, ~~[illegible]~~

ⓑ → Branch decomposition: define branches, measure lengths etc.



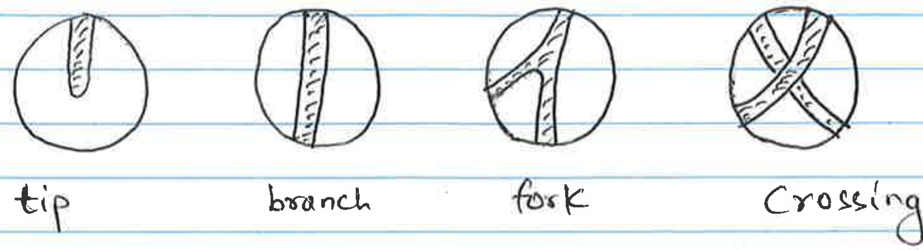
ⓒ Instead of branch lengths, compute persistence of points (a,b) (c,d) (e,f) (g,h)

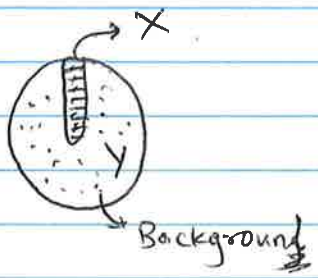
→ Persistence diagram of h for all $h \in S^1$



→ Generally roots are 3D: $h \in S^2$ uniformly sampled directions on a sphere.

(d)



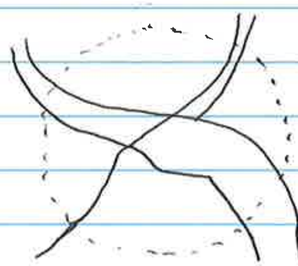


$$Y = B_r \setminus X$$

$$H_1(Y) = \begin{array}{ll} 1 & \text{tip} \\ 2 & \text{branch} \\ 3 & \text{fork} \\ 4 & \text{crossing} \end{array}$$

Application of PLH (3) Road network comparison

Consider GPS location data from taxi cabs. Because of noise in the data, two reconstructions would come up with two slightly different road networks.



→ look at persistent local homology, bottleneck / Wasserstein dist. betn two networks.

local signature at point x

$$S_X(x, r) := \mathrm{Dgm}(X, B_r(x)) \quad \Big| \quad PH(X_\alpha \cap B_r(x), X_\alpha \cap \partial B_r(x))$$

$$S_Y(x, r) := \mathrm{Dgm}(Y, B_r(x)) \quad \Big| \quad PH(Y_\alpha \cap B_r(x), Y_\alpha \cap \partial B_r(x))$$

$$\psi_r(X, Y) = W_\infty(S_X(x, \epsilon), S_Y(x, \epsilon)) \quad \Big| \quad \psi^*(X, Y) = \int_{r=r_1}^{r_2} \psi_r(X, Y)\, dr$$

ψ_r ↓ for fixed r

W_∞ ↓ Bottleneck dist.

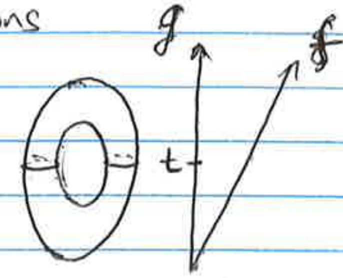
we can look at multiple r values and relation betn them.

Jacobi sets of multiple Morse functions

M : d-dim manifold, $d \geq 2$

$f, g : M \to \mathbb{R}$

eg



Jacobi set $\mathbb{J} = J(f,g) = \underbrace{Cl}_{\text{Closure}} \{ x \in M \mid x \text{ is critical point of } f_t \}$

$M_t = g^{-1}(t)$, f_t is restriction of f to M_t

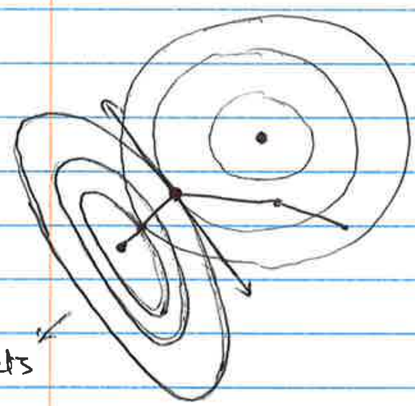
Alternatively: $\mathbb{J} = \{ x \in M \mid \nabla f(x) \parallel \nabla g(x) \}$

$\parallel$: parallel $\Rightarrow \nabla f(x) + \lambda \nabla g(x) = 0$

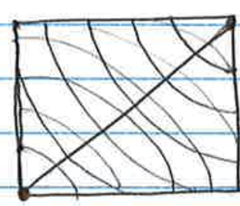
Or $\lambda \nabla f(x) + \nabla g(x) = 0$

for some $\lambda \in \mathbb{R}$.

level sets of g



level sets of f



Jacobi set consists of the diagonal