Local Homology (LH)
- Persistent Local Homology (PLLH)

⇒ Applications:
1. Root architecture
2. Road networks
3. Stratification learning / clustering

LH: X: topological space, let \( x_0 \in X \) be a point in X.

1. \( H_p(X, X - x) \): Relative homology of X with \( X - x \)
   (relative homology of space with the space minus the point).

2. \( \lim_{r \to 0} H_p(X, X \setminus U_r) \) where \( U_r \) is a nbd of \( x_0 \) of radius \( r \).

3. \( \lim_{r \to 0} H_p(X \cap U_r, X \cap U_r) \)

Step:
\[ \Rightarrow \text{rank } H_1(\cdot) = 1 \]

\( X \) is two 1-manifolds intersecting in a 0-dim point.

This is a stratified space.

i.e. \( X \) can be decomposed into manifold pieces \( X_0 \subseteq X, \ldots, X_d \subseteq X \)

Strata \( S_i = X_i \setminus X_{i-1} \)

\[ S_0 \sqcup S_1 \sqcup \ldots \sqcup S_d \]
Torus

Pinched torus

\Rightarrow local neighborhood

local neighborhood

Disk without boundary

Surface without boundary
Glue all 4 boundary points into one.

\[ H_1(\cdot) = 3 \]
\[ H_0(\cdot) = 1 \]

\[ H_2(\cdot) = 2 \]
\[ H_1(\cdot) = 1 \]

→ all three sheet boundaries glued into one point
   creates 2 voids
→ \[ H_2(\cdot) = 2 \]
   \[ H_1(\cdot) = 0 \]

→ We can use stratification learning as a pre-processing step and then apply manifold learning techniques.

Manifold: locally, every point looks similar \( \Rightarrow \) no strata

i.e. local homology will be same for all points.

Local homology can describe complexity of local neighborhood.
1. \( H_1(\times \cap B_y(x), \times \cap \partial B_z(x)) \)  
   rank 1

2. \( H_1(\times \cap B_y(y), \times \cap \partial B_z(y)) \)  
   rank 1

3. \( H_1(\times \cap B_y(x) \cap B_z(y), \times \cap \partial (B_y(x) \cap B_z(y))) \)  
   intersection of the two balls,  
   also has rank 1

There exist isomorphisms from 1 to 3 and from 2 to 3.

\[ \text{local homology at } x : \text{ rank } H_1 = 3 \]

\[ \text{local homology at } y : \text{ rank } H_1 = 1 \]

\[ \text{intersection of the two balls } \text{ rank } H_1 = 1 \]

\( \Rightarrow \) There is no isomorphism  
   i.e. kernel of a map from 1 to 3 is non trivial.
   3 loops mapping to 1 loop: 2 loops map to zero.

\# local homology transfer maps

\[ \text{PLH} : \quad H_p(\times \cap U_y, \times \cap \partial U_z) \quad \times \quad \times \]

\[ \alpha - \text{filtration} : \quad H_p(\times x \cap U_y, \times x \cap \partial U_z) \]

\[ \alpha' - \text{filtration} : \quad H_p(\times x' \cap U_y, \times x' \cap \partial U_z) \]

\( \alpha < \alpha' \)

\( \text{\# thickening of underlying space.} \)
Create a dummy vertex \( w \) then connect all vertices on the boundary to it and also create triangles.