CS 6170 Computational Topology Lecture 18

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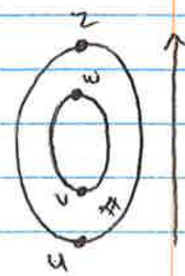
Mar 9

Review: Def: If we have a local coordinate system $(x_1, \dots, x_d)$ in the neighborhood of x ~~and~~ then x is a critical point iff all its partial derivatives are zero

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \dots = \frac{\partial f}{\partial x_d} = 0$$

eg. for $d=2$, $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0$



maxima

~~minima~~

minima

Saddle point

1st derivatives only tell you whether point is critical. to classify as minima/maxima or saddle → Hessian.

Def: The Hessian of f at $x \in M$ is the matrix of derivatives (2nd order partial derivatives)

$$H(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_d} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & & \vdots \\ \vdots & & \ddots & \\ \frac{\partial^2 f}{\partial x_d \partial x_1} & \cdots & & \frac{\partial^2 f}{\partial x_d \partial x_d} \end{bmatrix}$$

eg. for $d=2$

$$H(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} \end{bmatrix}$$

Def: A critical point is non-degenerate if the Hessian is non-singular i.e. $\det(H(x)) \neq 0$.

eg. Degenerate Critical points:

1. lying down torus

local maxima: all points on red ring (non-unique)
local minima: all points on blue ring (non-unique)

2. Monkey Saddle: $f(x_1, x_2) = x_1^3 - 3x_1x_2^2$ then $(0,0)$ is degenerate

3. $f(x) = x^3$, $x = 0$

Morse Lemma: "behavior of a function near critical points"

⇒ Let u be a non-degenerate critical point of $f: M \to \mathbb{R}$. There are local coordinates at u as $u = (0, 0, \dots, 0)$ s.t.

$$f(x) = f(u) - x_1^2 - x_2^2 - \dots - x_q^2 + x_{q+1}^2 + \dots + x_d^2$$

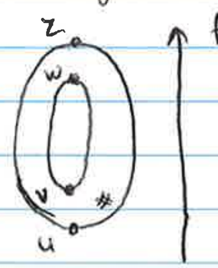
for any point $x = (x_1, \dots, x_d)$ in small neighborhood of u.

Def: The number of negative coefficients in the quadratic polynomial is the index of the critical point.

→ index is used to classify critical points into $d+1$ types.

→ eg. index of u in Lemma statement $= q$ (x_1 to x_q have −ve coefficients)

→ eg.



z: local maxima: index = 2

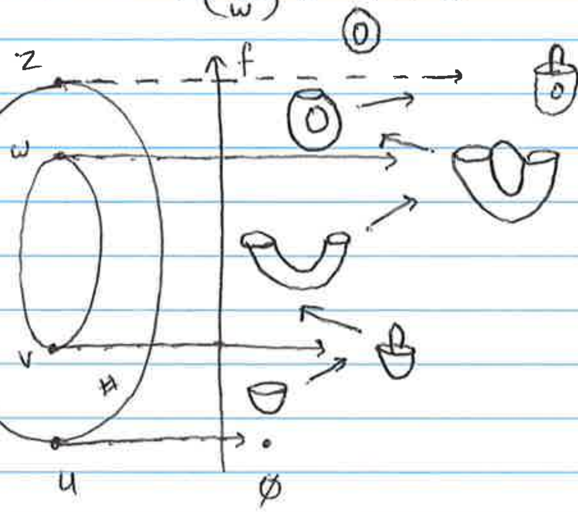
v, w: Saddles: index = 1

u: local minima: index = 0

$f(x) = f(u) + x_1^2 + x_2^2$ in nbd. of u

$f(x) = f(z) - x_1^2 - x_2^2$ in nbd. of z

$f(x) = f\binom{v}{w} \mp x_1^2 \pm x_2^2$ in nbd of v/w (signs alternate)



Attaching handles:

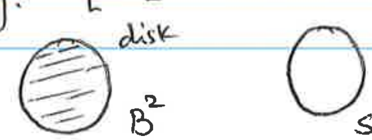
v is a critical point with index q.

$M_b \simeq M_a$ with a q-handle attached

B^q: q-dimensional unit ball

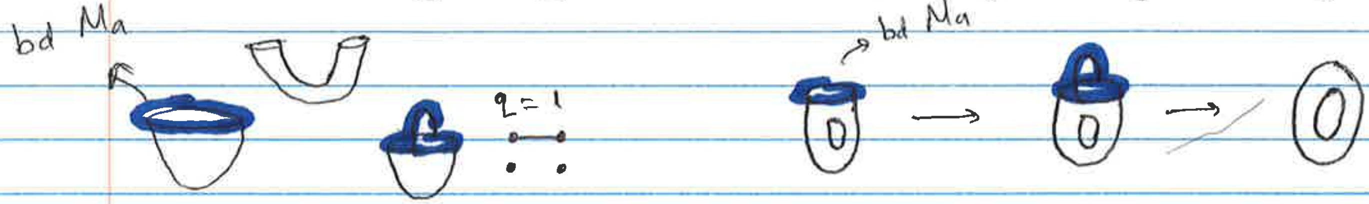
S^{q-1}: boundary of B^q

eg. $q = 2$



Let $g: S^{q-1} \to \cancel{B}\ \text{bd}\ M_a$ be a continuous map.

To attach a handle to M_a, we identify each point $x \in S^{q-1}$ with its image $g(x) \in \text{bd}\ M_a$ (boundary of M_a)



Claim: A consequence of Morse Lemma is that all non-degenerate critical points are isolated.

Def: Morse function: A Morse function is a smooth function on a manifold $f: M \to \mathbb{R}$ such that

(a) All critical points are non-degenerate

(b) The critical points have distinct function values

(ⓑ can sometimes be dropped)

→ All "height" functions on a sphere are Morse functions.

Morse Inequality: M: a d-dim manifold $f: M \to \mathbb{R}$

C_q = number of critical points with index q

① Weak version: $C_q \geq \beta_q(M)$ for all q $\qquad$ β_q: q^{th} Betti number.

Eg. For torus $C_0 = 1$, $C_1 = 2$, $C_2 = 1$

$\beta_0 = 1 \quad \beta_1 = 2 \quad \beta_2 = 1$

② Strong version: $$\sum_{q=0}^{j} (-1)^{j-q} C_q \geq \sum_{q=0}^{j} (-1)^{j-q} \beta_q(M)$$

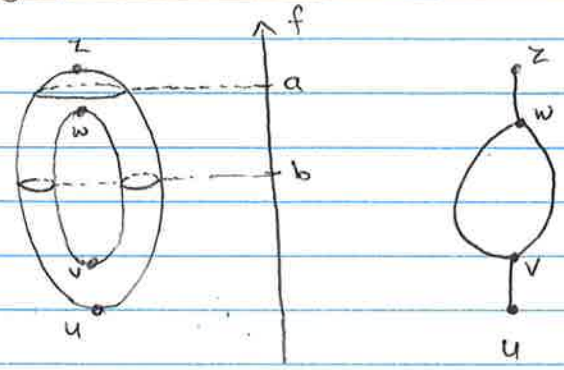
eg. for torus: $C_0 - C_1 + C_2 = 0 = \beta_0 - \beta_1 + \beta_2 = \chi(M)$

$\chi(M)$: Euler characteristic of the manifold.

When $j = d$ in strong version: equality holds.

Reeb graph of Morse function

Example 1 :



skeleton representation of the underlying space with respect to function f

idea : look at all level sets and shrink the connected component of level set to point

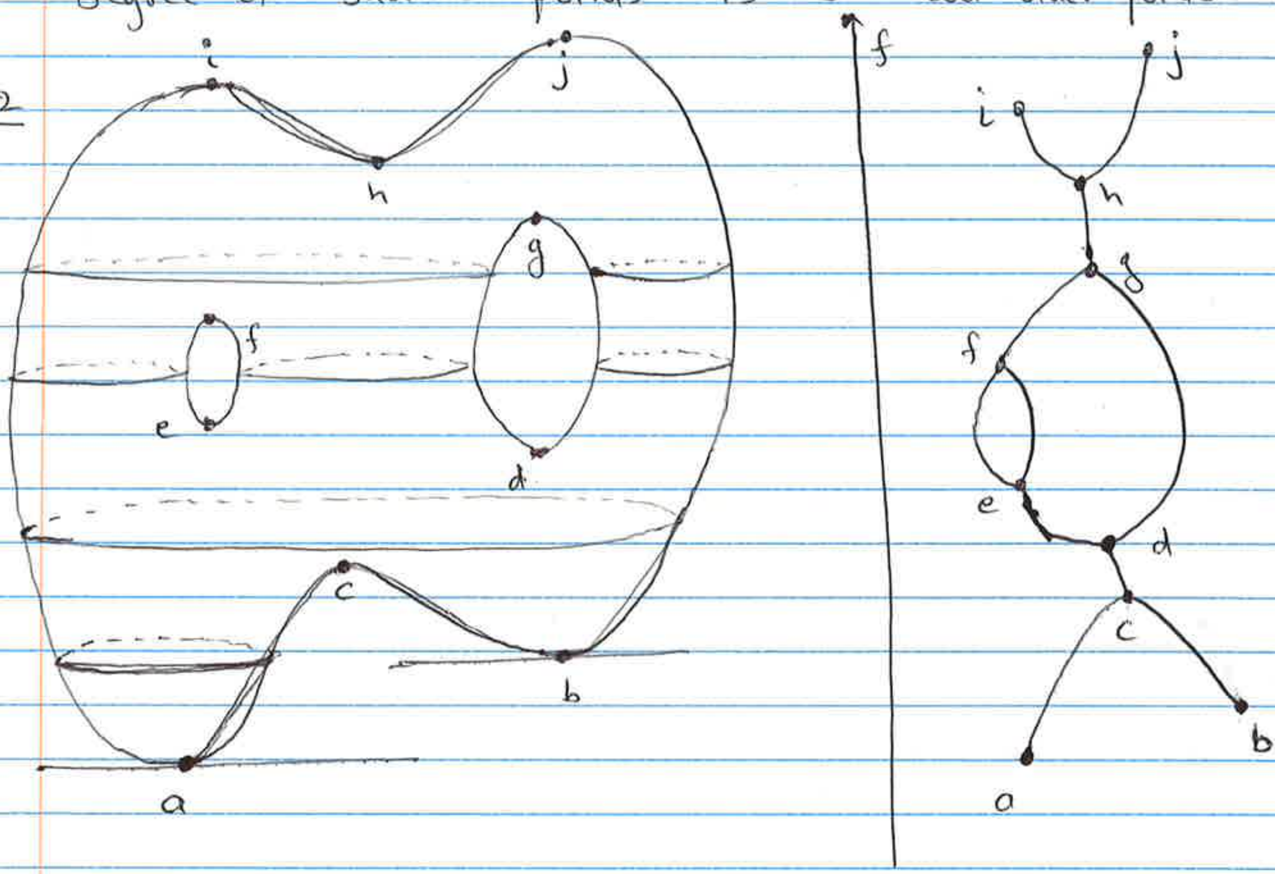
→ Reeb graph : $R(\mathbb{X}, f)$

→ Connected component of level set is sometimes called a contour.

→ Degree of local minima / maxima in Reeb graph is 1

→ Degree of saddle points is 3 . all other points : degree 2 .

Example 2



① Query for number of connected components

② Useful in shape classification / graphics

Reeb graph is a generic version of contour tree
(when no loops are present : reeb graph ⇒ contour tree)