Mar 7

* Scalar / Real valued function analysis
  e.g. Given $\mathbb{X}$: point cloud (high dimensional)
  analysing some function $f: \mathbb{X} \to \mathbb{R}$ [eg. GIS : elevation f^n]

* Multi-variate, vector valued function analysis
  $f: \mathbb{X} \to \mathbb{R}^d$ or $(f_1, f_2, \dots f_d) : \mathbb{X} \to \mathbb{R}^d$
  where f_i are real valued / scalar functions.

→ Tools: Contour trees, Morse-Smale Complex, Reeb graph, Reeb space
  Mapper etc. : Topological structures.

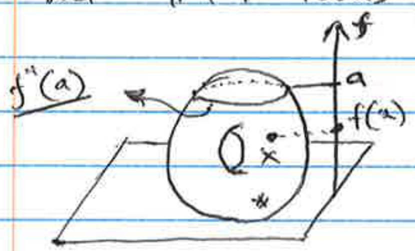
Morse function [Morse Theory : Intro. to Morse Theory – Yukio Yatsumota]

→ Goal is to study simple, real valued functions on a manifold.
→ Next step : general smooth functions, piecewise linear (PL) functions.

$\mathbb{M}$: manifold, $f: \mathbb{M} \to \mathbb{R}$

example: Let $\mathbb{M}$: torus, 2-manifold, $f: \mathbb{M} \to \mathbb{R}$ is "height" function

level set of a



$f(x)$: height / dist. from the plane for every point x on the surface of the torus.

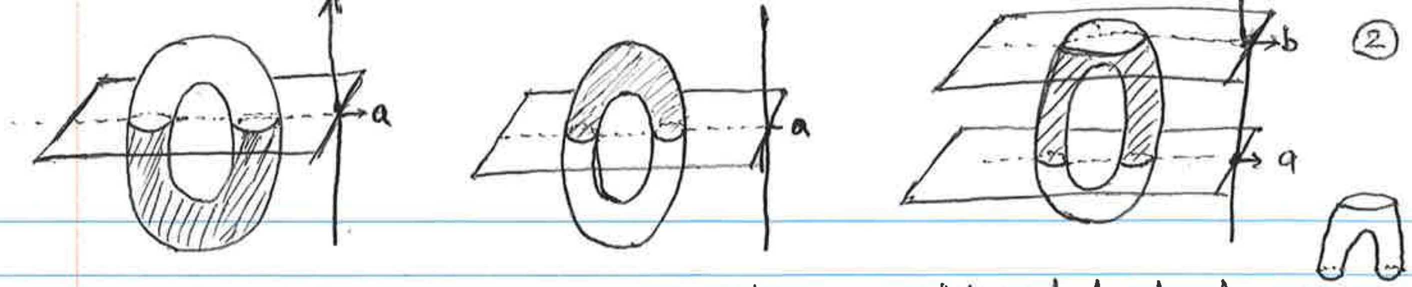
Def: level set: it is the pre-image of $a \in \mathbb{R}$
i.e.
$$f^{-1}(a) = \{x \in \mathbb{M} \mid f(x) = a\}$$

Def: Sub-level set: $\mathbb{M}_a = f^{-1}(-\infty, a] = \{x \in \mathbb{M} \mid f(x) \leq a\}$

all points that have height at most a

Def: Super-level set: $\mathbb{M}^a = f^{-1}[a, \infty) = \{x \in \mathbb{M} \mid f(x) \geq a\}$

Def: interval level set: $f^{-1}[a,b] = \{x \in \mathbb{M} \mid a \leq f(x) \leq b\}$



Sub level set | Super level set | interval level set | (pair of pants)

→ sub-level / super-level sets can be used to build a filtration.

$f : M \to \mathbb{R}$, for $a_1 \le a_2 \le \dots \le \dots$ for $a_i \in \mathbb{R}$

then $M_{a_1} \to M_{a_2} \to \dots$ gives us a filtration

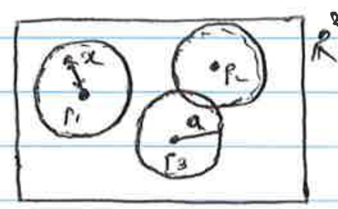
and $H(M_{a_1}) \to H(M_{a_2}) \to \dots$ give mappings from homology of $M_{a_i} \to M_{a_j}$

eg.: ① point-cloud data (sensor networks)

② function on point cloud data (eg. height function)

→ Sensor network data has an implicit distance function

→ we look at all points that are at most 'r' distance away from the point cloud.

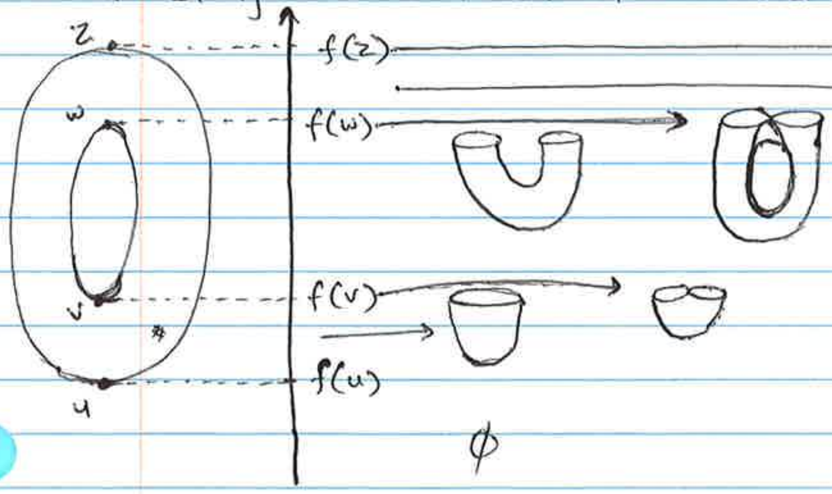


Let $p \in P$ be points in the point cloud

$f : \mathbb{R}^2 \to \mathbb{R} \Rightarrow f(x) = \inf \|x - p\|_2$

$f^{-1}[\infty, a] \Rightarrow$ sub-level set for a : all points within distance "a" from any of the points in P

\# Study the evolution of sublevel sets

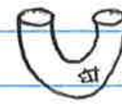
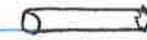


Review: Two spaces X and Y are of same homotopy type or are homotopy equivalent if there exist continuous maps $f : X \to Y$, $g : Y \to X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$

$\boxed{X \simeq Y}$

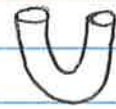
→ at u, single point $\simeq$ single point.

→ between u and v, →  $\simeq$ disk.

→ at v  $\simeq$  $\simeq$  $\simeq$ 

disk attached a 1-handle.

cylinder

→ between v, w →  → $\simeq$ Cylinder

→ at w →  $\simeq$  → cylinder attached a 1-handle

→ between w, z →  $\simeq$  capped torus.

→ at z → torus.

<u>Critical Points</u>: Let M: d-dimensional manifold
(locally: looks like open ball in $\mathbb{R}^d$)

<u>Def</u>: A critical point of function $f: M \to \mathbb{R}$
is a point $x \in M$ s.t. the derivative of f at x is 0.

→ if we have local co-ordinate system $(x_1, x_2, \ldots, x_d)$ in nbd of x
then x is a critical point iff all its partial derivatives
are 0.

i.e. $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \cdots = \frac{\partial f}{\partial x_d} = 0$

<u>Def</u>: If x is a critical point, $f(x)$ is called critical value.

Torus: 4 critical points $\underbrace{(u, z)}_{\text{local min/maxima}}$. [$v, w$ are saddle points.]
also critical points.