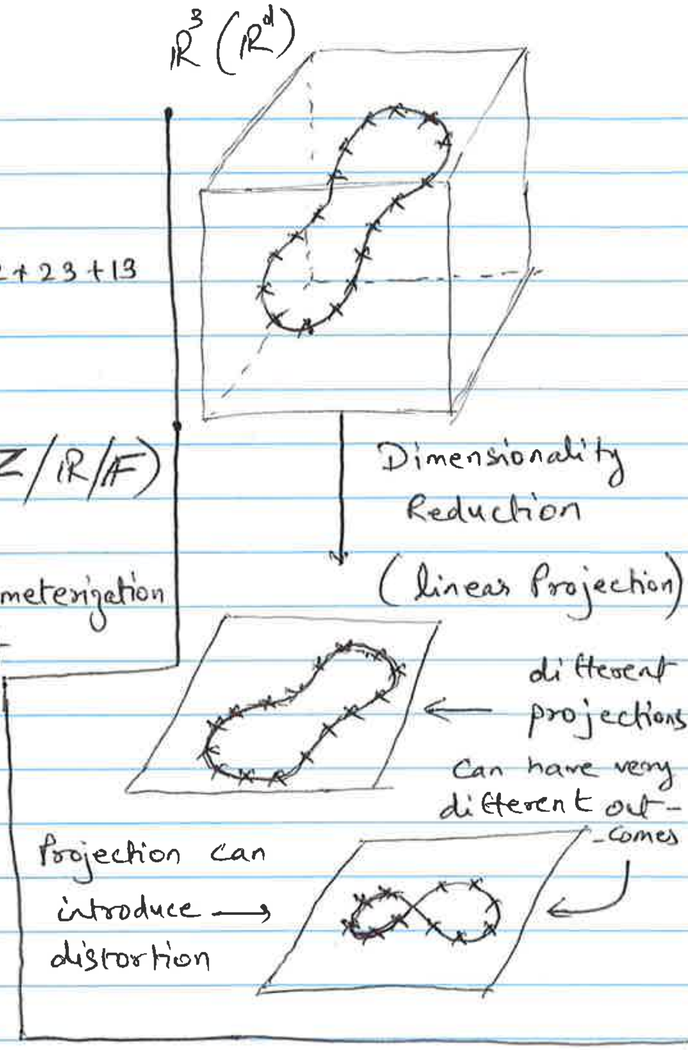
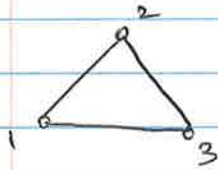


Homology Vs. Cohomology



→ Simplicial homology

$c \in C_p$, $c = 12 + 23 + 13$

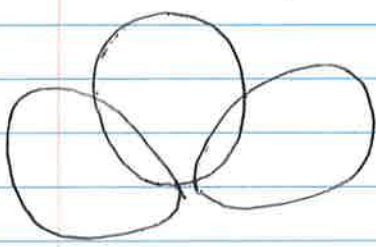
→ Simplicial Cohomology: $c^* \in C^p$

$c^* : c \longrightarrow \mathbb{Z}_2$ (Could be $\mathbb{Z} / \mathbb{R} / \mathbb{F}$)

→ Cohomology tries to assign a parameterization s.t. as parameter changes, ~~the~~ it traces the tunnel boundary

→ parameterize point cloud in high dim.

→ Every loop has an independent parameterization



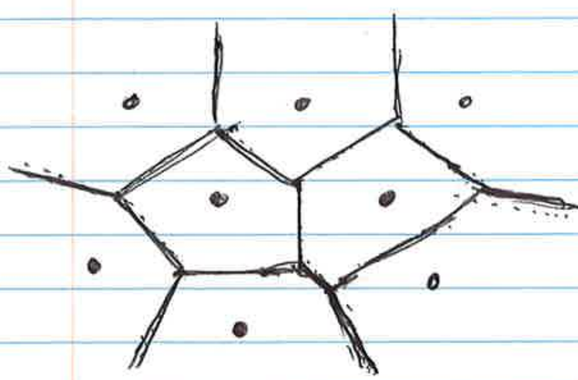
$\Rightarrow$ $f_1 : X \to S^1$ $[0, 1]$

$f_2 : X \to S^2$

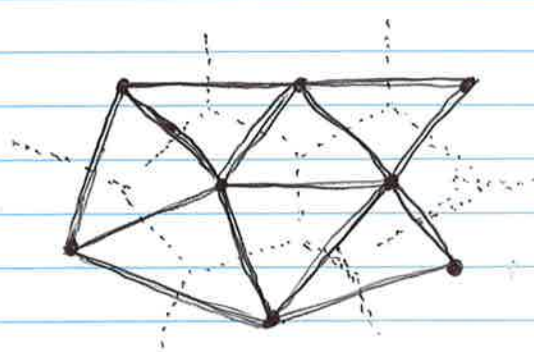
$f_3 : X \to S^3$

- Homology & Co-homology complement eachother (duality)

Duality Example: Voronoi diagram $\longleftrightarrow$ ~~the~~ Delaunay Complex



Voronoi diagram

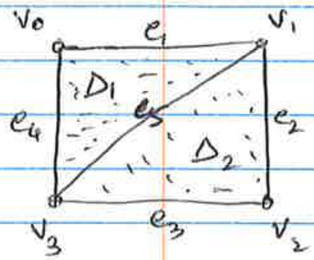


Delaunay Complex (triangulation)

homology $\longleftrightarrow$ Co-homology

"geometric" $\downarrow$ deal with objects

"algebraic" $\downarrow$ deal with functions



$K = \{ v_0, v_1, v_2, v_3, e_1, e_2, e_3, e_4, e_5, \Delta_1, \Delta_2 \}$

0-chain : denote as b eg. v_1, v_1+v_2,
$\hookrightarrow$ single vertex : elementary 0-chain.

1-chain : denote a eg. e_1, e_2+e_3, $e_1+e_2+e_4$
$\hookrightarrow$ single edge : elementary 1-chain

2-chain : denote C eg. Δ_1, $\Delta_1+\Delta_2$
$\hookrightarrow$ single triangle : elementary 2-chain.

Let
$b_1 = v_1$
$b_2 = v_1+v_2$
$b_3 = v_1+v_2+v_3$
$a_1 = e_1$
$a_2 = e_1+e_2+e_3+e_4+e_5$
$c_1 = \Delta_1$
$c_2 = \Delta_1+\Delta_2$

$\Rightarrow$ denote
0-cochain : β
1-cochain : α
2-cochain : γ

<u>0-Cochain</u> : $v_0^*, v_1^*, v_2^*, v_3^* \rightarrow$ elementary 0-Cochains $\rightarrow$ each one is a function.

$v_0^*(v_0) = 1$

$v_0^*(v_1) = 0 \quad v_0^*(v_2) = 0 \quad v_0^*(v_3) = 0$

$\cancel{\beta_0 = v_0^* + v_1^* \text{ then } \beta_0(v_0+v_2+v_3) = v_0^*(v_0+v_2+v_3) + v_1^*(v_0+v_2+v_3)}$

$\rightarrow$ We can have 0-cochain $\beta_0 = v_0^* + v_1^*$

<u>1-cochain</u> : $e_1^*, e_2^*, e_3^*, e_4^*, e_5^* \rightarrow$ elementary 1-Cochain

$e_1^*(e_1) = 1 \quad e_1^*(e_2) = 0$ | We can have 1-Cochains $\alpha_0^* = e_1^* + e_2^*$, $\alpha_1^* = e_1^* + e_5^*$

<u>2-Cochain</u> : $\Delta_1^*, \Delta_2^* \rightarrow$ elementary 2-cochains

We can have other 2-cochains eg. $\gamma_0^* = \Delta_1^* + \Delta_2^*$

$\Rightarrow$ Consider boundary operator in homology ∂

Let $c_2 = \Delta_1 + \Delta_2$ then $\partial c_2 = \partial(\Delta_1) + \partial(\Delta_2)$

$= (e_1 + e_4 + e_5) + (e_2 + e_3 + e_5)$

$= e_1 + e_4 + e_2 + e_3$

$\partial(\partial(c_2)) = 0$

$(v_0 + v_1) + (v_0 + v_3) + (v_1 + v_2) + (v_3 + v_2)$

$\boxed{\text{Boundary of boundary is always } 0}$

$\Rightarrow$ in Cohomology : coboundary δ

We can represent any cochain as linear combination of elementary cochains.

$\Rightarrow$ if $c = \sum_i g_i \sigma_i^* \longrightarrow \delta c = \sum_i g_i (\delta \sigma_i^*)$

$\sigma_i^* \longrightarrow v_i^*$ or e_i^* or Δ_i^* ...

$\delta \sigma^* = \sum_j \epsilon_j \tau_j^*$

p-simplex

(p+1)-simplex that have σ as a face

eg. $\delta e_5^* = \Delta_1^* + \Delta_2^*$

$\delta v_1^* = e_1^* + e_2^* + e_5^*$

$\boxed{\text{Coboundary of Coboundary is always } 0}$

$\delta(\delta v_1^*) = \delta e_1^* + \delta e_2^* + \delta e_5^* = \Delta_1^* + \Delta_2^* + (\Delta_1^* + \Delta_2^*) = \underline{\underline{0}}$

Eg. Let $\beta_0 = v_0^* + v_1^*$ $\quad \beta_0(v_0) = (v_0^* + v_1^*)(v_0) = v_0^*(v_0) + v_1^*(v_0)$

$= 1 + 0$

$\rightarrow$ All the co-chain functions are homomorphisms.

In homology we have Z_p : p-cycle group, B_p: p-boundary group

in co-homology we have Z^p : p-cocycle group, B^p: p-coboundary group

$\rightarrow \Delta_1^*$ is a co-cycle $\because \delta\Delta_1^* = 0$, $\delta\Delta_2^* = 0$ $\longrightarrow$ No higher dimension object exists in S.C.

$\rightarrow \Delta_1^*, \Delta_2^*$ are also co-boundaries because $\exists$ cocycle which has Δ_1^* or Δ_2^* as it's co boundary

$\delta e_1^* = \Delta_1^*$

$\delta e_2^* = \Delta_2^*$

eg.: 1-dim co-chain $a^* = e_1^* + e_2^* + e_5^*$ is a co-cycle.

$$\delta a^* = \delta e_1^* + \delta e_2^* + \delta e_5^* = \Delta_1^* + \Delta_2^* + (\Delta_1^* + \Delta_2^*) = 0$$

also $\delta v_1^* = e_1^* + e_2^* + e_5^* = a^* \Rightarrow a^*$ is also co-boundary.

eg: 0-cochain $\beta = v_0^* + v_1^* + v_2^* + v_3^*$ is a cocycle

$\because \delta\beta = \delta(v_0^*) + \delta(v_1^*) + \delta(v_2^*) + \delta(v_3^*) = 0 \Rightarrow$ each edge appears twice.

p-th Cohomology group $H^p = Z^p / B^p$

Consists of co-cycles that are not coboundaries.