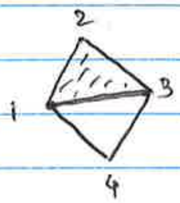
Feb 16

CS 6170 Computational Topology Lecture 12

Computing homology:

Review: Def: The p-th homology group is the p-th cycle group (Z_p) modulo p-th boundary group (B_p)

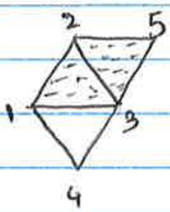


$$H_p(K) = H_p = Z_p / B_p$$ [group of cycles that don't bound]

$c = (13 + 34 + 14)$ $\quad c' = (12 + 23 + 13)$ $\quad c'' = (12 + 23 + 34 + 14)$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad = (12 + 23 + 13) + (13 + 34 + 14)$

$\nexists\, d \in C_2$ s.t. $\partial d = c$ $\quad$ Boundary of Δ 123.



$\rightarrow (12 + 25 + 35 + 34 + 14) = (13 + 34 + 14) +$ (Boundaries of two triangles)

$\in H_1$

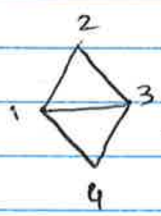
Def: A p-cycle is a p-chain with empty boundary $(Z_p = \ker \partial_p)$

Def: A p-boundary is a p-chain that is a boundary of p+1-chain
$c' = \partial d'$ where $d' \in C_{p+1}$, $c' \in C_p$ $(B_p = \operatorname{im} \partial_{p+1})$

Def: A p-th Betti number is the rank of H_p $(\beta_p = \text{rank } H_p)$

Def: Rank of a group G is the smallest cardinality of a generating set.

Def: A generating set of Group G is a subset s.t. every element in G can be expressed as a combination (under group operation) of finitely many elements of the subset & their inverses.



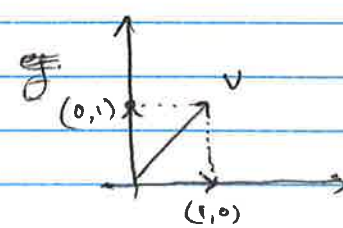
$\rightarrow$ rank $H_1 = 2$

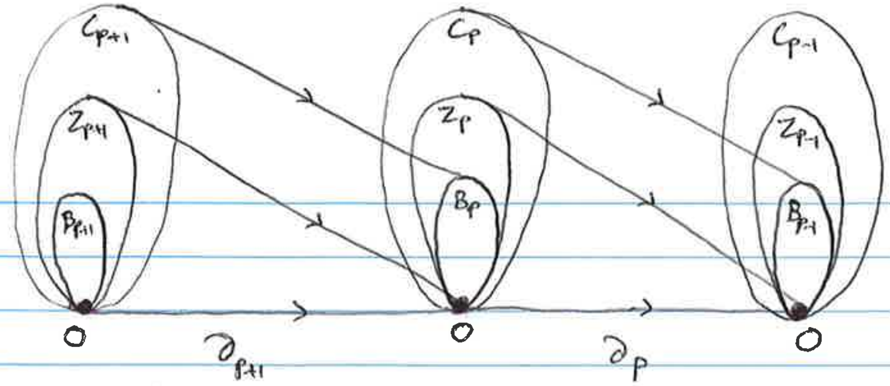
$$\left.\begin{aligned} c_1 &= 12 + 23 + 13 \\ c_2 &= 13 + 34 + 14 \\ c_3 &= 12 + 23 + 34 + 14 \end{aligned}\right\}$$ every cycle can be represented as combination of other two $\therefore$ Generator has 2 elements.

⊛ Generating set is not unique but the rank is unique.

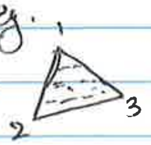
$\rightarrow$ This is related to basis of vector spaces.

e.g. in $\mathbb{R}^2 \rightarrow$ every vector can be represented as combination of the two unit vectors $(1,0)$ and $(0,1)$.





e.g.



$123 \in C_2$ $\quad \partial(123) = (12 + 23 + 13) \in C_1$ $\quad \partial(12) = 1 + 2 \in C_0$

H_p : Elements in Z_p but not in B_p

Reduced Homology : Consider the augmentation map $\varepsilon : C_0 \to \mathbb{Z}_2$ defined by $\varepsilon(u) = 1$ for every vertex u

$$\cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z}_2 \xrightarrow{0} 0 \to \cdots$$

if $p > 0$ $\quad \tilde{H}_p = \dfrac{\ker \partial_p}{\operatorname{im} \partial_{p+1}} = H_p$

if $p = 0$ $\quad \tilde{H}_0 = \dfrac{\ker \varepsilon}{\operatorname{im} \partial_1}$, $\quad \tilde{\beta}_p = \operatorname{rank}(\tilde{H}_p)$

$\boxed{\tilde{\beta}_0 = \beta_0 - 1}$, $\boxed{\tilde{\beta}_p = \beta_p}$ $\to$ if simplicial complex K is not empty.

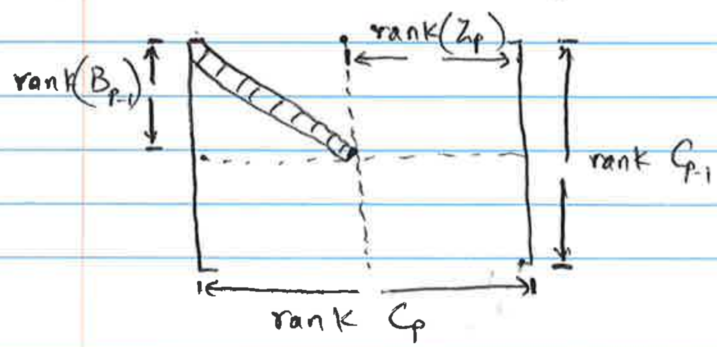
if $K = \phi \Rightarrow \tilde{\beta}_{-1} = 1$

Algorithm : p-th boundary matrix ∂_p

Col : p-simplices n_p

rows : $(p-1)$-simplices n_{p-1}

- Use row & col operations to reduce ∂_p to SNF (Smith Normal Form) N_p



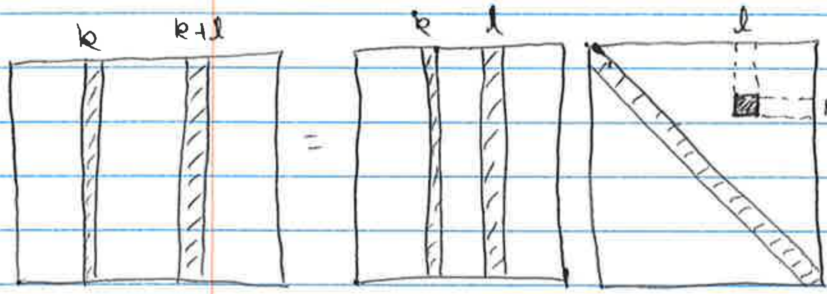
$H_p = Z_p / B_p$

$\therefore$ rank H_p = rank Z_p − rank B_p

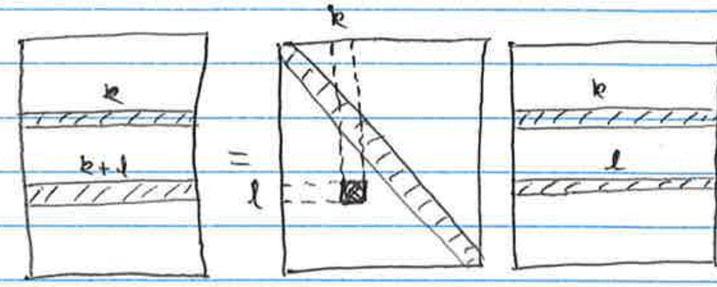
$\therefore$ Computing β_p requires reducing two boundary matrices ∂_p & ∂_{p+1}

$N_p = U_{p-1}\, \partial_p\, V_p$

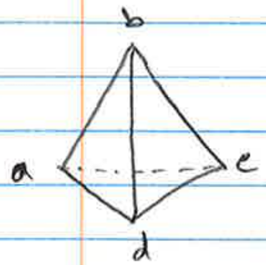
Col operations: (a) exchange col k & l
(b) add col k to l



row operations: (a) exchange rows k & l
(b) add row k to row l



tetrahedron : triangulated 3-ball



Reduced homology :

$\tilde{\beta}_0 = \beta_0 - 1 = 0$

$\tilde{\beta}_1 = \beta_1 = 0$

$\tilde{\beta}_2 = \beta_2 = 0$

$\partial_0 = \varepsilon$

	a	b	c	d
1	1	1	1	1

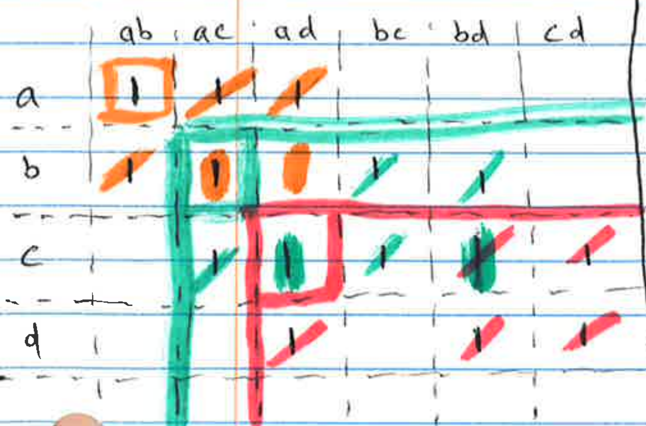
↓
dummy element

$N_0 =$

	a	b	c	d
1	1	0	0	0

∂_1

	ab	ac	ad	bc	bd	cd
a	1	1	1			
b	1			1	1	
c		1		1		1
d			1		1	1



1. make sure diagonal element is 1 [starting from first column] if not switch rows / columns
2. Zero out first row (column operations) except the first diagonal.
3. Zero out first column (row operations) except diagonal element.
4. Repeat 2 & 3 for submatrix.

$N_1 =$

	ab	ac	ad	bc	bd	cd
a	1					
b		1				
c			1			
d						

$\therefore$ rank $Z_1 = 3$

rank $B_0 = 3$

$\tilde{\beta}_0 = \text{rank } Z_0 - \text{rank } B_0 = 3 - 3 = 0$

$$\partial_2 \quad \begin{array}{c|cccc} & abc & abd & acd & bcd \\ \hline ab & \boxed{1} & 1 & & \\ ac & 1 & \boxed{1} & 1 & \\ ad & & 1 & 1 & \\ bc & 1 & & & 1 \\ bd & & 1 & \boxed{1} & 1 \\ cd & & & 1 & 1 \end{array}$$

This is not on the diagonal
* remember to switch rows / columns to ensure non-zero diagonal

rank $Z_2 = 1$, rank $B_1 = 3$ $\therefore \tilde{\beta}_1 = \beta_1 = \text{rank } Z_1 - \text{rank } B_1$
$= 3 - 3$
$= 0$.

$$\partial_3 \quad \begin{array}{c|c} & abcd \\ \hline abc & 1 \\ abd & 1 \\ acd & 1 \\ bcd & 1 \end{array} \Rightarrow N_3 = \begin{array}{c|c} & abcd \\ \hline abc & 1 \\ abd & 0 \\ acd & 0 \\ bcd & 0 \end{array} \Rightarrow \begin{array}{l} \text{rank } Z_3 = 0 \\ \text{rank } B_2 = 1 \end{array}$$

$\therefore \tilde{\beta}_2 = \beta_2 = \text{rank } Z_2 - \text{rank } B_2$
$= 1 - 1 = 0$.

if tetrahedron was hollow,
∂_3 wouldn't exist
→ assume rank $B_2 = 0$ $\therefore \tilde{\beta}_2 = \text{rank } Z_2 = 1$.

→ if not computing reduced homology, ∂_0 doesn't exist.
but rank Z_0 = number of vertices $\therefore$ we can still compute Betti numbers.