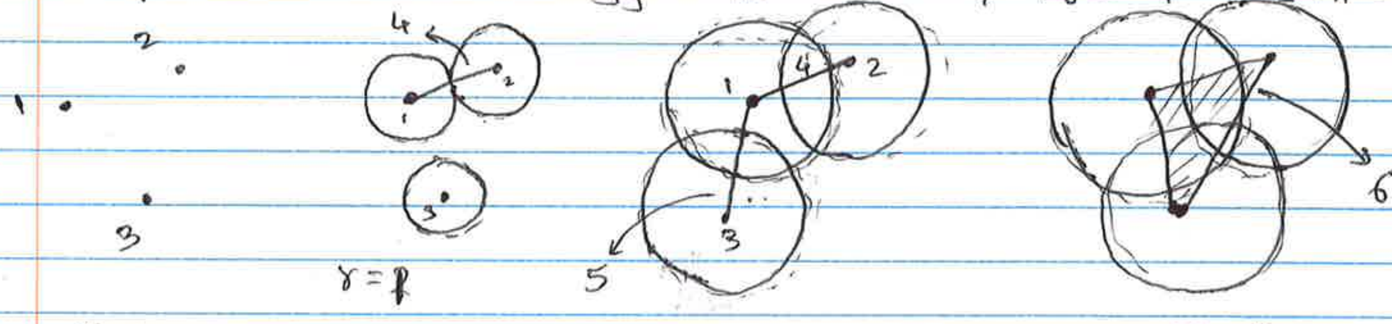
Feb 14

For computation, whatever the underlying space, we ~~need~~ [redacted] use a simplicial complex as a combinatorial representation of that space.

- Compute Homology of K : Simplicial Complex
- Compute Persistent homology : filtration $\phi = K_0 \subseteq K_1 \ldots \subseteq K_n = K$



$\gamma = 0$ $\gamma = 1$ $\gamma = 2.5$ $\gamma = 3$

→ In the filtration : number simplices as they appear in the filtration. Vertices are numbered arbitrarily since all were present at $\gamma = 0$.

$K_0 = \phi \xrightarrow{\subseteq} K_1 \xrightarrow{\subseteq} K_2 \xrightarrow{\subseteq} K_3 \xrightarrow{\subseteq} K_4 \xrightarrow{\subseteq} K_5 \xrightarrow{\subseteq} K_6 \xrightarrow{\subseteq} K_7$

$\{1\}$, $\{1,2\}$, $\{1,2,3\}$, $\{1,2,3,4\}$, $\{1,2,3,4,5\}$, $\{1,2,3,4,5,6\}$, $\{1,2,3,4,5,6,7\}$

$[t_2, t_4]$, $[t_3, t_5]$

Betti numbers (dim 0, dim 1)

$(0,0)$ $(1,0)$ $(2,0)$ $(3,0)$ $(2,0)$ $(1,0)$ $(1,1)$ $(1,0)$

$[t_1, \infty)$, $[t_6, t_7]$

→ Appearance of edge 4 in K_4 reduces β_0 by 1. i.e. one of the components is destroyed. According to our arbitrary ordering of vertices, we consider the younger component gets destroyed (2 in this case)

→ Component (vertex) 2 appeared in K_2 and destroyed in K_4

→ In persistent diagram, we have a point (0, 1) radii at which component appeared & got destroyed.

→ For complete ordering : edges come before triangles ∴ triangle is numbered 7.

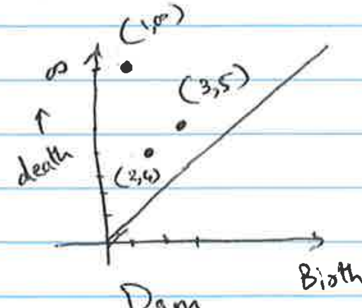
Persistent Diagram : Visualization of change in homology.

β_0 : $[t_1, \infty)$: $(1, \infty)$

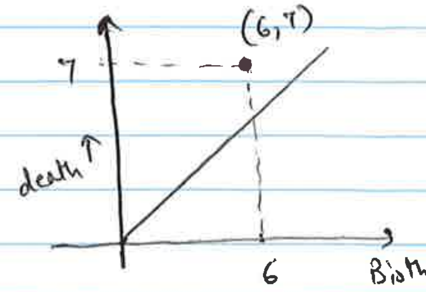
$[t_2, t_4]$ $(2, 4)$

$[t_3, t_5]$ $(3, 5)$

β_1 : $[t_6, t_7]$ $(6, 7)$



Dgm_0



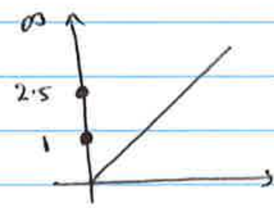
Dgm_1

- Persistence = death - birth
  (vertical distance from diagonal)

→ Appearance "time" & disappearance "time" may correspond to index of SC in filtration OR the radius of open balls corresponding to the SCs in filtration.

∴ For our first example :

→ the loop appears when edge 6 is added & disappears when triangle 7 appears !



Dgm_0

Dgm_1

In terms of radius, both simplices appear at $r = 3$

∴ the loop falls on the diagonal → 0 persistance.

→ long lived features are significant ∴ high persistence ⇒ significance.

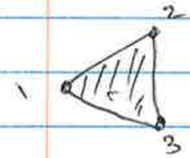
Suppose K is a simplicial complex. Let $f: K \to \mathbb{R}$.
if f is monotonic i.e. $f(\sigma) \le f(\tau)$ if σ is a face of τ then we have a <u>Compatible Ordering</u>: of simplices

$\sigma_1, \sigma_2, \sigma_3, \ldots \sigma_m$ s.t. if $i < j$ then $f(\sigma_i) \le f(\sigma_j)$

or if $\sigma_i \le \sigma_j$ (σ_i is face of σ_j) then $f(\sigma_i) \le f(\sigma_j)$

\# Think of f as appearance time. for a simplex to appear, all of its faces must be present already.

$m \times m$ boundary matrix ∂ : $\partial[i,j] = \begin{cases} 1 & \text{if } \sigma_i \text{ is a codim 1 face of } \sigma_j \\ 0 & \text{otherwise.} \end{cases}$



$\rightarrow$ 1, 2, 3, 12, 23, 13, 123

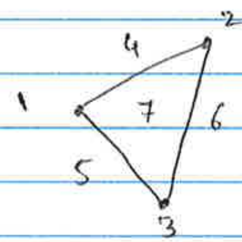
\# <u>Algorithm</u>: Column operations: Adding colums from left to right to reduce ∂ to 0-1 matrix R.

<u>Def</u>: $low(j)$: row index of the lowest 1 in column j

R is reduced when $low(j) \ne low(j_0)$ if $j \ne j_0$

Algo:

```
R = ∂
for j=1 to m do
    while ∃ j₀ < j s.t.
          low(j₀) = low(j) do
        add col j₀ to col j
    end while
end for
```



$4+5+6$
$\uparrow$
$5+6$
$\uparrow$

$$\partial = \begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & & & & 1 & 1 & \checkmark & \\ 2 & & & & \boxed{1} & & \boxed{\checkmark} & \\ 3 & & & & & \boxed{1} & \boxed{\checkmark} & \\ 4 & & & & & & & 1 \\ 5 & & & & & & & 1 \\ 6 & & & & & & & \boxed{1} \\ 7 & & & & & & & \end{array}$$

$R = \partial V$ where V encodes all the column operations.

After reduction:

$$
\underbrace{\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & & & & 1 & 1 & & \\
2 & & & & 1 & & & \\
3 & & & & & 1 & & \\
4 & & & & & & & 1 \\
5 & & & & & & & 1 \\
6 & & & & & & & 1 \\
7 & & & & & & &
\end{array}}_{R}
=
\underbrace{\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & & & & 1 & 1 & & \\
2 & & & & 1 & & 1 & \\
3 & & & & & 1 & 1 & \\
4 & & & & & & & 1 \\
5 & & & & & & & 1 \\
6 & & & & & & & 1 \\
7 & & & & & & &
\end{array}}_{\partial}
\underbrace{\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & & & & & & \\
 & 1 & & & & & \\
 & & 1 & & & & \\
 & & & 1 & & 1 & \\
 & & & & 1 & 1 & \\
 & & & & & 1 & \\
 & & & & & & 1
\end{array}}_{V}
$$

All three matrices are upper triangular.

→ Boundary matrix ∂ is always upper triangular
($\because$ all faces must appear before simplex)

→ Column operations only consider columns before the current
$\therefore$ V is also upper triangular.

dimension is the dim of row.

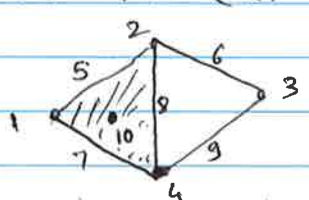
→ in R (after reduction) low $(4) = 2 \Rightarrow (2,4) \rightarrow 0$

low $(5) = 3 \Rightarrow (3,5) \longrightarrow 0$

low $(7) = 6 \Rightarrow (6,7) \longrightarrow 1$

$\because$ 2, 3 are vertices – dim 0

6 is an edge – dim 1



$$
\begin{array}{c|cccccccccc}
 & & & & & & & & +5\ +7 & +5\ +6\ +7 & \\
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
1 & \boxed{\star} & & & & 1 & & 1 & \times & \times & \\
2 & & & & & 1 & 1 & & \times & \times & \\
3 & & & & & & \boxed{1} & & & \times & \\
4 & & & & & & & \boxed{1} & \times & \times & \\
5 & & & & & & & & & & 1 \\
6 & & & & & & & & & & \\
7 & & & & & & & & & & 1 \\
8 & & & & & & & & & & \boxed{1} \\
9 & & & & & & & & & & \\
10 & & & & & & & & & &
\end{array}
$$

$\Rightarrow$ low $(5) = 2 \longrightarrow (2,5)$

low $(6) = 3 \longrightarrow (3,6)$

low $(7) = 4 \longrightarrow (4,7)$

low $(10) = 8 \longrightarrow (8,10)$

2, 3, 4 are vertices
$\therefore$ first three are dim 0 features. last one is dim 1 $\because$ 8 is an edge.

(★) if column is empty & does not appear in RHS $\Longrightarrow$ it created a feature that didn't die → col 1, col 9