Review: Simplicial Complex $K$: A collection of simplices such that if $\sigma \in K$, face of $\sigma \in K$ and for $\sigma_1, \sigma_2 \in K$, their intersection is a simplex also belongs to $K$.

Underlying space of $|K|$:

Def: A triangulation of a topological space $X$ is a simplicial complex $K$ together with a homeomorphism between $X$ and $|K|$.

Underlying space of triangulation ($\sigma \in K$) is homeomorphic to $X$.

Betti Numbers: (Homology in a nutshell)

$\beta_0$ or $b_0$: # of connected components
$\beta_1$ or $b_1$: # of tunnels / loops
$\beta_2$: # of voids
$\beta_k$: # of higher order voids

$\# \beta_k$: Rank of the $k^{th}$ homology group ($\text{rank } H_k$)

| $S^1$ (Circle) | 1 | 1 | 0 | $K^2$ (Klein bottle) | 1 | 1 | 0 |
| $S^2$ (Sphere) | 1 | 0 | 1 | 2 hole torus | 1 | 4 | 1 |
| $T^2$ (Torus) | 1 | 2 | 1 | $2g$-hole torus | 1 | 2g | 1 |
| Projective plane | 1 | 0 | 0 | | | |
Def: The genus of a connected orientable surface is an integer, representing the maximum number of cutting along simple closed curves without disconnecting the resulting manifold.

- Cut here
- We still have connected manifold

tube through sphere

# Groups, Abelian groups

Def: A group is a set $G$ with operation * such that

1. Closure: $a, b \in G$ then $a \cdot b \in G$
2. Associativity: $a, b, c \in G$ then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. Identity: $e \in G$ s.t. $e \cdot a = a \cdot e = a$ for any $a \in G$
4. Inverse: $\forall a \in G \ exists b \in G$ s.t. $a \cdot b = b \cdot a = e \rightarrow$ identity

Abelian group is a group with additional property that

5. $a, b \in G$ then $a \cdot b = b \cdot a$

Example: $(\mathbb{Z}, +)$: set of integers with addition operation

1. $a, b \in \mathbb{Z}, a + b \in \mathbb{Z}$
2. $a, b, c \in \mathbb{Z}, a + (b + c) = (a + b) + c$
3. Identity is $0 \in \mathbb{Z}: a + 0 = 0 + a = a$
4. Inverse is $-a \forall a \in \mathbb{Z}$
5. Abelian $a, b \in \mathbb{Z}, a + b = b + a$
Example: \((\mathbb{Z}_2, +)\): integers modulo 2 with addition
\[\mathbb{Z}_2 = \{0, 1\}\]
1 + 1 = 0 \(\text{ (2 mod 2)}\)
0 + 1 = 1 + 0 = 1 \(\in \mathbb{Z}\)

2. Identity is 0, 3. Inverse is ± element itself
0 + 0 = 0, 1 + 1 = 0

This is an Abelian group.

\# Homology: Let \(K\) be a simplicial complex, of dimension \(p\)

**Def.** modulo 2 coefficient \((\mathbb{Z}_2\text{ coeff.)}\) \(a \in \mathbb{Z}_2\) \(a = 0\) or \(a = 1\)

**Def.** A \(p\)-chain is a formal sum of \(p\)-simplices in \(K\)
\[c = \sum a_i \sigma_i\] where \(a_i \in \mathbb{Z}_2\) \((0\ or\ 1)\)

\[K = \{1, 2, 3, 12, 14, 24, 23, 34, 32, 124, 234\}\]

\[C_0: 0\text{-chain} = \sum a_i \sigma_i\text{, }1+2+3+4\text{, }1+4, 2+4\]
\[C_1: 1\text{-chain} = 12+23+34+41\]
\[C_2: 2\text{-chain} = 124+234\]

\[\rightarrow \text{Let } C_0 = 1+2+3, C_0' = 1+3+4 \text{ then } C_0 + C_0' = 1+2+3+1+3+4\]

\[\rightarrow \text{Let } C_1 = 12+23+34+4\text{, }14\text{ then } C_1 + C_1' = 2+4 \text{ (modulo 2)}\]
\[C_1' = 23+34+24\]
\[C_1 + C_1' = 12+24+14\]

\[\rightarrow \text{Chain addition: component-wise addition modulo 2}\]
\[c = \sum a_i \sigma_i\Rightarrow c + c' = \sum (a_i + b_i) \sigma_i\]
\[c' = \sum b_i \sigma_i\]

modulo 2 coefficients: \(a_i + b_i \in \{0, 1\}\)
Def: Chain group! The set of all $p$-chains with the addition operation form a group $(C_p, +)$ or $C_p(K)$.

- $C_0$: 0-chain group, $C_1$: 1-chain group, $C_2$: 2-chain group.

- $\forall c, c' \in C_p, c + c' \in C_p \quad \times$ Addition is component-wise sum modulo 2

- $0 + c = c + 0 = c$

- $c + c = 0$

Def: Boundary of a $p$-simplex is the sum of its $(p-1)$-dimensional faces.

$$\partial_p \sigma = \sum_{j=0}^{p-1} [u_0, \ldots, u_j, \ldots, u_p]$$

Boundary of a triangle is

- $[1, 2, 3] \rightarrow 12$
- $[1, 2, 3] \rightarrow 13$
- $[1, 2, 3] \rightarrow 23$

sum of its edges

- $[1, 2, 3] \rightarrow 12$
- $[1, 2, 3] \rightarrow 13$
- $[1, 2, 3] \rightarrow 23$

Let $C = 1 + 2 + 3 + 4$

$$\partial C = 1 + 2 + 3 + 4 = 1 + 4$$

- $C = 123$
- $\partial C = 12 + 23 + 13$
- $\partial(\partial C) = 1 + 2 + 2 + 3 + 1 + 3 = 0$

- Boundary of a boundary is always $0$. 

$\sigma = [u_0, \ldots, u_p]$
Def: A p-cycle is a p-chain with empty boundary ($\partial c = 0$)

Set of all p-cycles $Z_p = Z_p(K)$ is a group that is a subgroup of $C_p(K)$.

Def: A p-boundary is a p-chain that is boundary of a (p+1)-chain $c = \partial d$ where $d \in C_{p+1}$.

Set of all p-boundaries is a group $B_p = B_p(K)$.

$c \in B_p(K), \quad c = \partial (124) = 12 + 24 + 14$

$c \in Z_p(K), \quad c = 23 + 34 + 24, \quad c' = 12 + 24 + 14$

$c'' = 12 + 23 + 34 + 14$

Def: The p-th homology group is the p-th cycle group modulo the p-th boundary group $H_p = Z_p / B_p$.

"Cycles that don't bound" $\Rightarrow$ p-cycle that is not boundary of any (p+1)-chain.

$c \in (23 + 34 + 24)$ and $(12 + 23 + 34 + 14)$ belong to $Z_p$ but are not boundary of any 2-chain.

$(12 + 24 + 14) \rightarrow$ boundary of $(124)$.