Jan 31

Review: Homotopy Equivalence: $f: \mathbb{X} \to \mathbb{Y}$ is called a homotopy equivalence if there is a map

$$g: \mathbb{Y} \to \mathbb{X} \text{ such that } f \circ g \simeq 1, \quad g \circ f \simeq 1$$

$\simeq$ implies homotopic.

Def 1 A filtration is a sequence of simplicial complexes connected by inclusion

$$K_0 = \phi \subseteq K_1 \subseteq K_2 \ \ldots \subseteq K_m$$

[This will come up in persistent homology]

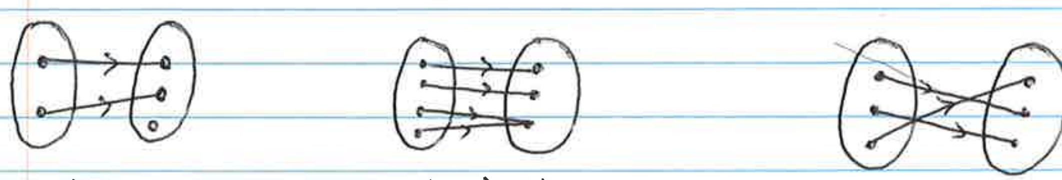
Def: Two topological spaces are homeomorphic or topologically equivalent if $\exists$ a continuous bijection from one space to the other whose inverse is also continuous

Def: A function $f: \mathbb{X} \to \mathbb{Y}$ is a homeomorphism if it satisfies following properties:

1. f is a bijection (one-to-one and on-to)
2. f is continuous
3. The inverse f^{-1} is continuous

Review: 1-to-1: "preserves distinction" (Injective)
Never maps two distinct elements to the same element

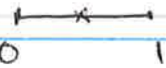
Onto: (surjective): Every element in range has a corresponding element in domain.



Injective

Surjective

bijective

example:  unit interval X

 Circle Y

X and Y are not homeomorphic

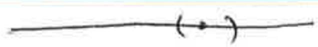
Can you define $f: X \to Y$ that is continuous & bijective ?

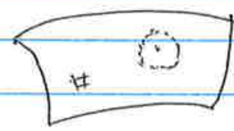
→ We can look at homology of the two spaces.

Dim-0 homology : Connected components : both X & Y have 1
Dim-1 homology : tunnels : Only Y has one, X doesn't.

→ Topologists care about core properties of shape that are unchanged through continuous deformations.

Manifolds : " ant's world "

1-manifold  locally 1-dimensional

2-manifold  locally 2-dimensional (plane)

3-manifold  ← solid snowball

→ The surface is a 2-manifold

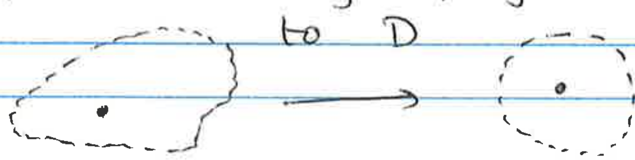
→ Open unit disk : $D = \{ x \in \mathbb{R}^2 \mid \|x\| < 1 \}$

Claim : D is homeomorphic to $\mathbb{R}^2$

$f: D \to \mathbb{R}^2$, $f(x) = \dfrac{x}{1 - \|x\|}$

if x is on the boundary of disk, f maps it to infinity

→ Open disk : Any topological space that is homeomorphic to D



Def: A 2-manifold (w/o boundary) is a topological space $\mathbb{M}$ whose points all lie in open disks.

(Intuitively, $\mathbb{M}$ locally looks like a plane)
→ a small open disk centered at any point on surface of a donut or a basketball → locally looks like plane.

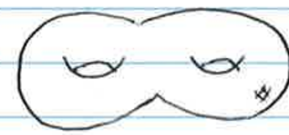
→ We get a 2-manifold with boundary by removing open disks from 2-manifold without boundary.

When we punch a hole on the surface of basketball, the surface left is a 2-manifold with boundary.

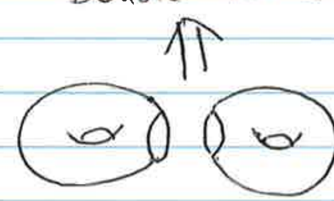
→ <u>without boundary:</u>

Sphere $\mathbb{S}^2$

Torus $\mathbb{T}^2$



Double Torus $\mathbb{T}^2 \# \mathbb{T}^2$



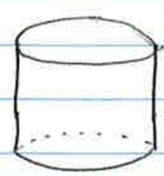
[Connected sum]

Two toruses, cut out open disks and glue together

→ <u>With boundary:</u>

Closed disk

1 boundary



Cylinder

2 boundaries

mobius Strip

1 - boundary

<u>Orientability</u> : mobius strip is an example of non-orientable surface : 2 sides locally, 1 side globally.

→ Torus : Orientable surface.

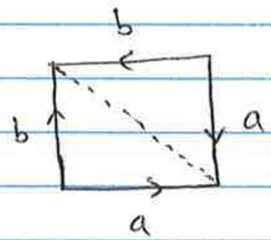
→ consider a small loop clutching the handle. moving the loop around does not change the orientation of the loop.

→ An oriented loop on surface of sphere : no matter how you move it, preserves the orientation.

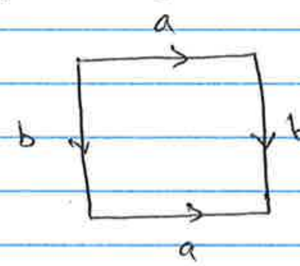
→ Oriented loop : take it around on mobius strip : changes Orientation.

<u>Def</u> : If all closed curves in a 2-manifold are orientation preserving then the 2-manifold is orientable.

→ Creating Compact 2-manifold by polygonal schema

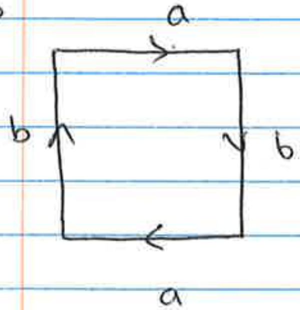


→ sphere



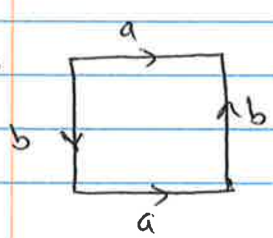
→ Torus

glue a, a together, b, b together preserving orientation.

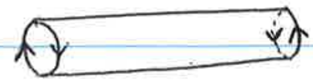
→ 

projective plane $\mathbb{RP}^2$: glue a disk to a mobius strip

Can't be embedded in $\mathbb{R}^3$ without self intersection.

→ 

→ Klein bottle : glue 2 mobius strips together.



[imagine joining the two ends of this tube but in such a way that the orientation of the boundries is preserved.]

<u>Theorem</u>: Classification theorem for compact 2-manifold.

The two infinite families

$$S^2, \ \mathbb{T}^2, \ \mathbb{T}^2 \# \mathbb{T}^2, \ \ldots$$

and

$$\mathbb{P}^2, \ \mathbb{P}^2 \# \mathbb{P}^2, \ \ldots$$

exhaust the compact 2-manifolds without boundary

<u>Def</u>: $\mathbb{M}$ is compact if for every cover of $\mathbb{M}$ by open set S, (open cover), we can find a finite ~~set~~ number of sets that cover $\mathbb{M}$

<u>Claim</u>: A subset of Euclidean space is compact if it is closed and bounded → [Contained in a ball of finite radius]

→ Think of patches covering a surface. If we can find a finite number of patches that cover the surface then it is bounded.

<u>Homology</u>: historically → Can 2 spaces be distinguished by examining their holes.

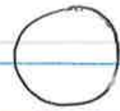
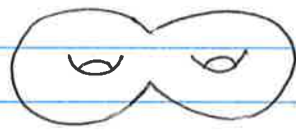
→ Betti Numbers:

β_0 : # connected components

β_1 : # of tunnels

β_2 : # of voids

We are talking about ~~numbes~~ number of independent cc / tunnels / voids i.e. set of cc / tunnels / voids that generate all other possible cc / tunnels / voids.

		β_0	β_1	β_2
Circle		1	1	0
Sphere		1	0	1
Torus		1	2	1
		1	4	1