Jan 26

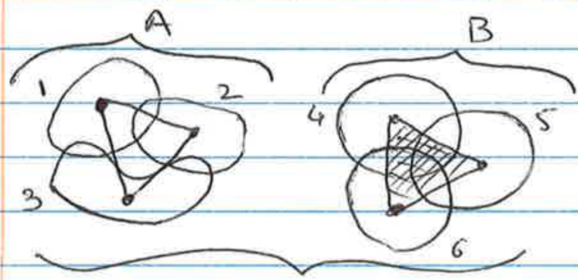
(1) Persistent homology → local homology
→ Cohomology (Dimensionality Reduction)
→ Advanced topics: persistent modules
Zig-zag persistence

(2) Topological Structures
[summarizing data]
→ Contour trees
→ Reeb graphs, Reeb space, mapper
→ Morse-Smale Complex

Nerves & Nerve Theorem [application: Mapper]

Def: Let F be a finite collection of sets. The nerve of F (denote $Nrv(F)$) $= \{X \subseteq F \mid \bigcap X \neq \phi\}$

i.e. set of subsets of F having non-empty intersection.



F: 6 sets

Sets 1, 2, 3 only intersect pairwise so the nerve is the three edges.
Sets 4, 5, 6 have common intersection ∴ the nerve is the triangle.

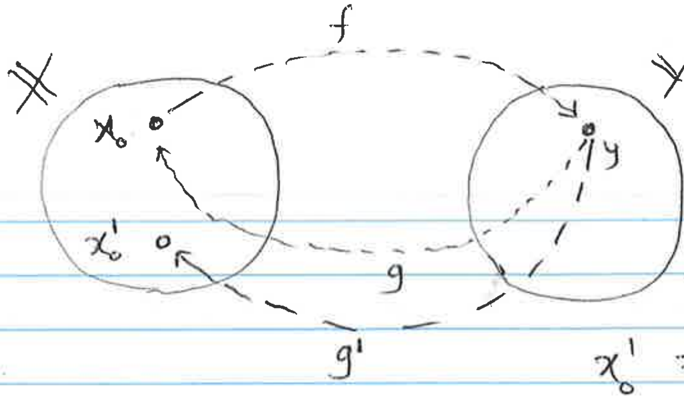
→ Nerve of a collection is an abstract SC
→ Usually the sets have some underlying geometry.

Aside: Convex Sets: Intersection of convex sets is convex.

Theorem: Nerve Theorem: F is a finite collection of closed convex sets in Euclidean space, Then the nerve of F and the union of sets in F have the same homotopy type.

Def: A map $f: X \to Y$ is called a homotopy equivalence if
(*) there is a function/map $g: Y \to X$ such that

$f \circ g \simeq I_Y$ and $g \circ f \simeq 1_X$ [$\simeq$ homotopic]



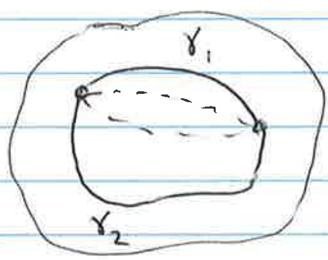
$x_0 = g \circ f = I_X$ [identity map]

$x_0' = g' \circ f \simeq I_X$ [homotopic equivalence]

<u>Def</u>: Two continuous functions $f, g : X \to Y$ are homotopic if one can be continuously deformed into the other

there exists a path from f to g

$H : X \times [0,1] \to Y \qquad X[0] = f, \quad X[1] = g$



$\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^2$

Here $\gamma_1 \simeq \gamma_2$

$\gamma_1[0] = \gamma_2[0]$

$\gamma_1[1] = \gamma_2[1]$

<u>Def</u>: Continuing def ⊛ from previous page:

Then X and Y are said to be homotopy equivalent or X and Y have the same homotopy type. $X \simeq Y$

<u>Čech Complex</u>: Convex sets are closed geometric balls

Let S: points in $\mathbb{R}^d$

$B_x(r)$: Closed ball of radius r centered at x

<u>Def</u>: The Čech complex of S at radius r is the nerve of the collection of balls. r is a parameter.

$$\check{C}ech(r) = \left\{ \sigma \subseteq S \;\middle|\; \bigcap_{x \in \sigma} B_x(r) \neq \emptyset \right\}$$

<u>Claim</u>: for $r_0 \leq r_1$

$$\check{C}ech(r_0) \subseteq \check{C}ech(r_1)$$

Vietoris - Rips Complex : $VR(r) = \{\sigma \subseteq S \mid diam(\sigma) \leq 2r\}$

→ ~~are closed balls~~.

diam is the diameter of set X making up σ

Claim : in $\mathbb{R}^2$, $\check{C}ech(r) \subseteq VR(r) \subseteq \check{C}ech(\sqrt{2}\, r)$

[Ref] : Coverage in Sensor Networks via persistent homology
– Vin de Silva, Robert Ghrist

Theorem [2.5] :

Let X be a set of points in $\mathbb{R}^d$ and $C_\epsilon(X)$ be the $\check{C}ech$ complex of the cover of X by balls of $\epsilon/2$ radius then there is a chain of inclusions :

$$VR_{\epsilon'}(X) \subseteq C_\epsilon(X) \subseteq VR_\epsilon(X) \quad \text{where} \quad \frac{\epsilon}{\epsilon'} \geq \sqrt{\frac{2d}{d+1}}$$

Delaunay Complex : for S: a point set in $\mathbb{R}^d$

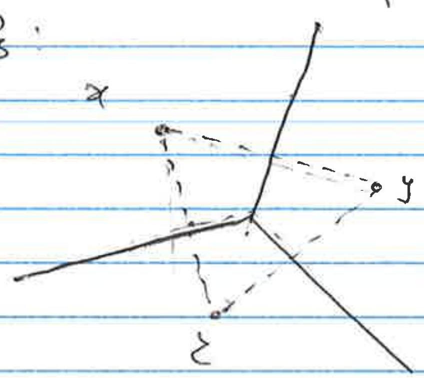
Def : A Voronoi cell of a point u in S is the set of points in $\mathbb{R}^d$ ~~st~~ for which u is the closest.

$$V(u) = \{x \in \mathbb{R}^d \mid \|x - u\| \leq \|x - v\| \ \forall\, v \in S\}$$

in $\mathbb{R}^2$, with only one point in S → $V(u) = \mathbb{R}^2$

if $S = \{u, v\}$ → two cells → half planes separated by the perpendicular bisector of segment uv.

$S = \{x, y, z\}$:



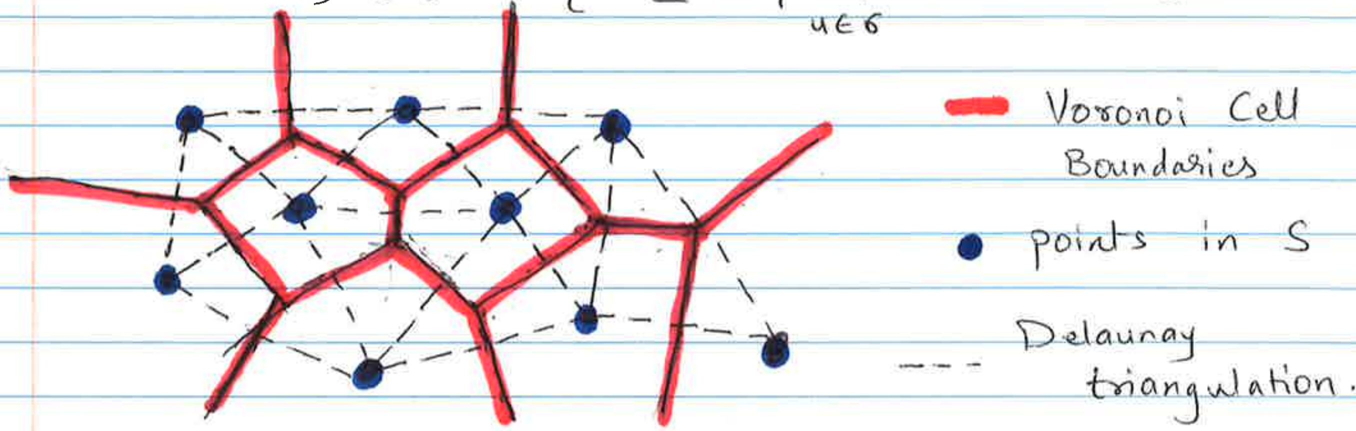
Def : $V(u)$ is the intersection of half spaces of points at least as close to u as to v for any v in S

→ it is a convex polyhedron in $\mathbb{R}^d$

→ half space is convex ; intersection of convex sets is convex.

Def: The Delaunay Complex / Delaunay triangulation of a set of points S in $\mathbb{R}^d$, is isomorphic to the nerve of the Voronoi diagram.

$$\text{Delaunay}(S) = \left\{ \sigma \subseteq S \;\middle|\; \bigcap_{u \in \sigma} V(u) \neq \phi \right\}$$

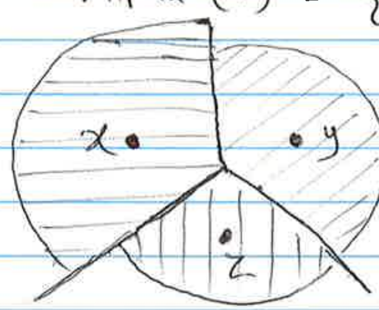


Let: $B_u(r)$: ball of radius r centered at $u \in S$

$R_u(r)$: $V(u) \cap B_u(r)$: intersection of ball with Voronoi cell

Def: Alpha Complex at radius r is

$$\text{Alpha}(r) = \left\{ \sigma \in S \;\middle|\; \bigcap_{u \in \sigma} R_u(r) \neq \phi \right\}$$



→ $R_y(r)$ → also called the alpha cell.

⊛ Think of soap bubbles. the common boundary betⁿ two bubbles is a flat surface.

→ Intersection with other balls → straight line boundaries.
non-intersecting cell boundaries : curved (circular)