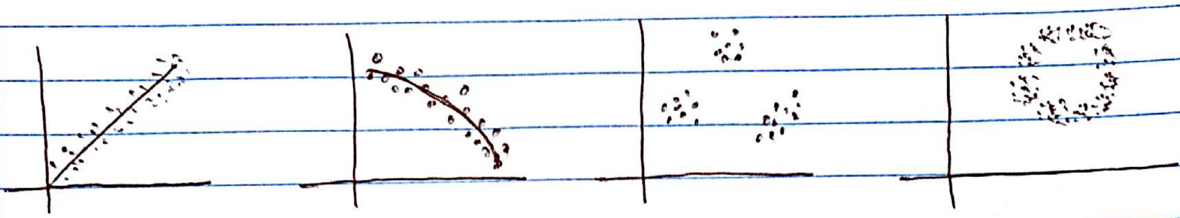
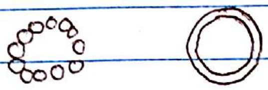
Computational Topology : Lecture 02

12 Jan 2017

Data has shape ; Shape matters ; shape has meaning



Shape can have different meaning at different scales



$\Rightarrow$ Depending on scale, data can be sampled from set of discs or from a single annulus.

→ Today we'll look at (a) Connectivity (b) Union-find

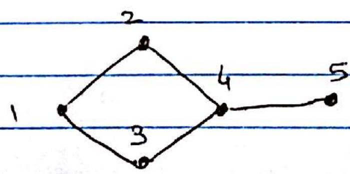
motivating data type: Graphs (Nodes and edges)

eg.

(i) Abstract Graphs: Social Networks, Biological networks (Gene Expression etc.)

(ii) Embedded Graphs: Graphs where nodes and edges have spatial meaning (co-ordinates)
- transportation networks, SLC road grid etc.

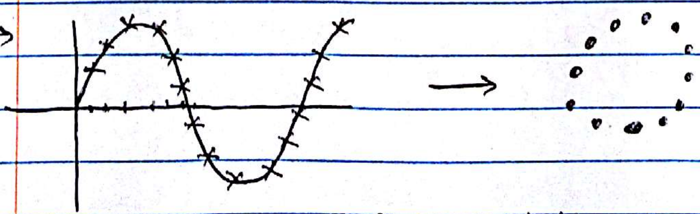
→ Abstract graphs are discrete graphs. Pair of sets $G = (V, E)$, where $E \subseteq V \times V$



$\Rightarrow$ $V = \{1, 2, 3, 4, 5\}$

$E = \{12, 13, 24, 34, 45\}$

→ Embedded graphs are continuous : Geometry matters.

→ 

x → points sampled from periodic signal

Delayed Window Embedding
moving window of consecutive pairs of points plotted as x,y co-ordinates maps periodic signal to circle.

②

$$\boxed{\text{Discrete Object}} \xleftrightarrow{\text{TDA}} \boxed{\text{Continuous Object}}$$

Def^n: Simple graph : ⓐ No multi-edges
ⓑ No self loops

$G = (V, E)$ with $|V| = n$ then $|E| \le \binom{n}{2}$

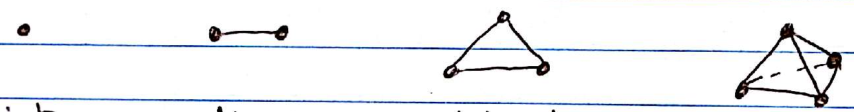
Def^n: Complete Graph (K_n) : Simple graph with all possible edges. ~~G~~ $K_n = (V, E)$: $|V| = n$, $|E| = \binom{n}{2}$

→ K_n is a regular graph of degree $(n-1)$.

Def^n: Regular graph : all vertices have same degree.

→ K_n represents edge of $(n-1)$ - simplex

K_1 K_2 K_3 K_4



point line triangle tetrahedron

Def^n: Given a graph $G = (V, E)$ a path $\gamma(u, v)$ is a sequence of vertices $\{u_0 = u, u_1, u_2, \dots u_k = v\}$ such that edge $u_i u_{i+1} \in E$ for all vertices u_i in sequence.

→ Simple Path : all vertices are distinct (No loops in the path)
→ Path length : ~~# vertices~~ Number of edges traversed (Number of hops)

Def^n: A simple graph is connected if there is a path between every pair of vertices.

A connected component (CC) of a graph is a maximal sub-graph that is connected.

③

→ Algorithms to test whether a graph is connected:
① DFS : depth first search ② BFS : Breadth first

→ We will see another algorithm called union-find.

Claim : Smallest connected graph is a tree. (n vertices, $n-1$ edges)
deleting any edge disconnects the tree.

Def^n : Spanning tree : Given graph $G = (V, E)$, spanning tree of G is a tree $T = (V, E')$ which is the connected subgraph of G with minimum number of edges

Def^n : Separation : Non-trivial partition of vertex set such that $V = U \vee W$ then no edges connect vertices from U to vertices in W.

Claim : Graph is connected $\iff$ it has spanning tree
Graph is connected $\iff$ it has no separation.

Topological Spaces

Def^n : A topological space is path connected if every pair of points is connected by a path eg. points on a disk

Def^n : A separation of topological space X is a partition $X = U \vee W$ with non-empty, open subsets.

→ Topological space is connected if there is no separation

→ path connected $\Rightarrow$ connected But connected $\not\Rightarrow$ path connected.
eg. Topologist's sin curve.