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Discrete Stratified Morse Theory

Algorithms and A User's Guide

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Abstract Inspired by the works of Forman on discrete Morse theory, which is a combinatorial adaptation to cell complexes of classical Morse theory on manifolds, we introduce a discrete analogue of the stratified Morse theory of Goresky and MacPherson. We describe the basics of this theory and prove fundamental theorems relating the topology of a general simplicial complex with the critical simplices of a discrete stratified Morse function on the complex. We also provide an algorithm that constructs a discrete stratified Morse function out of an arbitrary function defined on a finite simplicial complex; this is different from simply constructing a discrete Morse function on such a complex. We then give simple examples to convey the utility of our theory. Finally, we relate our theory with the classical stratified Morse theory in terms of triangulated Whitney stratified spaces.

Keywords Discrete Morse theory · Stratified Morse theory · Topological data analysis

1 Introduction

It is difficult to overstate the utility of classical Morse theory in the study of manifolds. A Morse function $f : \mathbb{M} \to \mathbb{R}$ determines an enormous amount of information about the manifold $\mathbb{M}$: a handlebody decomposition, a realization of $\mathbb{M}$ as a CW-complex whose cells are determined by the critical points of f, a chain complex for computing the integral homology of $\mathbb{M}$, and much more.

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With this as motivation, Forman developed discrete Morse theory on general cell complexes [13]. This is a combinatorial theory in which function values are assigned not to points in a space but rather to entire cells. Such functions are not arbitrary; the defining conditions require that function values generically increase with the dimensions of the cells in the complex. Given a cell complex with set of cells K, a discrete Morse function $f : K \to \mathbb{R}$ yields information about the cell complex similar to what happens in the smooth case.

While the category of manifolds is rather expansive, it is not sufficient to describe all situations of interest. Sometimes one is forced to deal with singularities, most notably in the study of algebraic varieties. One approach to this is to expand the class of functions one allows, and this led to the development of stratified Morse theory by Goresky and MacPherson [18]. The main objects of study in this theory are *Whitney stratified spaces*, which decompose into pieces that are smooth manifolds. Such spaces are triangulable.

The goal of this paper is to generalize stratified Morse theory to finite simplicial complexes, much as Forman did in the classical smooth case. Given that stratified spaces admit simplicial structures, and any simplicial complex admits interesting discrete Morse functions, this could be the end of the story. However, we present examples in this paper illustrating that the class of discrete stratified Morse functions defined here is much larger than that of discrete Morse functions. Moreover, there exist discrete stratified Morse functions that are nontrivial and interesting from a data analysis point of view.

Contributions. Throughout the paper, we hope to convey via simple examples the usability of our theory. It is important to note that our discrete stratified Morse theory is *not* a simple reinterpretation of discrete Morse theory; it considers a larger class of functions defined on any finite simplicial complex and has potentially many implications for data analysis. Our contributions are:

1. We describe the basics of a discrete stratified Morse theory and prove fundamental theorems that relate the topology of a finite simplicial complex with the critical simplices of a discrete stratified Morse function defined on the complex.
2. We provide an algorithm that constructs a discrete stratified Morse function on any finite simplicial complex equipped with a real-valued function.
3. We prove that given a stratified set S equipped with a triangulation T and a stratified Morse function $f : S \to \mathbb{R}$, there is an integer r such that the r-th barycentric subdivision of T supports a discrete stratified Morse function whose critical cells correspond to the critical points of f.
4. We demonstrate how to build a discrete stratified Morse function from a function defined on the vertices of a simplicial complex, based on a modification of the algorithm by King et al. [25].

An extended abstract of the present paper previously appeared as a conference paper [27], which gave preliminary results surrounding contributions 1 and 2 above. The current paper contains the following extensions that encompass improvements of and changes to the conference version as well as new results. In particular, we change the definition of a stratified simplicial complex (Definition 3.1) to be well-aligned with its continuous counterpart (e.g. Whitney stratification) that considers the condition of the frontier. Given this new definition, Theorem 3.3 and Corollary 3.1 discuss the change of homotopy type surrounding critical cells. We give new results that relate discrete Morse and discrete stratified Morse functions (Theorems 3.1 and 3.2). We further characterize the coarseness property of our algorithm in constructing stratified Morse functions from any real-valued function on a simplicial complex (Proposition 3.1). Finally, we discuss the applications of our theory to classical stratified Morse theory in discretizing a stratified Morse function (Theorem 5.3) and provide an algorithm to generate discrete stratified Morse functions from point data (Theorem 6.1).

A simple example. We begin with an example from Forman [15], where we demonstrate how a discrete stratified Morse function can be constructed from a function that is not a discrete Morse function. As illustrated in Figure 1, the function on the left is a discrete Morse function where the green arrows can be viewed as its discrete gradient vector field; function f in the middle is not a discrete Morse function, as the vertex $f^{-1}(5)$ and the edge $f^{-1}(0)$ both violate the defining conditions of a discrete Morse function. However, we can equip f with a stratification s by treating such violators as their own independent strata and taking care of boundary conditions, therefore converting f into a discrete stratified Morse function.

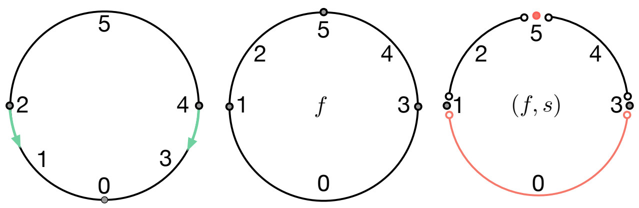

Fig. 1 The function on the left is a discrete Morse function. The function f in the middle is not a discrete Morse function; however, it can be converted into a discrete stratified Morse function on the right when it is equipped with an appropriate stratification s.

2 Preliminaries on Discrete Morse Theory

We review the most relevant definitions and results on discrete Morse theory and refer the reader to Appendix A for a review of classical Morse theory. Discrete Morse theory is a combinatorial version of Morse theory [13, 15]. It can be defined for any CW complex but in this paper we will restrict our attention to finite simplicial complexes.

Discrete Morse functions. Let K be any finite simplicial complex, where K need not be a triangulated manifold nor have any other special property [14].

When we write K we mean the set of simplices of K; by $|K|$ we mean the underlying topological space. Let $\alpha^{(p)} \in K$ denote a simplex of dimension p. Let $\alpha < \beta$ denote that simplex α is a face of simplex β. If $f : K \to \mathbb{R}$ is a function define $U(\alpha) = \{\beta^{(p+1)} > \alpha \mid f(\beta) \leq f(\alpha)\}$ and $L(\alpha) = \{\gamma^{(p-1)} < \alpha \mid f(\gamma) \geq f(\alpha)\}$. In other words, $U(\alpha)$ contains the immediate cofaces of α with lower (or equal) function values, while $L(\alpha)$ contains the immediate faces of α with higher (or equal) function values. Let $|U(\alpha)|$ and $|L(\alpha)|$ be their sizes.

Definition 2.1 A function $f : K \to \mathbb{R}$ is a *discrete Morse function* if for every $\alpha^{(p)} \in K$, (i) $|U(\alpha)| \leq 1$ and (ii) $|L(\alpha)| \leq 1$.

Forman showed that conditions (i) and (ii) are exclusive – if one of the sets $U(\alpha)$ or $L(\alpha)$ is nonempty then the other one must be empty ([13], Lemma 2.5). Therefore each simplex $\alpha \in K$ can be paired with at most one exception simplex: either a face γ with larger function value, or a coface β with smaller function value. Formally, this means that if K is a simplicial complex with a discrete Morse function f, then for any simplex α, either (i) $|U(\alpha)| = 0$ or (ii) $|L(\alpha)| = 0$ ([15], Lemma 2.4).

Definition 2.2 A simplex $\alpha^{(p)}$ is *critical* if (i) $|U(\alpha)| = 0$ and (ii) $|L(\alpha)| = 0$. A *critical value* of f is its value at a critical simplex.

Definition 2.3 A simplex $\alpha^{(p)}$ is *noncritical* if either of the following conditions holds: (i) $|U(\alpha)| = 1$; (ii) $|L(\alpha)| = 1$; as noted above these conditions can not both be true ([13], Lemma 2.5).

Given $c \in \mathbb{R}$, we have the *sublevel complex* $K_c = \cup_{f(\alpha) \leq c} \cup_{\beta \leq \alpha} \beta$. That is, K_c contains all simplices α of K such that $f(\alpha) \leq c$ along with all of their faces.

Results. We have the following two combinatorial versions of the main results of classical Morse theory.

Theorem 2.1 (DMT Theorem A, [14]) *Suppose the interval $(a, b]$ contains no critical value of f. Then K_b is homotopy equivalent to K_a. In fact, K_b simplicially collapses onto K_a.*

A key component in the proof of Theorem 2.1 is the following fact [13]: for a simplicial complex equipped with an arbitrary discrete Morse function, when passing from one sublevel complex to the next, the noncritical simplices are added in pairs, each of which consists of a simplex and a free face.

The next theorem explains how the topology of the sublevel complexes changes as one passes a critical value of a discrete Morse function. In what follows, $\dot{e}^{(p)}$ denotes the boundary of a p-simplex $e^{(p)}$.

Theorem 2.2 (DMT Theorem B, [14]) *Suppose $\sigma^{(p)}$ is a critical simplex with $f(\sigma) \in (a, b]$, and there are no other critical simplices with values in $(a, b]$. Then K_b is homotopy equivalent to the space obtained by attaching a p-cell $e^{(p)}$ along its entire boundary in K_a; that is, $K_b = K_a \cup_{\dot{e}^{(p)}} e^{(p)}$.*

The associated gradient vector field. Given a discrete Morse function $f : K \to \mathbb{R}$ we may associate a discrete gradient vector field as follows. Since any noncritical simplex $\alpha^{(p)}$ has at most one of the sets $U(\alpha)$ and $L(\alpha)$ nonempty, there is a unique face $\nu^{(p-1)} < \alpha$ with $f(\nu) \geq f(\alpha)$ or a unique coface $\beta^{(p+1)} > \alpha$ with $f(\beta) \leq f(\alpha)$. Denote by V the collection of all such pairs $\{\sigma < \tau\}$. Then every simplex in K is in at most one pair in V and the simplices not in any pair are precisely the critical cells of the function f. We call V the *gradient vector field associated to* f. We visualize V by drawing an arrow from α to β for every pair $\{\alpha < \beta\} \in V$. Theorems 2.1 and 2.2 may then be visualized in terms of V by collapsing the pairs in V using the arrows. Thus a discrete gradient (or equivalently a discrete Morse function) provides a collapsing order for the complex K, simplifying it to a complex L with potentially fewer cells but having the same homotopy type.

The collection V has the following property. By a *V-path*, we mean a sequence

$$\alpha_0^{(p)} < \beta_0^{(p+1)} > \alpha_1^{(p)} < \beta_1^{(p+1)} > \cdots < \beta_r^{(p+1)} > \alpha_{r+1}^{(p)}$$

where each $\{\alpha_i < \beta_i\}$ is a pair in V. Such a path is *nontrivial* if $r > 0$ and *closed* if $\alpha_{r+1} = \alpha_0$. Forman proved the following result.

Theorem 2.3 ([13]) *If V is a gradient vector field associated to a discrete Morse function f on K, then V has no nontrivial closed V-paths.*

In fact, if one defines a discrete vector field V to be a collection of pairs of simplices of K such that each simplex is in at most one pair in V, then one can show that if V has no nontrivial closed V-paths there is a discrete Morse function f on K whose associated gradient is precisely V.

We note here the following result that will be needed below. A proof may be found in [26, p. 99].

Lemma 2.1 *Suppose K' is the barycentric subdivision of K and let V be a discrete gradient vector field on K. Then there is a discrete gradient vector field V' on K' such that the critical cells of V and V' are in one-to-one correspondence. In fact, for a critical p-cell $\alpha \in K$ of V, one may choose a p-cell $\alpha' \in K'$ with $\alpha' \subset \alpha$ which is critical for V'.* □

3 A Discrete Stratified Morse Theory

Our goal is to describe a combinatorial version of stratified Morse theory. To do so, we need to: (a) define a discrete stratified Morse function; and (b) prove the combinatorial versions of the relevant fundamental results. Our results are very general as they apply to any finite simplicial complex K equipped with a real-valued function $f : K \to \mathbb{R}$. Our work is motivated by relevant concepts from (classical) stratified Morse theory [18], whose details are found in Appendix A.

3.1 Background

Open simplices. To state our main results, we need to consider open simplices (as opposed to the closed simplices of Section 2). Let $\{a_0, a_1, \cdots, a_k\}$ be a geometrically independent set in $\mathbb{R}^N$, a *closed k-simplex* $[\sigma]$ is the set of all points x of $\mathbb{R}^N$ such that $x = \sum_{i=0}^{k} t_i a_i$, where $\sum_{i=0}^{k} t_i = 1$ and $t_i \geq 0$ for all i [32]. An *open simplex* (σ) is the interior of the closed simplex $[\sigma]$.

A *simplicial complex* K is a finite set of open simplices such that: (a) If $(\sigma) \in K$ then all open faces of $[\sigma]$ are in K; (b) If $(\sigma_1), (\sigma_2) \in K$ and $(\sigma_1) \cap (\sigma_2) \neq \emptyset$, then $(\sigma_1) = (\sigma_2)$. For the remainder of this paper, we always work with a finite open simplicial complex K. Unless otherwise specified, we work with open simplices σ and define the boundary $\dot{\sigma}$ to be the boundary of its closure.

Stratified simplicial complexes. In the conference version of this paper [27], we worked with a weak notion of stratification. We have since discovered technical issues with that definition; work in progress seeks to find the most general setting in which our theory can be applied. In this paper, we employ Definition 3.1 to define a stratified simplicial complex.

Recall a subset S of a topological space Z is *locally closed* if it is the intersection of an open and a closed set in Z. For a topological space S, let $\overline{S}$ denote its closure, $\mathring{S}$ its interior.

Definition 3.1 A *stratification* of K is a finite collection of disjoint subsets $\mathcal{S} = \{S_i\}$ called *strata*, where each $|S_i|$ is a locally closed subset of $|K|$, such that $K = \bigcup S_i$, and which satisfies the condition of the frontier: $|S_i| \cap \overline{|S_j|} \neq \emptyset$ if and only if $|S_i| \subseteq \overline{|S_j|}$.

For the remainder of the paper, we abuse notation by writing S_i instead of its geometric realization $|S_i|$. Also, since we will be dealing exclusively with functions defined simplex-wise, when we write $f : S_i \to \mathbb{R}$, it will be understood that this is a function assigning a single value to each simplex in S_i.

Remark 3.1 Let $\overline{S_i} \setminus \mathring{S}_i$ be the frontier of a stratum S_i. The frontier condition of Definition 3.1 is equivalent to the statement that the frontier of each S_i is a union of strata. Each S_i is a union of (open) simplices; its connected components are called *strata pieces*. The condition of the frontier in Definition 3.1 yields a $\mathcal{P}$-decomposition as in [18, p. 36]. That is, it imposes a partial order $\mathcal{P} = (\mathcal{S}, \preceq)$ on the strata: $S_i \preceq S_j$ if and only if $S_i \subseteq \overline{S_j}$.

Lemma 3.1 *A minimal element in the partial order $\mathcal{P} = (\mathcal{S}, \preceq)$ is a subcomplex of K.*

Proof Suppose S_i is such a minimal element and suppose $\sigma \in S_i$. It suffices to show that $\partial\sigma \in S_i$ as well. Suppose $\tau < \sigma$. Then $\tau \in S_j$ for some j and so $\tau \in S_j \cap \overline{S_i}$. This implies that $S_j \subseteq \overline{S_i}$. But S_i is minimal in the order $\mathcal{P}$ and so $S_j = S_i$; that is, $\tau \in S_i$. □

A stratification gives an *assignment* from K to the set $\mathcal{S}$, denoted $s : K \to \mathcal{S}$. In our setting, each S_i is the union of finitely many open simplices (that may not form a subcomplex of K); and each open simplex σ in K is assigned to a particular stratum $s(\sigma)$ via the mapping s. Since these subspaces may not be simplicial complexes, we must modify Definition 2.1 as follows.

Definition 3.2 Suppose S_i is a stratum in $\mathcal{S}$. A function $f : S_i \to \mathbb{R}$ is a *discrete Morse function* if for every $\alpha^{(p)} \in S_i$, (i) $|U(\alpha)| \leq 1$, (ii) $|L(\alpha)| \leq 1$, and (iii) if one of the sets $U(\alpha)$ or $L(\alpha)$ is nonempty then the other must be empty.

Condition (iii) above is not necessary for functions defined on simplicial complexes, but the proof of that relies on the fact that all faces of a simplex are in the complex as well. This need not be true for the various strata and so we impose the condition here.

Stratum-preserving homotopies. If X and Y are two stratified spaces, we call a map $f : X \to Y$ *stratum-preserving* if the image of each connected component of a stratum of X lies in a unique component of a stratum of Y [16]. Such a map $f : X \to Y$ is a *stratum-preserving homotopy equivalence* if there exists a stratum-preserving map $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are stratum-preserving homotopic to the identity [16].

3.2 Discrete Stratified Morse Function

Discrete stratified Morse function. Let K be a simplicial complex equipped with a stratification s and a function $f : K \to \mathbb{R}$. We define

$$U_s(\alpha) = \{\beta^{(p+1)} > \alpha \mid s(\beta) = s(\alpha) \text{ and } f(\beta) \leq f(\alpha)\},$$
$$L_s(\alpha) = \{\gamma^{(p-1)} < \alpha \mid s(\gamma) = s(\alpha) \text{ and } f(\gamma) \geq f(\alpha)\}.$$

Definition 3.3 Given a simplicial complex K equipped with a stratification $s : K \to \mathcal{S}$, a function $f : K \to \mathbb{R}$ (equipped with s) is a *discrete stratified Morse function* if for every $\alpha^{(p)} \in K$, (i) $|U_s(\alpha)| \leq 1$, (ii) $|L_s(\alpha)| \leq 1$, and (iii) if one of these sets is nonempty then the other must be empty.

In other words, a discrete stratified Morse function is a pair (f, s) where $f : K \to \mathbb{R}$ is a discrete Morse function when restricted to each stratum $S_j \in \mathcal{S}$ (in the sense of Definition 3.2). We omit the symbol s whenever it is clear from the context.

Definition 3.4 A simplex $\alpha^{(p)}$ is *globally critical* if $|U(\alpha)| = |L(\alpha)| = 0$. A simplex $\alpha^{(p)}$ is *locally critical* if it is not globally critical and if $|U_s(\alpha)| = |L_s(\alpha)| = 0$. A *critical value* of f is its value at a critical simplex.

Definition 3.5 A simplex $\alpha^{(p)}$ is *globally noncritical* if $|U(\alpha)| + |L(\alpha)| = 1$. A simplex $\alpha^{(p)}$ is *locally noncritical* if it is not globally noncritical and $|U_s(\alpha)| + |L_s(\alpha)| = 1$.

The two conditions in Definition 3.5 mean that, within the same stratum as $s(\alpha)$: (i) there is a $\beta^{(p+1)} > \alpha$ with $f(\beta) \leq f(\alpha)$ or (ii) there is a $\gamma^{(p-1)} < \alpha$ with $f(\gamma) \geq f(\alpha)$; conditions (i) and (ii) cannot both be true.

A classical discrete Morse function $f : K \to \mathbb{R}$ is a discrete stratified Morse function with the trivial stratification $\mathcal{S} = \{K\}$. We will present several examples in Section 4 illustrating that the class of discrete stratified Morse functions is much larger.

Violators. The following definition is central to our algorithm in constructing a discrete stratified Morse function from any real-valued function defined on a simplicial complex.

Definition 3.6 Suppose K is a simplicial complex equipped with a real-valued function $f : K \to \mathbb{R}$. A simplex $\alpha^{(p)}$ is a *violator* of the conditions associated with a discrete Morse function if one of these conditions holds: (i) $|U(\alpha)| \geq 2$; (ii) $|L(\alpha)| \geq 2$; (iii) $|U(\alpha)| = 1$ and $|L(\alpha)| = 1$. These are referred to as type I, II and III violators; the sets containing such violators are not necessarily mutually exclusive.

Here is a useful fact about violators that we shall need later.

Lemma 3.2 *Suppose $(f, s) : K \to \mathbb{R}$ is a discrete stratified Morse function. If α is a violator for f, then either α is locally critical or α is a boundary simplex for the stratification; that is, either some face ν of α is in the frontier of $s(\alpha)$ or α is in the frontier of the stratum $s(\tau)$ of a coface τ.*

Proof By definition, $f|_{s(\alpha)}$ is a discrete Morse function on $s(\alpha)$. It is possible that α is critical for this restriction and since α is a violator it cannot be globally critical. Otherwise, there is either a face $\nu < \alpha$ with $f(\nu) \geq f(\alpha)$ or a coface $\tau > \alpha$ with $f(\alpha) \geq f(\tau)$, and this paired simplex (ν or τ) also lies in $s(\alpha)$. But α is a global violator. So in either case, there is another face $\nu' < \alpha$ or coface $\tau' > \alpha$ causing the violation. But $\nu', \tau' \notin s(\alpha)$ and hence either ν' belongs to the frontier of $s(\alpha)$ or α belongs to the frontier of $s(\tau')$ (and hence $s(\alpha) \subseteq \overline{s(\tau')}$). □

In addition to our first example of a discrete stratified Morse function shown in Figure 1, we give another example in one dimension higher in Figure 2. The function f on the left is not a discrete Morse function, as the vertex $f^{-1}(49)$ violates the conditions of a discrete Morse function. However, f can be converted into a discrete stratified Morse function on the right, when the violator $f^{-1}(49)$ is treated as its own stratum. The remaining simplices form another independent stratum where the green arrows denote the discrete gradient field in the stratum.

3.3 Back and Forth: Discrete Morse and Discrete Stratified Morse Functions

An honest discrete Morse function $f : K \to \mathbb{R}$ is a discrete stratified Morse function for the trivial stratification $\mathcal{S} = \{K\}$. More is true however.

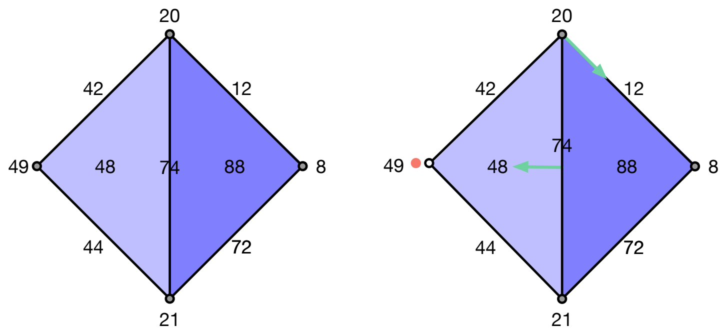

Fig. 2 The function f on the left is not a discrete Morse function; however, it can be converted into a discrete stratified Morse function on the right when it is equipped with an appropriate stratification s.

Lemma 3.3 *Suppose $f : K \to \mathbb{R}$ is a discrete Morse function and let $\mathcal{S} = \{S_i\}$ be a stratification of K, with $s : K \to \mathcal{S}$ the assignment map. Then (f, s) is a discrete stratified Morse function.*

Proof Since f is a discrete Morse function, for every simplex α the sets $U(\alpha)$ and $L(\alpha)$ satisfy the required conditions. In particular, if one of them is nonempty then the other is empty. Since $U_s(\alpha) \subseteq U(\alpha)$ and $L_s(\alpha) \subseteq L(\alpha)$, the conditions of Definition 3.3 hold. □

Lemma 3.3 is in contrast with the smooth case. Indeed, a Morse function on a manifold M may not be a stratified Morse function on an arbitrary stratification of M. For example, for a torus equipped with the standard height function h, choose a regular value c such that $h^{-1}(c)$ consists of two disjoint circles C_1 and C_2. Take the stratification of the torus consisting of a point on C_1, the circle C_1, and the complement of C_1. Then h is *not* a stratified Morse function with respect to this stratification since $h|_{C_1}$ is constant. However, a small perturbation of h is a stratified Morse function.

Lemma 3.3 is *not* true for discrete gradient vector fields, however. Suppose V is a discrete gradient on K associated to some function $f : K \to \mathbb{R}$ and suppose $\mathcal{S} = \{S_i\}$ is a stratification. It is entirely possible that a regular simplex α is paired with a simplex β with $s(\alpha) \neq s(\beta)$. That is, the vector field V may be orthogonal to the strata. We do have the following result.

Theorem 3.1 *Suppose $(f, s) : K \to \mathbb{R}$ is a discrete stratified Morse function with stratification $\mathcal{S} = \{S_i\}$. For each i, denote by V_i the discrete gradient vector field associated to $f|_{S_i}$, and let $V = \bigcup_i V_i$. Then V is a discrete gradient vector field on K.*

Proof It suffices to show that there are no closed V-paths. Suppose

$$\gamma := \{\alpha_0 < \beta_0 > \alpha_1 < \beta_1 > \cdots > \alpha_t < \beta_t > \alpha_0\}$$

is a closed V-path. Then γ is not contained in a single stratum piece. Say it lies in two strata pieces: $\{\alpha_0 < \beta_0 > \alpha_1 < \beta_1 > \cdots < \beta_k\} \subseteq S_1$, $\{\alpha_{k+1} < \beta_{k+1} > \cdots < \beta_u\} \subseteq S_2$, and $\{\alpha_{u+1} < \beta_{u+1} > \cdots < \beta_t > \alpha_0\} \subseteq S_1$. Note that γ must decompose this way since simplices can be paired only within the same stratum piece. Since $\beta_k > \alpha_{k+1}$, we have $\alpha_{k+1} \in \overline{S_1}$ and so by the frontier condition we have $S_2 \subseteq \overline{S_1}$. Also, since $\beta_u > \alpha_{u+1}$, we have $\alpha_{u+1} \in \overline{S_2}$ and again the frontier condition implies $S_1 \subseteq \overline{S_2}$. It follows that $\overline{S_1} = \overline{S_2}$ and since strata pieces are disjoint we conclude that $S_1 = S_2$; that is, γ lies in a single stratum piece, a contradiction. The general case of γ passing through multiple strata pieces follows inductively. □

Of course, the vector field V is *not* associated to the function f; that is, it is not the gradient vector field of f (more on this later). The gradient field V produced in Theorem 3.1 respects the strata in the sense that each pair $\{\alpha < \beta\}$ in V satisfies $s(\alpha) = s(\beta)$. The following result is a useful technical tool for us in the sequel.

Theorem 3.2 *Suppose $(f, s) : K \to \mathbb{R}$ is a discrete stratified Morse function with stratification $\mathcal{S} = \{S_i\}$, and let V be the induced discrete gradient vector field on K. If necessary, extend the partial order on $\mathcal{S}$ to a linear order and write the strata $S_1 < \cdots < S_n$. Then there is a discrete Morse function $g : K \to \mathbb{R}$ satisfying the following properties.*

1. *The gradient of g is V.*
2. *There are real numbers $a_1 < a_2 < \cdots < a_n$ such that $g^{-1}(-\infty, a_i] = \bigcup_{j \le i} S_j$ for $1 \le i \le n$.*

Proof There are infinitely many discrete Morse functions compatible with V; we need only construct one satisfying the second property. The standard way to construct discrete Morse functions with gradient V is to consider the Hasse diagram of K, modified by reversing arrows from β to α whenever $\{\alpha < \beta\}$ is a pair in V. This is an acyclic directed graph and a standard result in graph theory is that such graphs support functions on their vertices whose function values decrease along every directed path. Such a function yields a discrete Morse function on K with gradient V. We know that the minimal element S_1 is a subcomplex of K (Lemma 3.1); choose a discrete Morse function g_1 on S_1 compatible with V_1 and set $a_0 = \max_{\sigma \in S_1} g_1(\sigma)$. Assume inductively that we have constructed g_i, a discrete Morse function on $S_1 \cup \cdots \cup S_i$ satisfying the second property. We extend it to S_{i+1} as follows. Collapse the subgraph of the Hasse diagram corresponding to $S_1 \cup \cdots \cup S_i$ to a point. This is then a sink in this directed graph. Since $f|_{S_{i+1}}$ is a discrete Morse function, we can find a function g_{i+1} whose gradient agrees with V_{i+1} on S_{i+1} and which satisfies $g_{i+1}(\sigma) > a_i$ for all $\sigma \in S_{i+1}$. Set $a_{i+1} = \max_{\sigma \in S_{i+1}} g_{i+1}(\sigma)$. This completes the inductive step. □

We say that a function g satisfying the conclusions of Theorem 3.2 *separates* the strata.

3.4 Homotopy Type

In both smooth and discrete Morse theory, we have theorems about how the topology of the sublevel sets (or sublevel complexes, in the discrete case) vary as we move through increasing function values. The same is true in stratified Morse theory, where a neighborhood of a critical point consists of *Morse data*, which is a product of tangential and normal data (see Appendix A). Our definition of a discrete stratified Morse function is too loose to allow for such theorems as it stands. The issue is that we have no control on how the function values change as we cross from one stratum to another, as opposed to the smooth case where the function is continuous and so function values cannot vary too much in a neighborhood of a critical point.

We can still say something, however, in the form of Theorem 3.3 and Corollary 3.1. In the following, global and local noncritical pairs are derived based on the discrete gradient vector field V described in Theorem 3.1. A *global noncritical pair* involving a simplex α is obtained based on information pertaining to $U(\alpha)$ and $L(\alpha)$; while a *local noncritical pair* involving α is based upon the restriction of $U(\alpha)$ and $L(\alpha)$ to a particular stratum s, that is, $U_s(\alpha)$ and $L_s(\alpha)$.

Theorem 3.3 (Weak DSMT Theorem A) *Given a discrete stratified Morse function (f, s), performing a collapse of either a global noncritical pair or a local noncritical pair is a stratum-preserving homotopy equivalence.*

Proof We make use of the auxiliary discrete Morse function constructed in Theorem 3.2. Suppose $(f, s) : K \to \mathbb{R}$ is a discrete stratified Morse function with associated discrete gradient V. Let g be a discrete Morse function with gradient V which separates the strata. Then any noncritical pair, global or local, is simply a regular pair for g. By Theorem 2.1 we may collapse such a pair without changing the homotopy type of the complex. Moreover, since all noncritical pairs lie within a stratum, this homotopy equivalence is stratum-preserving. □

Describing what happens around a critical cell is much more complicated, but we can say the following. A consequence of Theorems 2.1 and 2.2 is that if the simplicial complex K has a discrete gradient vector field V, then K has the homotopy type of a CW-complex with one cell for each critical cell of V (of the same dimension). Theorem 3.1 then implies the following result.

Corollary 3.1 (Weak DSMT Theorem B) *Suppose $(f, s) : K \to \mathbb{R}$ is a discrete stratified Morse function and denote by V the discrete gradient vector field obtained as the union of the V_i associated with $f|_{S_i}$. Then K has the homotopy type of a CW-complex with one cell for each critical cell of V.*

3.5 Algorithm for Constructing Discrete Stratified Morse Functions

We give an algorithm to construct a discrete stratified Morse function from any real-valued function on a simplicial complex, $f : K \to \mathbb{R}$ as follows.

1. Make a single pass of all simplices in K, order the violators $\mathcal{V} = \{\sigma_1, \sigma_2, \ldots, \sigma_r\}$ by increasing dimension and by increasing function value within each dimension.
2. Initialize $\mathcal{S} = \emptyset$, $i = 0$.
3. $i = i + 1$. Obtain σ_i from $\mathcal{V}$ and add σ_i to $\mathcal{S}$ (as an independent stratum piece).
4. Consider $K_i = K \setminus \{\sigma_1, \ldots, \sigma_i\}$:
   - If the restriction of f to K_i, $f|_{K_i}$, is a discrete Morse function, then K_i may be further stratified to satisfy the frontier condition. Let J denote the set of indices $k \leq i$ such that $\sigma_k \in \overline{K_i}$ and add the following strata pieces to $\mathcal{S}$ (which may contain more than two strata pieces): the frontier $\overline{K_i} \setminus (\mathring{K}_i \cup \{\sigma_j\}_{j \in J})$ and $\mathring{K}_i$.
   - Otherwise, if $f|_{K_i}$ is not a discrete Morse function, then at least one σ_j with $j > i$ remains a violator.
5. Remove simplices that are no longer violators from the list $\mathcal{V}$, renumbering the remaining elements in $\mathcal{V}$ if necessary, and repeat the steps 3-4 above until no violators remain.

Before we prove the correctness of the above algorithm, we give a simple example to illustrate it step-by-step, as shown in Figure 3. If A is any subset of a topological space, we let A^c denote its complement and $\overline{A}$ denote its closure. Then the interior $\mathring{A}$ is the complement of the closure of the complement; that is, $\mathring{A} = \left(\overline{A^c}\right)^c$. For simplicity, in this example we represent each simplex by its function value; for instance, 89 represents the vertex $f^{-1}(89)$, and 46 represents the edge $f^{-1}(46)$. Given $f : K \to \mathbb{R}$ in Figure 3(a), the algorithm proceeds as follows:

1. Compute $\mathcal{V} = \{89, 7, 46, 75\}$. The violators are first sorted by dimension, then by function value: 89 is a 0-dimensional violator, while the rest in $\mathcal{V}$ are 1-dimensional.
2. Initialize $\mathcal{S} = \emptyset$, $i = 0$.
3. $i = i + 1 = 1$. Obtain $\sigma_i = \sigma_1 = 89$ from $\mathcal{V}$ and add it to $\mathcal{S}$. That is, $\mathcal{S} = \{\{89\}\}$. See Figure 3(b).
4. Consider $K_1 = K \setminus \{89\}$: $f|_{K_1}$ is not a discrete Morse function.
5. Remove simplex 7 in $\mathcal{V}$ that is no longer a violator. Now $\mathcal{V} = \{89, 46, 75\}$.
6. $i = i + 1 = 2$. Obtain $\sigma_i = \sigma_2 = 46$ from $\mathcal{V}$ and add it to $\mathcal{S}$. That is, $\mathcal{S} = \{\{89\}, \{46\}\}$. See Figure 3(c).
7. Consider $K_2 = K \setminus \{89, 46\}$: $f|_{K_2}$ is not a discrete Morse function.
8. $i = i + 1 = 3$. Obtain $\sigma_i = \sigma_3 = 75$ from $\mathcal{V}$ and add it to $\mathcal{S}$. That is, $\mathcal{S} = \{\{89\}, \{46\}, \{75\}\}$. See Figure 3(d).
9. Consider $K_3 = K \setminus \{89, 46, 75\} = \{7, 25, 58, 62, 66, 83, 97, 99\}$: $f|_{K_3}$ is now a discrete Morse function. We may need to further stratify K_3 to satisfy the frontier condition.
   (a) Compute $K_3^c = K \setminus K_3 = \{46, 75, 89\}$
   (b) Compute $\overline{K_3^c} = \{46, 58, 62, 75, 89, 97\}$

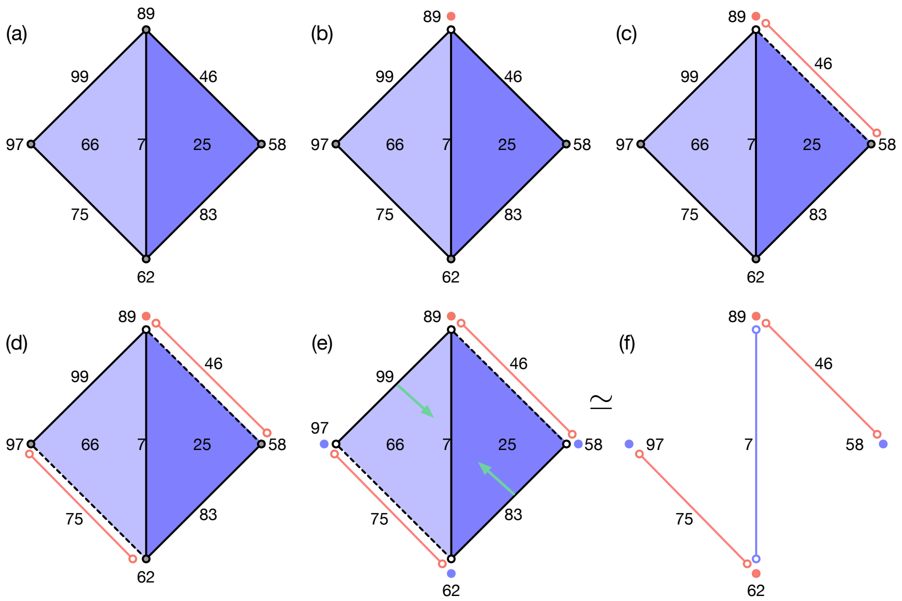

Fig. 3 An illustration of the algorithm for constructing discrete stratified Morse function from an arbitrary function defined on a simplicial complex.

(c) Compute the interior $\mathring{K}_3$ of K_3: $\mathring{K}_3 = \left(\overline{K_3^c}\right)^c = \{7, 25, 66, 83, 99\}$

(d) Compute the closure of K_3: $\overline{K_3} = K$; $J = \{1, 2, 3\}$.

(e) Compute the frontier of K_3: $\overline{K_3} \setminus \mathring{K}_3 = \{46, 58, 62, 75, 89, 97\}$

(f) Add connected components of $\mathring{K}_3$ and $\mathcal{S}' = \overline{K_3} \setminus (\mathring{K}_3 \cup \{\sigma_j\}_{j\in J})$ to $\mathcal{S}$ as strata pieces. In particular $\mathcal{S}' = \overline{K_3} \setminus (\mathring{K}_3 \cup \{89, 46, 75\}) = \{58, 62, 97\}$. Therefore $\mathcal{S} = \{\{89\}, \{46\}, \{75\}, \{46, 58, 62, 75, 89, 97\}, \{58\}, \{62\}, \{97\}\}$. See Figure 3(e).

Lemma 3.4 *The collection $\mathcal{S}$ satisfies the condition of the frontier and therefore meets the conditions of Definition 3.1.*

Proof First note that every simplex in K belongs to some strata piece; the strata pieces are obviously disjoint. Based on Lemma 3.2, if σ_k and σ_ℓ are distinct violators in $\mathcal{S}$, then $\sigma_k \cap \overline{\sigma_\ell} \neq \emptyset$ if and only if $\sigma_k \in \partial\sigma_j := (\overline{\sigma_\ell} \setminus \mathring{\sigma}_j) \subseteq \overline{\sigma_\ell}$. Similarly, if σ_k intersects the closure of the frontier strata piece, then it must lie in the boundary of one of the simplices in that strata piece. A violator in $\mathcal{S}$ cannot intersect the open strata piece $\mathring{K}_i$ by definition, and if it intersects its closure then it intersects the frontier strata piece and we are done. □

Theorem 3.4 *The function (f, s) associated to the stratification defined in the algorithm above is a discrete stratified Morse function.*

Proof We assume K is connected. If f itself is a discrete Morse function, then there are no violators in K. The algorithm produces the trivial stratification

$\mathcal{S} = \{K\}$ and since f is a discrete Morse function on the entire complex, the pair (f, s) trivially satisfies Definition 3.3.

If f is not a discrete Morse function, let $\mathcal{S} = \mathcal{V} \cup \{F\} \cup \{I\}$ denote the stratification produced by the algorithm, where $\mathcal{V}$ is the set of violators that form their own strata, F is the set of frontier strata pieces and I is the interior complementary to F. Since each violator $\alpha \in \mathcal{V}$ forms its own strata piece $s(\alpha)$, the restriction of f to $s(\alpha)$ is trivially a discrete Morse function in which α is a critical simplex.

Recall that the sets F and I are obtained as follows. We remove the collection $\mathcal{V}$ from K to obtain L and consider the restriction of f to this subspace. The function f is a discrete Morse function here, and since I is the interior of L, f restricts to a discrete Morse function on I. The set F is obtained as $\overline{L} \setminus (I \cup \mathcal{V})$. If σ is a simplex in F, then σ is not one of the violators for f that get removed (or it is not a violator at all in the first place). It follows that $f|_F$ is a discrete Morse function and we are done. □

Remark 3.2 When we restrict the function $f : K \to \mathbb{R}$ to one of the strata $S_i \in \mathcal{S}$, a non-violator σ that is regular globally (that is, σ forms a gradient pair with a unique simplex τ) may become a critical simplex for the restriction of f to S_i.

The algorithm is relatively efficient. We give a back-of-the-envelope argument below. Suppose K has n simplices and let c be the maximum number of codimension-1 faces and cofaces of any simplex in K (in other words, c could be considered as the maximum "degree" of a simplex in K). The first step of the algorithm takes $O(cn)$ steps to identify the set of violators $\mathcal{V}$ by checking for each simplex $\alpha^{(p)}$, its faces $\gamma^{(p-1)} < \alpha$ and cofaces $\beta^{(p+1)} > \alpha$. For the initial sorting of the r number of violators by dimension and by function values, it takes $O(r \log r)$. Then for each violator σ_i removed from the set $\mathcal{V}$, the algorithm must check the complex K_i for remaining violators by paying attention to simplicies adjacent to σ_i, which takes $O(c)$. For r violators, this requires $O(cr)$. Assuming all the list and set operations are $O(1)$ and c being a constant, then the algorithm runs in $O(n \log n)$ time, where the bottleneck is the sorting of violators.

3.6 Coarseness

Suppose f is a function on K and denote the set of stratifications $\mathcal{S}$ of the complex K on which f is a discrete stratified Morse function by $\Sigma(K, f)$. The set $\Sigma(K, f)$ is partially ordered by inclusion: $\mathcal{S} \leq \mathcal{S}'$ if each stratum piece $S_i \in \mathcal{S}$ is contained in some element of $\mathcal{S}'$. Generally, we wish to work with coarse stratifications; that is, we seek maximal elements of $\Sigma(K, f)$. Our algorithm in Section 3.5 does just that.

Proposition 3.1 *The stratification produced by the algorithm of Section 3.5 is a maximal element of* $\Sigma(K, f)$.

Proof The algorithm produces a stratification $\mathcal{S}$ consisting of some violators $\sigma_1, \sigma_2, \ldots, \sigma_\ell$ for the function f, the interior of $K_\ell = K \setminus \{\sigma_1, \ldots, \sigma_\ell\}$, and the frontier of K_ℓ (with the violators removed). Suppose there is a stratification $\mathcal{S}' \in \Sigma(K, f)$ with $\mathcal{S} \leq \mathcal{S}'$. Then there is some stratum piece $S \in \mathcal{S}'$ containing $\mathring{K}_\ell$. If they are not equal, then there is a simplex α in $S \setminus \mathring{K}_\ell$. If α is one of the violators for f then $f|_S$ cannot be a discrete Morse function on S, otherwise the algorithm would have terminated sooner. If α lies in the frontier of K_ℓ then S contains the entire frontier by definition and hence must be all of K_ℓ. But then $\mathcal{S}'$ would not satisfy the frontier condition. Thus, $\mathring{K}_\ell$ must be one of the elements of $\mathcal{S}'$. Similarly, the frontier of K_ℓ must also be an element of $\mathcal{S}'$. The violators are disjoint and so they must then also be elements of $\mathcal{S}'$. It follows that $\mathcal{S} = \mathcal{S}'$ and that $\mathcal{S}$ is maximal. □

Note that if f is a discrete Morse function on K, then the algorithm produces the trivial stratification $\mathcal{S} = \{K\}$, which is indeed maximal in $\Sigma(K, f)$.

4 Discrete Stratified Morse Theory by Example

We apply the algorithm described in Section 3.5 to a collection of examples to demonstrate the utility of our theory. For each example, given a function $f : K \to \mathbb{R}$ that is not necessarily a discrete Morse function, we equip f with a particular stratification s, thereby converting it to a discrete stratified Morse function (f, s). These examples help to illustrate that the class of discrete stratified Morse functions is much larger than that of discrete Morse functions.

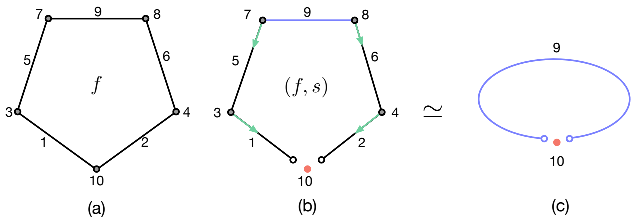

Fig. 4 Upside-down pentagon. (a): f is not a discrete Morse function. (b): (f, s) is a discrete stratified Morse function where violators removed by the algorithm are in red; critical simplicies (other than the violators) are in blue; the discrete gradient vector field is marked by green arrows. (c): the simplified simplicial complex by removing the Morse pairs following the discrete gradient vector field.

Example 1: upside-down pentagon. As illustrated in Figure 4 (a), $f : K \to \mathbb{R}$ defined on the boundary of an upside-down pentagon is not a discrete Morse function, as it contains a set of violators: $\mathcal{V} = \{f^{-1}(10), f^{-1}(1), f^{-1}(2)\}$, since $|U(f^{-1}(10))| = 2$ and $|L(f^{-1}(1))| = |L(f^{-1}(2))| = 2$, respectively.

By following the algorithm in Section 3.5, we would first remove the violator $f^{-1}(10)$ and check to see if what remains is a discrete Morse function. We see that this is indeed the case: we have four Morse pairs illustrated by green arrows in Figure 4 (b): $(f^{-1}(3), f^{-1}(1))$, $(f^{-1}(4), f^{-1}(2))$, $(f^{-1}(7), f^{-1}(5))$, and $(f^{-1}(8), f^{-1}(6))$. The resulting discrete stratified Morse function (f, s) is a discrete Morse function when restricted to each stratum. Recall that a simplex is critical for (f, s) if it is neither the source nor the target of a discrete gradient vector. The critical values of (f, s) are therefore 9 and 10.

One of the primary uses of classical discrete Morse theory is *simplification*. In this example, we can collapse a portion of each stratum following the discrete gradient field (illustrated by green arrows, see Section 2). Removing the Morse pairs simplifies the original complex as much as possible without changing its homotopy type, and the resulting simplification yields a complex with one vertex and one edge, see Figure 4 (c).

Example 2: pentagon. For our second pentagon example, f can be made into a discrete stratified Morse function (f, s) by making $f^{-1}(0)$ (a type II violator) and $f^{-1}(9)$ (a type I violator) their own strata following the algorithm in Section 3.5 (Figure 5). The critical values of (f, s) are $0, 1, 3, 7, 8$ and 9. It is important to note that $f^{-1}(1)$ and $f^{-1}(3)$ are considered critical as they form their own strata pieces; however they are not the violators removed by the algorithm. The simplicial complex can be reduced to one with fewer cells by canceling the Morse pairs, as shown in Figure 5 (d).

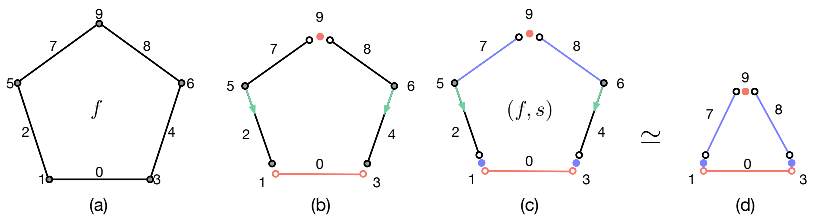

Fig. 5 Pentagon. (a): f is not a discrete Morse function. (b): an intermediate simplicial complex after removing violators in red. (c): separating simplicies $f^{-1}(1)$ and $f^{-1}(3)$ from (b) results in a stratification that satisfies the frontier condition. There are six strata pieces associated with the discrete stratified Morse function (f, s). (d): the simplified simplicial complex.

Example 3: split octagon. The split octagon example (Figure 6) begins with a function f defined on a triangulation of a stratified space that consists of two 0-dimensional and three 1-dimensional strata pieces (Figure 6(a)). The set of violators to be considered is $\mathcal{V} = \{f^{-1}(0), f^{-1}(10), f^{-1}(24), f^{-1}(30), f^{-1}(31)\}$. However, after removing $f^{-1}(30)$, then $f^{-1}(31)$, the rest of the simplicies in $\mathcal{V}$ are no longer violators and the restriction of f to what is left is a discrete Morse function (Figure 6(b)). The result of canceling Morse pairs yields a simpler complex shown in Figure 6(c).

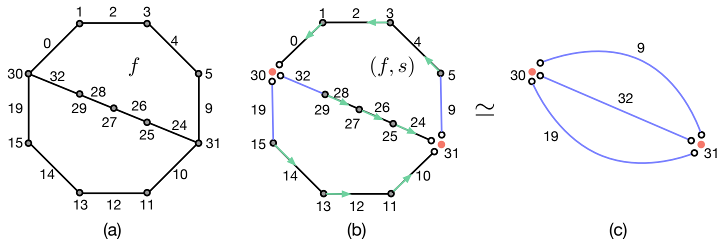

Fig. 6 Split octagon. (a) f is defined on the triangulation of a stratified space. (b) the resulting discrete stratified Morse function (f, s). (c) the simplified complex.

Example 4: tetrahedron. In Figure 7(a), the values of the function f defined on the simplices of a tetrahedron are specified for each dimension. For each simplex $\alpha \in K$, we list the elements of its corresponding $U(\alpha)$ and $L(\alpha)$ in Table 1. We also classify each simplex in terms of its criticality in the setting of classical discrete Morse theory. According to Table 1 the violators $\mathcal{V}$ have function values of $10, 14$ (type I), 6 (type II), $7, 8, 11$ and 12 (type III).

We describe our algorithm step by step, the intermediate results (strata pieces) are illustrated in Figure 7(b). For simplicity, a simplex α is represented by its function value $f(\alpha)$. First, initialize $\mathcal{S} = \emptyset$. Second, remove the vertex 10, then 7 is no longer a violator, remove it from the list $\mathcal{V}$; now $\mathcal{S} = \{10\}$. Third, remove the vertex 14, then $8, 11, 12$ are no longer violators, remove them from the list $\mathcal{V}$; $\mathcal{S} = \{10, 14\}$. Fourth, 6 is the only remaining violator, add it to $\mathcal{S} = \{10, 14, 6\}$. Finally, let $C = K \setminus \{10, 14, 6\}$. Then $\overline{C} = K \setminus \{6\}$ and $\mathring{C} = C \setminus \{2, 8, 11, 13\}$. Add $\mathring{C}$ and $\overline{C} \setminus (\mathring{C} \cup \{14\})$ to $\mathcal{S}$. $\mathcal{S}$ now contains 5 strata pieces. Besides vertices 10 and 14 and triangle 6, $\mathcal{S}$ also contains a strata piece $\{1, 2, 3, 8, 11\}$ that is homotopy equivalent to an open 1-manifold; vertex 3 and edge 2 forms a Morse pair. The last strata piece in $\mathcal{S}$ is $\{4, 5, 7, 9, 12, 13\}$, which is topologically a punctured disc; in particular, there are two Morse pairs, $(12, 9)$ and $(7, 5)$.

As an alternative to the algorithm described in Section 3.5, we show in Figure 7(c) that we could obtain a different stratification by changing the ordering of the violators to be removed. As in (b), $\mathcal{V} = \{10, 14, 6, 7, 8, 11, 12\}$. First, initialize $\mathcal{S} = \emptyset$. Second, remove the vertex 14, then $8, 11, 12$ are no longer violators, remove them from $\mathcal{V}$; now $\mathcal{S} = \{14\}$. Third, remove the edge 7, then 10 is no longer a violator, remove it from $\mathcal{V}$; $\mathcal{S} = \{14, 7\}$. Fourth, 6 is the only remaining violator, add it to $\mathcal{S} = \{14, 7, 6\}$. Finally, let $C = K \setminus \{14, 7, 6\}$. Then $\overline{C} = K \setminus \{6\}$ and $\mathring{C} = C \setminus \{1, 2, 3, 8, 11\}$. $\mathcal{S}$ now contains 5 strata pieces in (c) that are slightly different from (b). Note that the stratifications in (b) and (c) are incomparable in the set $\Sigma(K, f)$.

Example 5: split solid square. As illustrated in Figure 8, the function f defined on a split solid square is not a discrete Morse function; there are three type I violators $f^{-1}(9)$, $f^{-1}(10)$, and $f^{-1}(11)$. Making these violators their own strata (in the order of increasing function value following the algorithm in

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$U(\alpha)$	$\emptyset$	$\emptyset$	$\{2\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{5\}$	$\{6\}$	$\emptyset$	$\{4,7\}$	$\{6\}$	$\{9\}$	$\emptyset$	$\{8,11,12\}$
$L(\alpha)$	$\emptyset$	$\{3\}$	$\emptyset$	$\{10\}$	$\{7\}$	$\{8,11\}$	$\{10\}$	$\{14\}$	$\{12\}$	$\emptyset$	$\{14\}$	$\{14\}$	$\emptyset$	$\emptyset$
Type	C	R	R	R	R	II	III	III	R	I	III	III	C	I

Table 1 Tetrahedron. For simplicity, a simplex α is represented by its function value $f(\alpha)$ (as f is 1-to-1). In terms of criticality for each simplex: C means critical; R means regular; I, II and III correspond to type I, II and III violators.

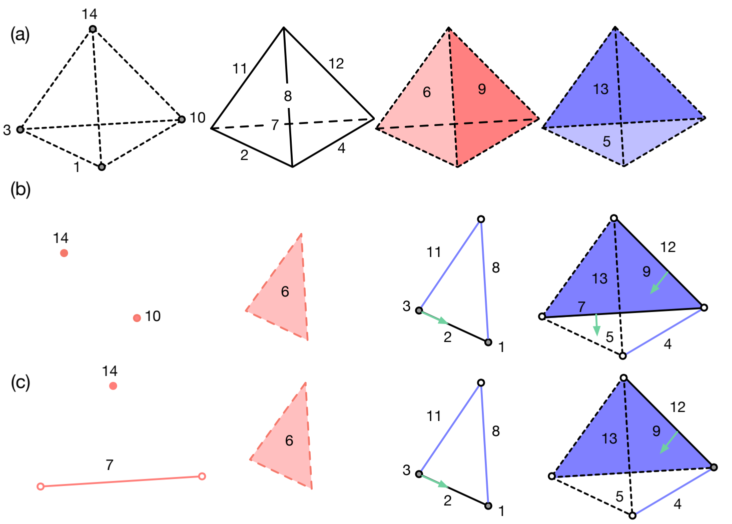

Fig. 7 Tetrahedron. (a) f is defined on the simplices of increasing dimensions. (b) the resulting discrete stratified Morse function (f,s) is shown by individual strata pieces; using the algorithm in Section 3.5. Not all simplices are shown. (c) An alternative stratification.

Section 3.5) helps to convert f into a discrete stratified Morse function (f,s). In this example, all simplices are considered critical for (f,s). For instance, consider the open 2-simplex $f^{-1}(4)$, we have $L(f^{-1}(4)) = \{f^{-1}(11)\}$ and $U(f^{-1}(4)) = \emptyset$; with the stratification s in Figure 8 (right), $L_s(f^{-1}(4)) = \emptyset$ and so 4 is not a critical value for f but it is a critical value for (f,s). Since every simplex is critical for (f,s), there is no simplification to be done.

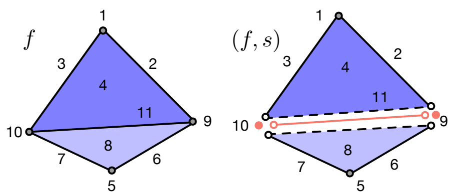

Fig. 8 Split solid square. Every simplex is critical for (f,s).

5 Applications to Triangulations of Stratified Spaces

5.1 Background on Whitney Stratifications and Triangulations

Whitney stratifications. We review relevant background on Whitney stratifications; the primary reference for the material in this section is [28]. For simplicity, we assume all manifolds are smooth (i.e., of class C^∞). If $x, y \in \mathbb{R}^n$ with $x \neq y$, we define the *secant* $\overline{xy}$ to be the line through the origin in $\mathbb{R}^n$ parallel to the line joining x and y. If $x \in \mathbb{R}^n$, we identify the tangent space $T_x\mathbb{R}^n$ with $\mathbb{R}^n$ in the standard way.

Let M be a smooth manifold without boundary and let Z be a subset of M. A *stratification* $\mathcal{S} = \{S_i\}_{i \in \mathcal{P}}$ of Z is a cover of Z by pairwise disjoint smooth submanifolds of M which lie in Z; these submanifolds S_i are called *strata* (whose connected components are referred to as *strata pieces*); where $\mathcal{P}$ is some poset. The stratification $\mathcal{S}$ is *locally finite* if each point of M has a neighborhood which meets finitely many strata. We say $\mathcal{S}$ satisfies the condition of the *frontier* if the strata in $\mathcal{S}$ satisfy $S_i \cap \overline{S_j} \neq \emptyset$ if and only if $S_i \subseteq \overline{S_j}$; or equivalently, if for each stratum S_i of $\mathcal{S}$ its frontier $(\overline{S_i} \setminus S_i) \cap Z$ is a union of strata (compare with Definition 3.1).

Definition 5.1 Let X and Y be submanifolds of a smooth manifold M. We say that X is *Whitney regular* over Y if whenever $\{x_i\} \subset X$ and $\{y_i\} \subset Y$ are sequences of points both converging to some point $y \in Y$, the lines $\ell_i = \overline{x_i y_i}$ converge to a line $\ell \in \mathbb{R}^n$, and the tangent spaces $T_{x_i}X$ converge to a space $T \subseteq \mathbb{R}^n$, then

(A) $T_yY \subseteq T$, and
(B) $\ell \subseteq T$.

Remark 5.1 Convergence here should be thought of as taking place in a small neighborhood of y identified with $\mathbb{R}^n$ via a coordinate chart. Also, Condition B above implies Condition A [28].

Proposition 5.1 *[28, Proposition 2.7] Suppose $y \in \overline{X \setminus Y}$ and (X, Y) satisfies condition B at y. Then* $\dim Y < \dim X$.

Definition 5.2 A stratification $\mathcal{S} = \{S_i\}$ is a *Whitney stratification* if it is locally finite, satisfies the condition of the frontier, and if whenever $j \leq i$, S_i is Whitney regular over S_j.

Remark 5.2 Let $\mathcal{S}$ be a Whitney stratification of a subset Z of a manifold M and let S_i, S_j be strata. Proposition 5.1 implies that if $i \leq j$ then $\dim S_i < \dim S_j$.

Here is a useful way of constructing stratified spaces [24]. A stratified set of type 0 is a smooth manifold. To construct a stratified set $X^{(k+1)}$ of type $k+1$, take a stratified set $X^{(k)}$ of type k, a smooth manifold K, a smooth submanifold L of codimension 0 in ∂K, and an "attaching" map $\alpha : L \to X^{(k)}$, and then set $X^{(k+1)} = X^{(k)} \cup_\alpha K$. These attaching maps are not arbitrary continuous maps; they must be proper, continuous, and as close as possible to a smooth fiber bundle. The strata are then the various $X_k = X^{(k)} \setminus X^{(k-1)}$. For the pinched torus in Figure 9, we begin with $X^{(0)}$ as the pinch point. To build $X^{(1)}$ we take the closed interval $K = [0, 1]$, $L = \{0, 1\}$, and $\alpha : L \to X^{(0)}$ the obvious map. To build $X^{(2)}$, we take K to be the disjoint union of a disc and a square, L to be the disjoint union of the boundary circle and boundary square, and $\alpha : L \to X^{(1)}$ to be the map identifying the circle via the identity and the square via the map that first yields a wedge of two circles and then collapses one to the base point.

Triangulating stratified sets. By a *triangulation* of a set Z we mean a finite simplicial complex K and a homeomorphism $h : |K| \to Z$, where $|K|$ denotes the geometric realization of K. Any smooth manifold is triangulable, for example.

Suppose we have a compact set Z with Whitney stratification $\mathcal{S} = \{S_i\}$. As above, we may think of Z as being built up by the pieces S_i in such a way that when $i \leq j$, we have $S_i \subseteq \partial \overline{S_j}$ (this is essentially the condition of the frontier). We now have the following theorem (see Theorem 2.1 of [24] or Proposition 5 of [17]).

Theorem 5.1 *[24, Theorem 2.1] A compact Whitney stratified set Z admits a triangulation by a finite simplicial complex so that each $\overline{S_i}$ is triangulated as a subcomplex.*

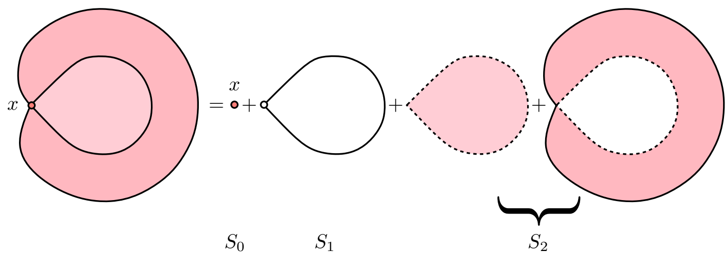

Fig. 9 The pinched torus as a stratified space, whose stratification is formed by strata S_0, S_1, and S_2.

In addition, whenever $i \leq j$, we have that S_i is triangulated as a subcomplex of $\partial \overline{S_j}$. So, for example, in the pinched torus of Figure 9 we have that the pinch point $x := S_0$ is a vertex in the triangulation of $\overline{S_1}$, and that $\overline{S_1}$ is a subcomplex of the closure $\overline{S_2}$ of the two disjoint discs. The important idea here is that one may think of building the triangulation from the bottom up by first triangulating the 0-stratum, then extending that to a triangulation of the 1-stratum, and so on, noting that at each stage, the lower-dimensional (closed) stratum is a subcomplex of the boundary of the next (closed) stratum.

5.2 Applications to Classical Stratified Morse Theory

Suppose Z is a Whitney stratified subset of a smooth manifold M with stratification $\mathcal{S} = \{S_i\}$. A *stratified Morse function* $f : S \to \mathbb{R}$ is, roughly speaking, a function that restricts to a Morse function on each stratum (see Appendix A for the formal definition). In this section, we investigate the following obvious question. Suppose $f : Z \to \mathbb{R}$ is a stratified Morse function. Is there a triangulation of Z and a discrete stratified Morse function on that triangulation that "mirrors" the behavior of f? That is, can we find a discrete stratified Morse function and a bijection between its critical cells and the critical points of the function f?

Comparing classical (smooth) and discrete Morse theory. To answer this question, we first need to address it in the classical nonstratified case. This has been solved satisfactorily by Benedetti [4,5]. Suppose M is a smooth d-manifold with boundary (possibly empty) and $f : M \to \mathbb{R}$ is a Morse function. Denote by c_i the number of critical points of f of index i. We call the d-tuple $\mathbf{c} = (c_0, c_1, \ldots, c_d)$ the *Morse vector* of the function f and we say that M *admits* $\mathbf{c}$ *is a Morse vector*. A classical theorem of Morse asserts that the manifold M is homotopy equivalent to a cell complex with c_i cells of dimension i.

Similarly, if K is a d-dimensional simplicial complex with a discrete Morse function $g : K \to \mathbb{R}$ having c_i critical cells of dimension i, we call $\mathbf{c} = (c_0, c_1, \ldots, c_d)$ the *discrete Morse vector* of the function g and say that K *admits* $\mathbf{c}$ *as a discrete Morse vector*. If K is a triangulation of a manifold M with boundary, we say the function g is *boundary critical* if all the cells in the subcomplex triangulating ∂M are critical for g. Forman proved the analogue of Morse's theorem: the complex K has the homotopy type of a cell complex with c_i cells of dimension i. We recall Theorem 2.28 of [5] below.

Theorem 5.2 *[5, Theorem 2.28] If a smooth d-manifold M (with boundary) admits* $\mathbf{c}$ *as a Morse vector, then for any PL triangulation T of M, there exists an integer r so that the r-th barycentric subdivision of T admits*

(a) *a discrete Morse function with c_i critical i-faces, and*
(b) *a boundary-critical discrete Morse function with c_{d-i} critical interior i-faces.*

The statement (b) in Theorem 5.2 is related to duality. If $f : M \to \mathbb{R}$ is a Morse function on M with Morse vector $\mathbf{c} = (c_0, c_1, \ldots, c_d)$, then the function $-f : M \to \mathbb{R}$ is also a Morse function but with Morse vector $\mathbf{c}^* = (c_d, c_{d-1}, \ldots, c_0)$. In the discrete case, the negative of a discrete Morse function on a complex K is *not* a discrete Morse function. However, in the case of a triangulation T of a manifold, one may consider the dual block complex T^* with a corresponding dual function f^*, yielding an analogous result.

Discretizing a stratified Morse function. Suppose Z is a compact set with stratification $\mathcal{S} = \{S_i\}$ and that $f : Z \to \mathbb{R}$ is a stratified Morse function. Let d denote the dimension of the top stratum. Set $d_i = \dim S_i$ and denote by

$\mathbf{c}^i = (c_0^i, \ldots, c_{d_i}^i)$ the Morse vector of $f|_{S_i}$. According to Theorem 5.1, there is a triangulation T of Z so that each closed stratum $\overline{S_i}$ is triangulated as a subcomplex T_i. This leads to our main result in this section relating discrete stratified Morse theory to (classical) stratified Morse theory.

Theorem 5.3 *There exists an integer r such that the r-th barycentric subdivision of T admits a discrete stratified Morse function F satisfying the following:*

(a) *the stratification of T is given by the various $T_i \setminus T_{i-1}$, $i = 0, \ldots, d$; and*
(b) *the restriction of F to the i-th stratum has discrete Morse vector $\mathbf{c}_i^* = (c_{d_i}^i, \ldots, c_0^i)$.*

Proof Keeping in mind the discussion at the end of Section 5.1, we proceed as follows. The 0-stratum S_0 is a smooth manifold. By Theorem 5.2 we may choose r_0 so that the r_0-th subdivision of T_0 admits a (boundary critical) discrete Morse function with discrete Morse vector $\mathbf{c}_0^*$. (In this case we could also find a discrete Morse function with vector $\mathbf{c}_0$ since S_0 has no boundary, but this is not true moving forward). We now proceed inductively. Suppose the result is true for stratum $i \geq 0$, and consider the r-th subdivision of T, where $r = r_0 + \cdots + r_i$. This means that we have stratified T_i by the various $T_i \setminus T_{i-1}$ and we have a discrete stratified Morse function F_i satisfing condition (b) above on T_i. We know that $S_i \subseteq \overline{S}_{i+1}$; in fact, it lies inside the boundary of $\overline{S}_{i+1}$. Again by Theorem 5.2, there is an integer r_{i+1} so that the r_{i+1}-th subdivision of T_{i+1} has a boundary critical discrete Morse function with vector $\mathbf{c}_{i+1}^*$. Observe that this requires subdivision of the subcomplex T_i, but by Lemma 2.1, this subdivision of T_i supports a discrete Morse function with the same Morse vector. This completes the proof. □

6 Generating Discrete Stratified Morse Functions from Point Data

In this section, we are interested in generating discrete stratified Morse functions from point cloud data – a natural question relevant to data analysis. Consider the following scenario. Suppose K is a simplicial complex and that f is a function defined on the 0-skeleton K_0 of K. Such functions arise naturally in data analysis where one has a sample of function values on a space. Algorithms exist to build discrete Morse functions on K extending f (see, for example, [25]). However, existing algorithms are oblivious of additional structure in K. What if K arises from a triangulated Whitney stratified space? More generally, what if we are given a stratification of K and we want to extend f to a function on K that respects the stratification? In our framework, we may take this input and generate a discrete stratified Morse function which will not be a global discrete Morse function in general, but which will allow us to preserve interesting information about the underlying complex.

Formally speaking, **given a simplicial complex K equipped with an injective function on its vertices $f : K_0 \to \mathbb{R}$, can we extend f to a discrete stratified Morse function $\tilde{f}$ on K?**

An algorithm to extend f on K_0 to a discrete Morse function f on K was presented in [25]. In this section, we extend the work of King et al. [25] to the setting of discrete stratified Morse theory. Let us first review the algorithm of [25]. Since the function f is injective, we may order the vertices. We begin with the vertex with smallest function value and proceed as follows. Given a vertex v, consider the lower link K_v of v. If K_v is empty then we know that v is a local minimum and so we make v critical. Otherwise, we restrict f to K_v and iteratively run the algorithm on K_v. During this iteration we take the extra step of canceling all possible gradient paths; that is, if there is a unique gradient path between two critical cells we reverse it to eliminate those critical cells. We then find the critical vertex w in K_v with smallest function value and pair v with the edge $[v, w]$ (this makes sense as it should be the steepest edge away from v). For each regular pair $\sigma < \tau$ in K_v we then pair $v * \sigma$ with $v * \tau$, and for each critical cell $\alpha \neq w$ in K_v we make $v * \alpha$ critical. The resulting discrete vector field has no directed loops and is therefore a discrete gradient.

To bring this into the stratified setting, we begin by assuming that we already have a stratification $\mathcal{S} = \{S_i\}$ of the complex K; let $s : K \to \mathcal{S}$ be the associated assignment map. Extend the partial order on $\mathcal{S}$ to a linear order if necessary and write the strata as $S_0 < S_1 < \cdots < S_n$. Given the function f on K_0, consider the function maxf on K defined by setting $\mathrm{maxf}(\sigma) = \max_{v \in \sigma} f(v)$. We then proceed as follows.

1. The stratum S_0, being minimal in the order, is a subcomplex of K by Lemma 3.1. Use the algorithm of [25] to generate a discrete Morse function f_0 on S_0 extending the restriction of f to the vertices of S_0. We may choose such an extension to be arbitrarily close to the function maxf ([25], Theorem 3.4).
2. Assume inductively that we have defined an extension f_i on S_i, $i \geq 0$, that is a discrete stratified Morse function on S_i. The algorithm of [25] works on $S_{i+1} \setminus S_i$ to generate a discrete Morse function on this space, with the following modification. Simplices adjacent to the boundary of S_{i+1} may not be considered by the algorithm if the lower link of a vertex is empty. We therefore declare that all simplices that do not get considered remain unpaired (critical).
3. In the end we obtain a discrete stratified Morse function $\tilde{f} : K \to \mathbb{R}$ extending f.

Remark 6.1 This algorithm potentially leaves many simplices σ having a face $\tau < \sigma$ with $s(\sigma) \neq s(\tau)$ critical. That is, the simplices in each stratum having a face in the stratum's frontier could be left unpaired by the algorithm. To address this, we could implement a greedy pairing among such simplices in each stratum, pairing as many as possible until the creation of a cycle is forced. There may be other approaches to decreasing the number of such critical simplices.

It is not clear that we can choose $\tilde{f}$ to be arbitrarily close to maxf on all of K. Indeed, if the values of f on lower strata are much larger than on higher strata it may not be possible to find such an extension in the inductive step. Moreover, in the inductive step, it could happen that a vertex in S_{i+1} has an empty lower link, either because all its neighbors lie in S_{i+1} and have higher values or because some of its neighbors lie in a lower stratum and are therefore not considered by the algorithm of [25]. This will force the vertex to be critical and in the latter case the adjacent simplices will be made critical as well, therefore making it impossible to keep associated function values close to the function maxf.

We do have the following curious result, however.

Theorem 6.1 *We may choose an extension $\tilde{f} : K \to \mathbb{R}$ of f that is a discrete Morse function on all of K.*

Proof The algorithm of [25] actually generates a discrete gradient vector field from the function $f : K_0 \to \mathbb{R}$. There is then a great deal of flexibility in choosing an extension $\tilde{f}$. Observe the following: in the inductive step we actually first generate a discrete gradient on $S_{i+1} \setminus S_i$ which happens to leave some cells on the boundary critical (i.e., some simplices having a face in S_i remain unpaired). We know that the union of these gradients is a discrete gradient on all of K (Theorem 3.1) and we may then choose a discrete Morse function $\tilde{f}$ extending $f : K_0 \to \mathbb{R}$ compatible with this gradient. □

Now, if we are not given a stratification of K, there are several ways we could proceed. We can choose some extension of f to K, such as the function maxf or the piecewise linear extension of f (take the average value of the vertices of a simplex). Employing the algorithm of Section 3.5 yields a stratification on which the extension is a discrete stratified Morse function. We could stop there, or we could discard the chosen extension and implement the algorithm above. Another approach is to use the algorithm of [33] to produce the coarsest stratification of K into cohomology manifolds, or the approach of [6] to obtain a stratification of K based on homology, and then proceed using the algorithm above. It is not clear which method is preferable; this will be the subject of future research.

7 Discussions and Future Work

We end our paper by providing some food for thought. All of this is the focus of current research and the results will be presented elsewhere.

Topology of sublevel sets with constraints on the function. While we have laid the foundations for the study of a discrete version of stratified Morse theory and provided some examples and basic results, much work remains to be done. As mentioned above, our definition of a discrete stratified Morse function is too loose to allow for analogues of the classical sublevel set theorems in the smooth case. As Theorem 3.2 makes clear, we can have such functions that separate the strata and so there need not be a relation between the values of a discrete stratified Morse function in one stratum and the values in an adjacent one. In essence, there is no semblance of continuity in our most general setting.

If we wish to obtain results concerning sublevel sets, we will be forced to impose additional conditions on either our stratifications or our functions (or both). The fact that we are working with simplicial complexes helps out a bit. Indeed, in this setting, we are aided by the fact that we can talk about the *open star* $\mathrm{St}\,(\sigma)$ and *link* $\mathrm{Lk}\,(\sigma)$ of a simplex σ; these are the simplicial analogues of neighborhood and boundary of a neighborhood, respectively. In turn, if we have a function $f : K \to \mathbb{R}$ we then have the notion of the *lower link* $\mathrm{Lk}^-(\sigma)$ of a simplex. It consists of those simplices α in $\mathrm{Lk}\,(\sigma)$ with $f(\alpha) < f(\sigma)$. The classical sublevel set theorems are often described in terms of these objects. In particular, in the case of stratified Morse theory, the normal Morse data has the homotopy type of the cone on $\mathrm{Lk}^-(\sigma)$.

So, to prove analogues of these theorems in our context we need to have some control over the behavior of our function $f : K \to \mathbb{R}$ around a local critical simplex. To this end, the following condition seems necessary.

Separation. If σ is a critical cell then the closed star $\overline{\mathrm{St}}(\sigma)$ contains no other critical cells.

This is analogous to the notion that critical points of a (stratified) Morse function are separated. We also want some form of the following.

Continuity. Denote by $s(\sigma)$ the stratum containing the critical cell σ and by $\overline{\mathrm{St}}_{s(\sigma)}(\sigma)$ the closed star of σ inside the stratum $s(\sigma)$. Then in a component X of $\overline{\mathrm{St}}(\sigma) - \overline{\mathrm{St}}_{s(\sigma)}(\sigma)$ if $\alpha \in X$ we have $f(\alpha) > f(\sigma)$ or $f(\alpha) < f(\sigma)$ depending on whether X intersects the upper link or lower link of σ, respectively.

With these conditions on a discrete stratified Morse function in place, we are optimistic that we can prove sublevel set theorems. There is also the matter of trying to understand the discrete analogue of the normal Morse data (see Appendix A). Since we are working with arbitrary simplicial complexes, it is not at all clear what the proper notion of "normal slice" is, and so we must seek alternative formulations (see Theorem A.5). This is work in progress that will be presented elsewhere. A related notion that also needs further study is that of the ordinary discrete Morse theory on open simplicial complexes. Note that a stratum is a union of open simplices, and some of the boundary faces of a given simplex may lie in a different stratum. The restriction of a discrete stratified Morse function to such a stratum is a discrete Morse function (on that stratum), but we still need to prove analogues of Forman's theorems in this context. If a critical cell lies away from the frontier of such a stratum, then Forman's theorems still apply, but what if a critical cell has one of its faces in a different stratum? What is the proper statement about how the homotopy type of the space changes as we pass the critical value? As these questions appear to be rather subtle, we will defer them for now.

One other issue to explore, as suggested by a referee, is whether we should develop a gradient-based version of discrete stratified Morse theory. That is, by analogy with the classical "smooth" setting we defined a notion of discrete stratified Morse function, but perhaps we could begin by setting up the notion of a discrete stratified gradient. Such a theory could perhaps help us identify the proper analogue of the normal Morse data in this arena. It seems that we would need to allow flows to bifurcate in some way, and then pair a p-simplex with multiple $(p+1)$-simplices, treating the whole collection as some sort of critical object. This may be related to work of Batko et al. [2] on discrete dynamical systems.

Implementations. In Section 3.5, we give an algorithm that constructs a discrete stratified Morse function on any finite simplicial complex equipped with an arbitrary real-valued function. We recently released an open-source visualization tool that implements this algorithm for 2-dimensional simplicial complexes embedded in the plane. The tool provides an interactive demo for exploring the algorithmic process and for performing homotopy-preserving simplification of the resulting stratified complex [37]. In fact, many examples presented in this paper can be recreated using the tool. Implementing this algorithm in higher dimension and the algorithm in Section 6 is left for future work.

Filtration-preserving reductions of complexes in persistent homology and parallel computation. As discrete Morse theory is useful for providing a filtration-preserving reduction of complexes in the computation of both persistent homology [9,31,36] and multi-parameter persistent homology [1], we hope that discrete stratified Morse theory could offer a new perspective on these computations. First, given any real-valued function defined on a simplicial complex, $f : K \to \mathbb{R}$, our algorithm generates a stratification of K such that the restriction of f to each stratum is a discrete Morse function. Applying *Morse pairing* to each stratum reduces K to a smaller complex of the same homotopy type. Second, if such a reduction can be performed in a filtration-preserving way with respect to each stratum, it would lead to an alternative computation of persistent homology in the setting where the function is not required to be Morse. Finally, since discrete Morse theory can be applied independently to each stratum of K, we can design a parallel algorithm that computes persistent homology pairings by strata and uses the stratification, which captures relations among strata pieces, to combine the results. Such reductions may not be asymptotically faster in comparison to existing approaches; nevertheless, it is a direction worth investigation.

Applications in imaging and visualization. Discrete Morse theory can be used to construct discrete Morse complexes in imaging (e.g. [8,36]), as well as Morse-Smale complexes (MSCs) [11,12] in visualization (e.g. [19,21]). In addition, it plays an essential role in the visualization of scalar fields and vector fields (e.g. [34,35]). Since discrete stratified Morse theory leads naturally to stratification-induced domain partitioning where discrete Morse theory becomes applicable, we envision our theory to be applicable for the analysis and visualization of large complex data.

Consider MSCs as an example. They are effective for identifying, ordering, and selectively simplifying features of data across a wide range of applications such as combustion [20] and battery design [22,23]. A MSC [12] describes the topology of a function based on its induced gradient flow by clustering the points in the domain into regions of monotone gradient flows, where each region is associated with a pair defined by a minimum (source) and a maximum (sink) of the function. Using discrete Morse theory, the complex is given by a combinatorial gradient field where simplification is done implicitly by changing the flow in the field [19]. Inspired by previous work [7] that approximates a MSC from point cloud data, we may consider the problem of defining and visualizing a generalized MSC derived from a discrete stratified Morse function, referred to as a *generalized MSC*.

A generalized MSC satisfies the Morse-Smale condition only locally, in spirit similar to a discrete stratified Morse function, which satisfies the Morse conditions locally within each stratum. Intuitively, suppose we are given an arbitrary function f on a simplical complex K, then we can generate a discrete stratified Morse function on K with a stratification s such that the restriction of f each stratum is Morse. Assuming in addition f that is also Morse-Smale when restricted to each strata piece, this means that we could compute a MSC for each strata and glue the resulting complexes together to form a generalized MSC. Such a complex will enjoy some key properties: they are easy to implement (and may handle high-dimensional data) by following the standard algorithms in computing MSC for each strata piece; they give simple interpretations of data associated with arbitrary functions (provided with a reasonable stratification); they expand the applicability of MSCs in visualization without requiring the functions on the point cloud data to be Morse; and they also lead to implicit, local simplification schemes that change the flow in a combinatorial field.

There is some connection between the above setup and previous work [21, 36]. To compute the Morse complex of a 2D or 3D grayscale digital image, Robins et al. [36] proposed an algorithm that decomposes a multidimensional image into (cubical) lower-stars by grayscale value, computes the discrete Morse matching separately and merges the results. The Morse matching is obtained by performing simple homotopy expansions from one subcomplex to the next, introducing critical cells only when such an expansion cannot be found. The resulting vector field is optimal in the sense that the number of critical cells is minimal. In contrast to our algorithm, the algorithm of [36] partitions the domain into regions (Morse cells) with uniform flow behaviors (points belong to the same partition when their gradient flows terminate at the same local maxima), while we partition the domain in a way that respect the underlying stratification. A similar algorithm was used by Gyulassy et al. [21] to compute MSC in parallel. The algorithm [21] relies on a divide-and-conquer approach that divides the dataset into *parcels* where the discrete gradient and MSC are computed locally on the boundary and in the interior of each parcel. In particular, "the boundary flow is fixed such that any flow passing through the boundary must pass through critical points restricted to the boundary" [21]. When two parcels are glued back together, the gradient flow on the boundary of parcels and the MSC on the new interior need to be updated to maintain consistency [21]. Furthermore, certain artifacts that have resulted from merging are removed by simplifying the ϵ-persistence pairs of the MSC. In comparison, the strata pieces from our framework can be considered as special types of *parcels* in the setting of [21]. Such parcels have potential advantages in the sense that the gradient flows on their boundaries remain consistent (and do not need to be updated) when the parcels are merged together and no artifacts are generated during the merging process. From an implementational perspective, these parcels do not necessarily induce more scalable implementations for computing MSC; however they do give rise to a different partition of the domain that respects its existing stratified structure.

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Conflict of interest

The authors declare that they have no conflict of interest.

Data availability statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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A Preliminaries on classical and stratified Morse theory

For completeness, we include here a review of the basics of (stratified) Morse theory. Given a topological space X, studying the relation between the critical points of a Morse function (or a stratified Morse function) on X and the topology of X requires more care in the smooth setting in comparison with the discrete setting. Most of our review originates from the seminal work of Goresky and MacPherson [18].

A.1 Classical Morse theory

Let X be a compact, differentiable d-manifold and $f : X \to \mathbb{R}$ a smooth real-valued function on X. For a given value $a \in \mathbb{R}$, let $X_a = f^{-1}(-\infty, a] = \{x \in X \mid f(x) \leq a\}$ denote the *sublevel set.* Morse theory studies the topological changes in X_a as a varies.

Morse functions. A point $x \in X$ is *critical* if the derivative at x equals zero. The value of f at a critical point is a *critical value.* All other points are *regular points* and all other values are *regular values* of f. A critical point x is *non-degenerate* if the Hessian, the matrix of second partial derivatives at the point, is invertible. The *Morse index* of the non-degenerate critical point x is the number of negative eigenvalues in the Hessian matrix, denoted as $\lambda(x)$.

Definition A.1 $f : X \to \mathbb{R}$ is a *Morse function* if all critical points are non-degenerate and its values at the critical points are distinct.

Results. We now review two fundamental results of classical Morse theory (CMT).

Theorem A.1 (CMT Theorem A) (*[18], p. 4; [10], p. 129*) *Let $f : X \to \mathbb{R}$ be a differentiable function on a compact smooth manifold X. Let $a < b$ be real numbers such that $f^{-1}[a, b]$ is compact and contains no critical points of f. Then X_a is diffeomorphic to X_b.*

On the other hand, let f be a Morse function on X. We consider two regular values $a < b$ such that $f^{-1}[a, b]$ is compact but contains one critical point u of f, with index λ. Then X_b has the homotopy type of X_a with a λ-cell (or λ-handle, the smooth analogue of a λ-cell) attached along its boundary ([18], page 5; [10], page 129). We define *Morse data* for f at a critical point u in X to be a pair of topological spaces (A, B) where $B \subset A$ with the property that as a real value c increases from a to b (by crossing the critical value $f(u)$), the change in X_c can be described by gluing in A along B [18] (page 4). Morse data measures the topological change in X_c as c crosses critical value $f(u)$. We have the second fundamental result of Morse theory,

Theorem A.2 (CMT Theorem B) (*[18], p. 5; [29], p. 77*) *Let f be a Morse function on X. Consider two regular values $a < b$ where $f^{-1}[a, b]$ is compact and contains one critical point u of f, with index λ. Then X_b is diffeomorphic to the space $X_a \cup_B A$, where $(A, B) = (D^\lambda \times D^{d-\lambda}, (\partial D^\lambda) \times D^{d-\lambda})$ is the Morse data, d is the dimension of X, λ is the Morse index of u, D^k denotes the closed k-dimensional disk, and ∂D^k is its boundary.*

A.2 Stratified Morse Theory

Morse theory can be generalized to certain singular spaces, in particular to Whitney stratified spaces [18, 30].

Stratified Morse function. Let X be a compact d-dimensional Whitney stratified space embedded in some smooth manifold $\mathbb{M}$. A function on X is *smooth* if it is the restriction to X of a smooth function on $\mathbb{M}$. Let $\bar{f} : \mathbb{M} \to \mathbb{R}$ be a smooth function. The restriction f of $\bar{f}$ to X is *critical* at a point $x \in X$ iff it is critical when restricted to the particular manifold piece which contains x [3]. A *critical value* of f is its value at a critical point.

Definition A.2 f is a *stratified Morse function* if ([3], [18] page 13):

1. All critical values of f are distinct.
2. At each critical point u of f, the restriction of f to the stratum S containing u is non-degenerate.
3. The differential of f at a critical point $u \in S$ does not annihilate (destroy) any limit of tangent spaces to any stratum S' other than the stratum S containing u.

Condition 1 and 2 imply that f is a Morse function when restricted to each stratum in the classical sense. Condition 2 is a non-degeneracy requirement in the tangential directions to S. Condition 3 is a non-degeneracy requirement in the directions normal to S [18] (page 13).

Results. Now we state the two fundamental results of stratified Morse theory.

Theorem A.3 (SMT Theorem A) (*[18], p. 6*) *Let X be a Whitney stratified space and $f : X \to \mathbb{R}$ a stratified Morse function. Suppose the interval $[a, b]$ contains no critical values of f. Then X_a is diffeomorphic to X_b.*

Theorem A.4 (SMT Theorem B) (*[18], p. 8 and p. 64*) *Let f be a stratified Morse function on a compact Whitney stratified space X. Consider two regular values $a < b$ such that $f^{-1}[a, b]$ is compact but contains one critical point u of f. Then X_b is diffeomorphic to the space $X_a \cup_B A$, where the Morse data (A, B) is the product of the normal Morse data at u and the tangential Morse data at u.*

To define tangential and normal Morse data, we have the following setup. Let X be a Whitney stratified subset of some smooth manifold $\mathbb{M}$. Let $f : X \to \mathbb{R}$ be a stratified Morse function with a critical point u. Let S denote the stratum of X which contains the critical point u. Let N be a *normal slice* at u, that is, $N = X \cap N' \cap B^{\mathbb{M}}_{\delta}(u)$, where N' is a sub-manifold of $\mathbb{M}$ which is traverse to each stratum of X, intersects the stratum S in a single point u, and satisfies $\dim S + \dim N' = \dim \mathbb{M}$. $B^{\mathbb{M}}_{\delta}(u)$ is a closed ball of radius δ in $\mathbb{M}$ based on a Riemannian metric on $\mathbb{M}$. By Whitney's condition, if δ is sufficiently small then $\partial B^{\mathbb{M}}_{\delta}(u)$ will be transverse to each stratum of X, and to each stratum in $X \cap N'$, fix such a $\delta > 0$ [18] (page 40).

The *tangential Morse data* for f at u is the pair

$$(P, Q) = (D^{\lambda} \times D^{s-\lambda}, (\partial D^{\lambda}) \times D^{s-\lambda}),$$

where λ is the (classical) Morse index of f restricted to S, $f|S$, at u, and s is the dimensional of stratum S [18] (page 65).

The *normal Morse data* is the pair

$$(J, K) = (N \cap f^{-1}[v - \varepsilon, v + \varepsilon], N \cap f^{-1}(v - \varepsilon)),$$

where $f(u) = v$ and $\varepsilon > 0$ is chosen such that $f|N$ has no critical values other than v in the interval $[v - \varepsilon, v + \varepsilon]$ [18] (page 65).

The *Morse data* is the topological product of the tangential and the normal Morse data, where the product of pairs is defined as $(A, B) = (P, Q) \times (J, K) = (P \times J, P \times K \cup Q \times J)$.

Theorem A.4 corresponds to the Main theorem of [18] (page 65), which has the following homotopy consequences. Suppose X is a Whitney stratified space, $f : X \to \mathbb{R}$ is a proper stratified Morse function, and $[a, b]$ contains no critical values except for a single isolated critical value $v \in (a, b)$ which corresponds to a critic point p in some stratum $\mathbb{S}$ of X. λ is the Morse index of $f|_{\mathbb{S}}$ at the point p.

Theorem A.5 (SMT Homotopy Consequences) (*[18] (Section 3.12, p. 68)* *The space X_b has the homotopy type of a space which is obtained from X_a by attaching the pair*

$$(D^{\lambda}, \partial D^{\lambda}) \times (\text{cone}(l^{-}), l^{-}).$$

Here, l^{-} is the *lower half link* of X where $l^{-} = N \cap f^{-1}(v - \varepsilon) \cap B^{\mathbb{M}}_{\delta}$.