Convergence between Categorical Representations of Reeb Space and Mapper

Elizabeth Munch
University at Albany – SUNY

Bei Wang*
University of Utah

Abstract

The Reeb space, which generalizes the notion of a Reeb graph, is one of the few tools in topological data analysis and visualization suitable for the study of multivariate scientific datasets. First introduced by Edelsbrunner et al. [9], it compresses the components of the level sets of a multivariate mapping and obtains a summary representation of their relationships. A related construction called the mapper [21], and a special case of mapper called the Joint Contour Net [2] have been shown to be effective in visual analytics. Mapper and JCN are intuitively regarded as discrete approximations of the Reeb space, however without formal proofs or approximation guarantees. An open question has been proposed by Dey et al. [7] as to whether the mapper converges to the Reeb space in the limit.

In this paper, we are interested in developing the theoretical understanding of the relationship between the Reeb space and its discrete approximations to support its use in practical data analysis. Using tools from category theory, we formally prove the convergence between the Reeb space and mapper in terms of an interleaving distance between their categorical representations. Given a sequence of refined discretizations, we prove that these approximations converge to the Reeb space in the interleaving distance; this also helps to quantify the approximation quality of the discretization at a fixed resolution.

*This work was partially support by NSF IIS-1513616.

1 Introduction

Motivation and prior work. Multivariate datasets arise in many scientific applications, ranging from oceanography to astrophysics, from chemistry to meteorology, from nuclear engineering to molecular dynamics. Consider, for example, combustion or climate simulations where multiple physical measurements (e.g. temperature and pressure) or concentrations of chemical species are computed simultaneously. We model these variables mathematically as multiple continuous, real-valued functions defined on a shared domain, which constitute a multivariate mapping $f : \mathbb{X} \to \mathbb{R}^d$, also known as a multi-field. We are interested in understanding the relationships between these real-valued functions, and more generally, in developing efficient and effective tools for their analysis and visualization.

Recently, topological methods have been developed to support the analysis and visualization of scalar field data with widespread applicability. In particular, a great deal of work for scalar topological analysis has been focused on computing the Reeb graph [20]. The Reeb graph contracts each contour (i.e. component of a level set) of a real-valued function to a single point and uses a graph representation to summarize the connections between these contours. When the domain is simply connected, this construction forms a contour tree [23]. The Reeb graph has been shown to be effective in many applications, including volume rendering [24], shape comparison [12], and data simplification and exploratory visualization [3]. From a computational perspective, both randomized [10] and deterministic [17] algorithms exist that compute the Reeb graph for a function defined on a simplicial complex K in time $O(m \log m)$, where m is the total number of vertices, edges and triangles in K. Recent work by de Silva et al. [6] has shown that the data of a Reeb graph can be stored in a category-theoretic object called a cosheaf, which opens the way for defining a metric for Reeb graphs known as the interleaving distance.

Unlike for real-valued functions, very few tools exist for studying multivariate data topologically as the situation becomes much more complicated. The most notable examples of these tools are the Jacobi set [8] and the Reeb space [9]. The Jacobi set analyzes the critical points of a real-valued function restricted to the intersection of the level sets of other functions. On the other hand, the Reeb space, a generalization of the Reeb graph, compresses the components of the level sets of the multivariate mapping (i.e. $f^{-1}(c)$, for $c \in \mathbb{R}^d$) and obtains a summary representation of their relationships. These two concepts are shown to be related as the image of the Jacobi sets under the mapping corresponds to certain singularities in the Reeb space. An algorithm has been described by Edelsbrunner et al. [9] to construct the Reeb space of a generic piecewise-linear (PL), $\mathbb{R}^d$-valued mapping defined on a combinatorial manifold up to dimension 4. Let n be the number of $(d-1)$-simplices in the combinatorial manifold. Assuming d is a constant, the running time of the algorithm is $O(n^d)$, polynomial in n [18].

A related construction called mapper [21] takes as input a multivariate mapping and produces a summary of the data by using a cover of the range space of the mapping. Such a summary converts the mapping with a fixed cover into a simplicial complex for efficient computation, manipulation, and exploration [14, 16]. When the mapping is a real-valued function (i.e. $d = 1$) and the cover consists of a collection of open intervals, it is stated without proof that the mapper recovers the Reeb graph precisely as the scale of the cover goes to zero [21]. A similar combinatorial idea has also been explored with the α-Reeb graph [5], which is another relaxed notion of a Reeb graph produced by a cover of the range space consisting of open intervals of length at most α. Recently, Dey et al. [7] extended mapper to its multiscale version by considering a hierarchical family of covers and the maps between them. At the end of their exposition, the authors raised an open question in understanding the continuous object that the mapper converges to as the scale of the cover goes to zero, in particular, whether the mapper converges to the Reeb space. In addition, Carr and Duke [2] introduced a special case of mapper called the Joint Contour Net (JCN) together with its efficient computation, for a PL mapping defined over a simplicial mesh involving an arbitrary number of real-valued functions. Based on a cover of the range space using d-dimensional intervals, the JCN quantizes the variation of multiple variables simultaneously by considering connected components of interval regions (i.e. $f^{-1}(a, b)$) instead of the connected components of level sets (i.e. $f^{-1}(c)$). It can be computed in time $O(km\alpha(km))$, where m is the size of the input mesh, k is the total number of quantized interval regions, and α is the slow-growing inverse Ackermann function [2]. The authors stated that the JCN can be considered as a discrete approximation that converges in the limit to the Reeb space [2], although this statement was supported only by intuition and lacked approximation guarantees.

Contributions. In this paper, we are interested in developing theoretical understandings between the Reeb space and its discrete approximations to support its use in practical data analysis. Using tools from category theory, we formally prove the convergence between the Reeb space and mapper in terms of an interleaving distance between their categorical representations (Theorem 4.1). Given a sequence of refined discretizations, we prove that these approximations converge to the Reeb space in the interleaving distance; this also helps to quantify the approximation quality of the discretization at a fixed resolution. Such a result easily generalizes to special cases of mapper such as the JCN. Our work extends and generalizes the tools from the categorical representation of Reeb graphs [6] to a new categorical framework for Reeb spaces. In particular, we provide for the first time the definition of the interleaving distance for Reeb spaces (Definition 5.1). We demonstrate that such a distance is an extended pseudometric (Theorem 5.2) and it provides a simple and formal language for structural comparisons. Finally in the settings of Reeb graphs (when $d = 1$), we demonstrate that the mapper converges to the Reeb graph geometrically on the space level (Corollary 8.1). We further provide an algorithm for constructing a continuous representation of mapper geometrically from its categorical representation.

2 Topological Notions

We now review the relevant background on the Reeb space [9, 18] and mapper [7, 21]. In theory, we assume the data given is a compact topological space $\mathbb{X}$ with an $\mathbb{R}^d$-valued function, $f : \mathbb{X} \to \mathbb{R}^d$, often denoted $(\mathbb{X}, f)$. In practice, we assume the data we work with is a multivariate PL mapping f defined over a simplicial mesh; more restrictively (for easier exposition of our algorithms and proofs), we consider a generic, PL mapping f from a combinatorial manifold[1] to $\mathbb{R}^d$.

Reeb Space. Let $f : \mathbb{X} \to \mathbb{R}^d$ be a generic, continuous mapping[2]. Intuitively, the Reeb space of f parametrizes the set of components of preimages of points in $\mathbb{R}^d$ [9]. Two points $x, y \in \mathbb{X}$ are equivalent, denoted by $x \sim_f y$, if $f(x) = f(y)$ and x and y belong to the same path connected component of the preimage, $f^{-1}(f(x)) = f^{-1}(f(y))$. The *Reeb space* is the quotient space obtained by identifying equivalent points, that is, $\mathcal{R}(\mathbb{X}, f) = \mathbb{X}/\sim_f$, together with the quotient topology inherited from $\mathbb{X}$. A powerful analysis tool, the *Reeb graph*, can be considered a special case in

[1] A *combinatorial s-manifold* $\mathbb{M}$ is the geometric realization of a simplicial complex for which the closed star of each vertex can be mapped by a homeomorphism onto a combinatorial s-ball in $\mathbb{R}^s$ such that each simplex of $\mathbb{M}$ is mapped affinely to a simplex in $\mathbb{R}^s$ [19]. Assuming the function values for a multivariate mapping are only known on the vertices of $\mathbb{M}$, we use barycentric coordinates to extend the function to the higher-order simplices by PL interpolation, creating a PL mapping $f : |\mathbb{M}| \to \mathbb{R}^d$ defined over the underlying space $|\mathbb{M}|$. The mathematical assumption of the continuity of f is then preserved. For simplicity, let $\mathbb{X} = |\mathbb{M}|$.

[2] For simplicity, assume f is a PL mapping defined on a combinatorial manifold.

this context when $d = 1$. Reeb spaces have been shown to have triangulations and canonical stratifications into manifolds for nice enough starting data [9].

Mapper. An open cover of a topological space $\mathbb{X}$ is a collection $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of open sets for some indexing set A such that $\bigcup_{\alpha \in A} U_\alpha = \mathbb{X}$. In this paper, we will always assume that each U_α is path-connected and a cover means a finite open cover. We define a finite open cover $\mathcal{U}$ to be a *good* cover if every finite nonempty intersection of sets in $\mathcal{U}$ is contractible. Given a cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of $\mathbb{X}$, let $\mathrm{Nrv}(\mathcal{U})$ denote the simplicial complex that corresponds to the *nerve* of the cover $\mathcal{U}$, $\mathrm{Nrv}(\mathcal{U}) = \left\{\sigma \subseteq A \mid \bigcap_{\alpha \in \sigma} U_\alpha \neq \emptyset\right\}$. Given a (potentially multivariate) continuous map $f : \mathbb{X} \to \mathbb{Y}$ where $\mathbb{Y}$ is equipped with a cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$, we write $f^*(\mathcal{U})$ as the cover of $\mathbb{X}$ obtained by considering the path connected components of $f^{-1}(U_\alpha)$ for each α, referred to as the *pullback cover* of $\mathbb{X}$ (induced by $\mathcal{U}$ via f). Given such a function f, its *mapper* M is defined to be the nerve of such a pullback cover, $M(\mathcal{U}, f) := \mathrm{Nrv}(f^*(\mathcal{U}))$ [21]. Intuitively, considering a real-valued function $f : \mathbb{X} \to \mathbb{R}$ and a cover $\mathcal{U}_\varepsilon$ of $\mathrm{image}(f) \subseteq \mathbb{R}$ consisting of intervals of length at most ε, the corresponding mapper $M(\mathcal{U}_\varepsilon, f)$ can be thought of as a relaxed Reeb graph that has been conjectured to converge to the Reeb graph of f as ε tends to zero [7, 21], although no formal proofs have been previously provided.

3 Categorical Notions

Here, we give a brief introduction to the necessary concepts from category theory. A more thorough treatment can be found, for example, in [15].

Category and opposite category. Category theory can be thought of as a generalization of set theory in the sense that the item of study is still a set (technically a proper class), but now we are additionally interested in studying the relationships between the elements of the set. Mathematically, a *category* is an algebraic structure that consists of mathematical *objects* with a notion of *morphisms* (colloquially referred to as *arrows* for the most of the remaining paper) between the objects. Thus, the data of a category consists of two pieces: the objects and the arrows. A category has the ability to compose the arrows associatively, and there is an identity arrow for each object. Examples are abundant and those important to our exposition are: the category of topological spaces (as the objects) with continuous functions between them (as the arrows), denoted as **Top**; the category of sets with set maps, denoted as **Set**; the category of open sets in $\mathbb{R}^d$ with inclusion maps, denoted as $\mathbf{Open}(\mathbb{R}^d)$; the category of vector spaces with linear maps, denoted as **Vect**; and the category of real numbers with inequalities connecting them, denoted as $\mathbf{R}$. In addition, any simplicial complex K induces a category $\mathbf{Cell}(K)$ where the objects are the simplices of K, and there is a morphism $\sigma \to \tau$ if σ is a face of τ. In-

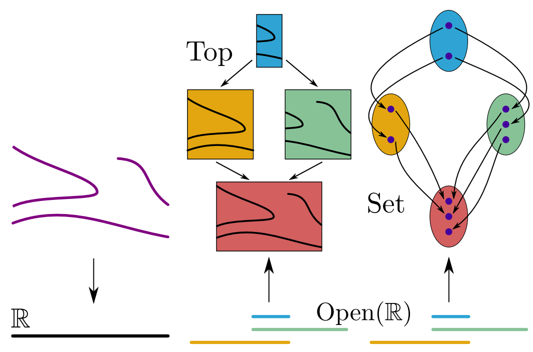

Figure 1: The data of a Reeb graph (on the left) can be stored as a functor. First, we give the middle functor $f^{-1} : \mathbf{Open}(\mathbb{R}) \to \mathbf{Top}$ which sends each open set I to the topological space $f^{-1}(I)$; and sends each inclusion map between open sets $I \subseteq J$ to an inclusion map $f^{-1}(I) \to f^{-1}(J)$. Then the Reeb graph information is represented by composing this functor with the functor $\pi_0 : \mathbf{Top} \to \mathbf{Set}$, producing a functor on the right $\pi_0 f^{-1} : \mathbf{Open}(\mathbb{R}) \to \mathbf{Set}$. Via π_0, the inclusion maps on the topological spaces become set maps.

tuitively, we could think of a category as a big (probably infinite) directed multi-graph with extra underlying structures (due to the associativity and identity axioms obeyed by the arrows): the objects are the nodes, and each possible arrow between the nodes is represented as a directed edge. One common example used extensively throughout this paper is the idea of a *poset category*, which is a category $\mathbf{P}$ in which any pair of elements $x, y \in \mathbf{P}$ has at most one arrow $x \to y$. Categories such as $\mathbf{Open}(\mathbb{R}^d)$ and $\mathbf{R}$ are poset categories since there is exactly one arrow $I \to J$ between open sets if $I \subseteq J$ and exactly one arrow $a \to b$ between real numbers if $a \leq b$. We often abuse notation and denote arrows in this category by the relation providing the poset structure, e.g. $I \subseteq J$ instead of $I \to J$ and $a \leq b$ instead of $a \to b$. In the graph description, a poset category can be thought of as a directed graph which is not a multigraph.

The *opposite category* (or dual category) $\mathcal{C}^{op}$ of a given category $\mathcal{C}$ is formed by reversing the arrows (morphisms), i.e. interchanging the source and target of each arrow.

Functor. A *functor* is a map between categories that maps objects to objects and arrows to arrows. A functor $F : \mathcal{C} \to \mathcal{D}$ for categories $\mathcal{C}$ and $\mathcal{D}$ maps an object x in $\mathcal{C}$ to an object $F(x)$ in $\mathcal{D}$, and maps an arrow $f : x \to y$ of $\mathcal{C}$ to an arrow $F[f] : F(x) \to F(y)$ of $\mathcal{D}$ in a way that respects the identity and composition laws. In the above graph allegory, a functor is a map between graphs which sends nodes (objects) to nodes and edges (arrows) to edges in a way that is compatible with the structure of the graphs. An example of a functor is the homology functor $H_p : \mathbf{Top} \to \mathbf{Vect}$ which sends a topological space $\mathbb{X}$ to its p-th singular homology group $H_p(\mathbb{X})$ (a vector space assuming field coefficients), and sends any continuous map $f : \mathbb{X} \to \mathbb{Y}$ to the linear map between homology groups, $H_p[f] := f_* : H_p(\mathbb{X}) \to H_p(\mathbb{Y})$. Another functor used extensively in this paper is $\pi_0 : \mathbf{Top} \to \mathbf{Set}$ which sends a topological space $\mathbb{X}$ to a set $\pi_0(\mathbb{X})$ where each element represents a path connected component of $\mathbb{X}$, and sends a map $f : \mathbb{X} \to \mathbb{Y}$ to a set map $\pi_0[f] := f_* : \pi_0(\mathbb{X}) \to \pi_0(\mathbb{Y})$.

Natural transformation. In addition, we can make any collection of functors of the form $F : \mathcal{C} \to \mathcal{D}$ into a category by defining arrows between the functors. A *natural transformation* $\varphi : F \Rightarrow G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$ is a family of arrows φ in $\mathcal{D}$ such that (a) for each object x of $\mathcal{C}$, we have $\varphi_x : F(x) \to G(x)$, an arrow of $\mathcal{D}$; and (b) for any arrow $f : x \to y$ in $\mathcal{C}$, $G[f] \circ \varphi_x = \varphi_y \circ F[f]$, that is, the diagram of Figure 2 commutes. Any collection of functors $F : \mathcal{C} \to \mathcal{D}$ can thus be turned into a category, with the functors themselves as objects and the natural transformations as arrows, notated as $\mathcal{D}^{\mathcal{C}}$. This notation is used heavily throughout this paper where always $\mathcal{D} = \mathbf{Set}$. If for every object x of $\mathcal{C}$, the arrow φ_x is an isomorphism in $\mathcal{D}$, then φ is a *natural isomorphism* (equivalence) of functors. Two functors F and G are *(naturally) isomorphic* if there exists a natural isomorphism from F to G.

$$\begin{tikzcd} F(x) \arrow[r, "\varphi_x"] \arrow[d, "F[f]"'] & G(x) \arrow[d, "G[f]"] \\ F(y) \arrow[r, "\varphi_y"] & G(y) \end{tikzcd}$$

Figure 2: The diagram for a natural transformation.

Categorical Reeb graph. For a real-valued function $f : \mathbb{X} \to \mathbb{R}$, the data of its corresponding Reeb graph can be stored as a functor $F := \pi_0 f^{-1} : \mathbf{Open}(\mathbb{R}) \to \mathbf{Set}$, defined by sending each open set I to a set $F(I) := \pi_0 f^{-1}(I)$ that contains all the path connected components of $f^{-1}(I)$; and by sending an inclusion $I \subseteq J$ to a set map $F[I \subseteq J] : F(I) \to F(J)$ induced by the inclusion $f^{-1}(I) \subseteq f^{-1}(J)$. This is illustrated in Figure 1. The objects $F(I)$ store the connected components sitting over any open set; the information from the arrows $F(I) \to F(J)$ gives the information needed to glue together all of this data. This construction produces a categorical representation of the Reeb graph, referred to as the *categorical Reeb graph*. It was used in [6] to define the interleaving distance for Reeb graphs which we generalize to Reeb spaces in Section 5.

Colimit. The final category theoretic notion necessary for our results are colimits. The *cocone* (N, ψ) of a functor $F : \mathcal{C} \to \mathcal{D}$ is an object N of $\mathcal{D}$ along with a family of ψ of arrows $\psi_x : F(x) \to N$ for every object x of $\mathcal{C}$, such that for every arrow $f : x \to y$ in $\mathcal{C}$, we have $\psi_y \circ F[f] = \varphi_x$. We say that a cocone (N, ψ) factors through another cocone (L, φ) if there exists an arrow $u : L \to N$ such that $u \circ \varphi_x = \psi_x$ for every x in $\mathcal{C}$. The *colimit* of $F : \mathcal{C} \to \mathcal{D}$, denoted as $\operatorname{colim} F$, is a cocone (L, φ) of F such that for any other cocone (N, ψ) of F, there exists a unique arrow $u : L \to N$ such that (N, ψ) factors through (L, φ). In other words, the diagram of Figure 3 commutes. The colimit is universal; in particular, this means that if the colimit (L, φ) factors through another cocone (M, δ), then L is isomorphic to M and the isomorphism is given by the unique arrow $u' : M \to L$ that defines it. We will use this property in the proof of Lemma 7.1.

$$\begin{tikzcd}
F(x) \arrow[d, "F[f]"'] \arrow[dr, "\varphi_x"] \arrow[drr, bend left, "\psi_x"] & & \\
 & L \arrow[r, dashed, "u"] & N \\
F(y) \arrow[ur, "\varphi_y"] \arrow[urr, bend right, "\psi_y"'] & &
\end{tikzcd}$$

Figure 3: Defining a colimit.

Because we often wish to consider these colimits over a full subcategory $\mathcal{A} \subseteq \mathcal{C}$, we will denote the restriction as $\operatorname{colim}_{A \in \mathcal{A}} F(A)$. The properties of a colimit also imply that if we have nested subcategories $\mathcal{A} \subseteq \mathcal{B}$ $(\subseteq \mathcal{C})$, then there is a unique map $\operatorname{colim}_{A \in \mathcal{A}} F(A) \to \operatorname{colim}_{B \in \mathcal{B}} F(B)$ since we can consider $\operatorname{colim}_{B \in \mathcal{B}} F(B)$ as cocone over $\mathcal{A}$.

4 Main Results Overview

The main focus of this paper is to provide a convergence result between the continuous Reeb space and the discrete mapper. We define their distance as the interleaving distance between their corresponding categorical representations and emphasize that neither the Reeb space nor the interleaving distance must ever be computed for this result. Instead, we provide a theoretical bound on the distance which requires only knowledge of the quantization resolution of the cover. To define the desired distance measure, we use the diagram in Figure 4 as our roadmap. The remainder of this section is dedicated to describing the various categories at the nodes of the diagram as well as the functors that connect them.

$$\begin{tikzcd}
\mathbb{R}^d\text{-}\mathbf{Top} \arrow[d, "\mathcal{C}_K"] \arrow[dr, bend left, "\mathcal{C}"] & \\
\mathbf{Set}^{\mathbf{Cell}(K)^{\mathrm{op}}} \arrow[r, "\mathcal{P}_K"] & \mathbf{Set}^{\mathbf{Open}(\mathbb{R}^d)}
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Data} \arrow[d] \arrow[dr, bend left] & \\
\text{Categorical mapper} \arrow[r] & \text{Categorical Reeb space}
\end{tikzcd}$$

Figure 4: The diagram for connecting categorical representations of the Reeb space and the mapper.

Data. In our context, data comes in the form of a topological space $\mathbb{X}$ with an $\mathbb{R}^d$-valued mapping, called an $\mathbb{R}^d$-space. We store such data in the category $\mathbb{R}^d$-**Top**. Specifically, an object of $\mathbb{R}^d$-**Top** is a pair consisting of a topological space $\mathbb{X}$ with a continuous map $f : \mathbb{X} \to \mathbb{R}^d$, denoted as $(\mathbb{X}, f)$. An arrow in $\mathbb{R}^d$-**Top**, $\nu : (\mathbb{X}, f) \to (\mathbb{Y}, g)$, is a function-preserving map; that is, it is a continuous map on the underlying spaces $\nu : \mathbb{X} \to \mathbb{Y}$ such that $g \circ \nu(x) = f(x)$ for all $x \in \mathbb{X}$. Note that many nice constructions such as PL functions on simplicial complexes or Morse functions on manifolds are objects in $\mathbb{R}^d$-**Top**.

Categorical Reeb space. Recall the categorical representation of a Reeb graph is a functor $\mathbf{Open}(\mathbb{R}) \to \mathbf{Set}$. In order to define a categorical representation of the Reeb space, we need a higher dimensional analogue of $\mathbf{Open}(\mathbb{R})$, namely, $\mathbf{Open}(\mathbb{R}^d)$. $\mathbf{Open}(\mathbb{R}^d)$ is a category with open sets $I \subseteq \mathbb{R}^d$ as objects, and a unique arrow $I \to J$ if and only if $I \subseteq J$; that is, $\mathbf{Open}(\mathbb{R}^d)$ is a poset category. The data of the Reeb space can be stored as a functor $\pi_0 f^{-1} : \mathbf{Open}(\mathbb{R}^d) \to \mathbf{Set}$, defined by sending each open set I to a set $\pi_0 f^{-1}(I)$ representing the path connected components of $f^{-1}(I)$; and by sending the inclusion arrow $I \subseteq J$ to a set map $\pi_0 f^{-1}(I) \to \pi_0 f^{-1}(J)$ induced by the inclusion $f^{-1}(I) \subseteq f^{-1}(J)$. These functors, referred to as the *categorical Reeb spaces*, become objects of the category of functors $\mathbf{Set}^{\mathbf{Open}(\mathbb{R}^d)}$.

Reeb space construction. Constructing a Reeb space from the data is now represented by the functor $\mathcal{C} : \mathbb{R}^d\text{-}\mathbf{Top} \to \mathbf{Set}^{\mathbf{Open}(\mathbb{R}^d)}$ in Figure 4. In particular, $\mathcal{C}$ maps an object $(\mathbb{X}, f)$ in $\mathbb{R}^d$-$\mathbf{Top}$, representing the data, to a functor $F : \mathbf{Open}(\mathbb{R}^d) \to \mathbf{Set}$ in $\mathbf{Set}^{\mathbf{Open}(\mathbb{R}^d)}$, representing its corresponding Reeb space. The functor $\mathcal{C}$ restricts to the Reeb graph construction when $d = 1$ [6]. In addition, from the generalized persistence module framework [1], we can also extend the idea of the interleaving distance between Reeb graphs (in the case $d = 1$) to these categorical Reeb spaces (in the case $d \geq 1$). The definition of functor $\mathcal{C}$ and the Reeb space interleaving distance are covered in Section 5.

Categorical mapper and its construction. Instead of working with continuous objects, we can instead consider discrete ones, by choosing a quantization represented by a cell complex K. Given a cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ for image$(f) \subseteq \mathbb{R}^d$, let $K = \mathrm{Nrv}(\mathcal{U})$. Through the machinery detailed in Section 6, we create a categorical representation of the mapper (referred to as the *categorical mapper*) as a functor $F : \mathbf{Cell}(K)^{\mathrm{op}} \to \mathbf{Set}$ (an object of $\mathbf{Set}^{\mathbf{Cell}(K)^{\mathrm{op}}}$); and such a construction is represented by the $\mathcal{C}_K$ functor[3].

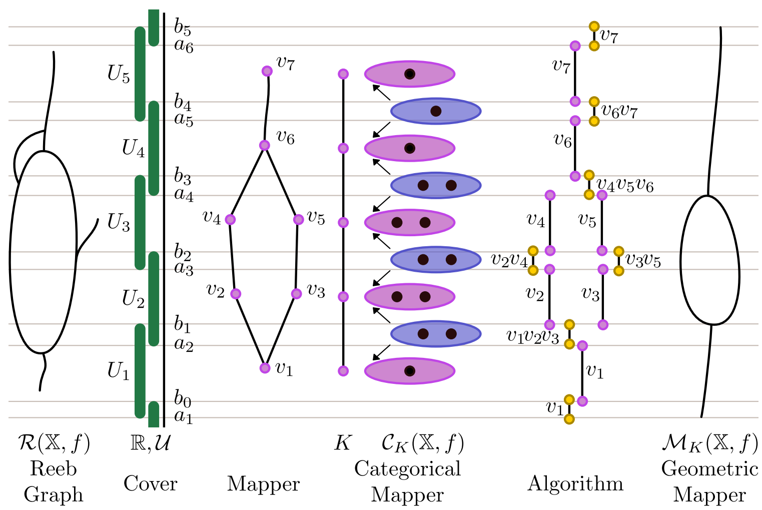

Figure 5: An example of a Reeb space for $d = 1$ (a Reeb graph), denoted as $\mathcal{R}(\mathbb{X}, f)$, is shown on the left. Its associated data $(\mathbb{X}, f)$ is an object in $\mathbb{R}^d$-$\mathbf{Top}$ with function f given by height. A cover $\mathcal{U}$ is shown by the green intervals, and the corresponding mapper is shown to its right. The mapper data is equivalently stored as the $C_K(\mathbb{X}, f)$ functor defined on simplicial complex $K = \mathrm{Nrv}(\mathcal{U})$. The geometric representation of this data, $\mathcal{M}_K(\mathbb{X}, f) := \mathcal{DP}_K\mathcal{C}_K(\mathbb{X}, f)$ is shown at the far right. Corollary 8.1 asserts that the interleaving distance between the leftmost and rightmost graphs is bounded by $\varepsilon = \mathrm{res}(\mathcal{U})$.

Comparing Reeb space and mapper. It should be noted that the Reeb space and the mapper are inherently different objects. The Reeb space comes equipped with an $\mathbb{R}^d$-valued function, while there is no such function built into the mapper even though its construction is highly dependent on the functions chosen to partition the data set [21]. In particular, the two objects are in completely different categories.

[3] A related but slightly different categorical mapper was introduced by Stovner [22], as a functor from the category of covered topological spaces to the category of simplicial complexes.

So, to compare these objects, we study the image of the categorical mapper under the functor $\mathcal{P}_K$, which turns the categorical mapper (a discrete object) into a continuous one comparable with the categorical Reeb space. In particular, for data given as $(\mathbb{X}, f)$ in $\mathbb{R}^d$-**Top**, we compare its image in $\mathbf{Set}^{\mathbf{Open}(\mathbb{R}^d)}$ via the functor $\mathcal{C}$, to its image in $\mathbf{Set}^{\mathbf{Open}(\mathbb{R}^d)}$ via the functor $\mathcal{P}_K\mathcal{C}_K$. Symbolically, following Figure 4, we are comparing $\mathcal{P}_K\mathcal{C}_K(\mathbb{X}, f)$ to $\mathcal{C}(\mathbb{X}, f)$. This relationship and the construction of functor $\mathcal{P}_K$ are covered in Section 7.

We then prove our main result, the categorical convergence theorem below.

Theorem 4.1 (Convergence between Categorical Reeb Space and Categorical Mapper)**.** *Given a multivariate function $f : \mathbb{X} \to \mathbb{R}^d$ defined on a compact topological space*[4]*, the data is represented as an object $(\mathbb{X}, f)$ in $\mathbb{R}^d$-**Top**. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a good cover of $f(\mathbb{X}) \subseteq \mathbb{R}^d$, K be the nerve of the cover and* $\mathrm{res}(\mathcal{U})$ *be the resolution of the cover, that is, the maximum diameter of the sets in the cover* $\mathrm{res}(\mathcal{U}) = \sup\{\mathrm{diam}(U_\alpha) \mid U_\alpha \in \mathcal{U}\}$*. Then*

$$d_I(\mathcal{C}(\mathbb{X}, f), \mathcal{P}_K\mathcal{C}_K(\mathbb{X}, f)) \leq \mathrm{res}(\mathcal{U}).$$

Theorem 4.1 states that for increasingly refined covers, the image of the categorical mapper converges to the categorical Reeb space in the interleaving distance. In other words, the distance between the mapper and the Reeb space is bounded above by the resolution of the quantization. Thus, we can make approximation guarantees about the accuracy of the mapper based on a property of the chosen quantization.

Summary. The various categorical representations can be summarized in Figure 4, some of which are illustrated in Figure 5 for the case when $d = 1$. The initial data received is an object $(\mathbb{X}, f)$ in $\mathbb{R}^d$-**Top**. Then we can either construct its categorical Reeb space through the functor $\mathcal{C}$, or construct its categorical mapper using the functor $\mathcal{C}_K$. In order to compare these two objects in the same category, we push the mapper along using the $\mathcal{P}_K$ functor, and then compute the distance between $\mathcal{C}(\mathbb{X}, f)$ and $\mathcal{P}_K\mathcal{C}_K(\mathbb{X}, f)$ in $\mathbf{Set}^{\mathbf{Open}(\mathbb{R}^d)}$. We should stress before we continue that this diagram does not necessarily commute. In a way, the above distance is measuring how far the diagram is from being commutative. Making no assumptions about $\mathcal{U}$, Theorem 4.1 states that the interleaving distance between the results of the two paths in the diagram is bounded by the resolution of $\mathcal{U}$. Furthermore in Section 8, for the special case when $d = 1$, we can turn our categorical convergence theorem, Theorem 4.1, into the geometric convergence theorem, Corollary 8.1. Finally, we provide an algorithm for producing a geometric representation of the image of categorical mapper, $\mathcal{P}_K\mathcal{C}_K(\mathbb{X}, f)$.

5 Interleaving Distance between Reeb Spaces

As described in Section 4, we start by generalizing the categorical Reeb graph to the categorical Reeb space. Given the data received as a topological space $\mathbb{X}$ equipped with an $\mathbb{R}^d$-valued function $f : \mathbb{X} \to \mathbb{R}^d$, denoted as $(\mathbb{X}, f)$, we define the functor $\mathcal{C} : \mathbb{R}^d\text{-}\mathbf{Top} \to \mathbf{Set}^{\mathbf{Open}(\mathbb{R}^d)}$ as follows: $\mathcal{C}$ maps an object $(\mathbb{X}, f)$ in $\mathbb{R}^d$-**Top** to a functor $\mathcal{C}(\mathbb{X}, f) := \pi_0 f^{-1} : \mathbf{Open}(\mathbb{R}^d) \to \mathbf{Set}$ in $\mathbf{Set}^{\mathbf{Open}(\mathbb{R}^d)}$, and an arrow $\nu : (\mathbb{X}, f) \to (\mathbb{Y}, g)$ to a natural transformation $\mathcal{C}[\nu]$ induced by the inclusion $\nu f^{-1}(I) \subseteq g^{-1}(I)$. The functor $\mathcal{C}$ turns the given data into the categorical representation of the Reeb space, and the functoriality of π_0 makes it a well-defined functor.

Our first goal is to define the interleaving distance for these categorical Reeb spaces. Denote the ε-thickening of a open set $I \in \mathbf{Open}(\mathbb{R}^d)$ to be the set $I^\varepsilon := \{x \in \mathbb{R}^d \mid \|x - I\| < \varepsilon\}$.

[4]For simplicity, we assume a combinatorial s-manifold; however this is not necessary for the proof.

Using this, we can define a thickening functor $T_\varepsilon : \mathbf{Open}(\mathbb{R}^d) \to \mathbf{Open}(\mathbb{R}^d)$ by $T_\varepsilon(I) := I^\varepsilon$, and $T_\varepsilon[I \subseteq J] := \{I^\varepsilon \subseteq J^\varepsilon\}$. Let $\mathcal{S}_\varepsilon$ be the functor from $\mathbf{Set}^{\mathbf{Open}(\mathbb{R}^d)}$ to itself defined by $\mathcal{S}_\varepsilon(\mathcal{F}) := \mathcal{F}T_\varepsilon$, for every functor $\mathcal{F} : \mathbf{Open}(\mathbb{R}^d) \to \mathbf{Set}$. Given the two functors $\mathcal{F}$ and $\mathcal{S}_{2\varepsilon}(\mathcal{F})$, both of which are defined on $\mathbf{Open}(\mathbb{R}^d) \to \mathbf{Set}$, there is an obvious natural transformation $\eta : \mathcal{F} \Rightarrow \mathcal{S}_{2\varepsilon}\mathcal{F}$ defined by $\eta_I = \mathcal{F}[I \subseteq I^{2\varepsilon}]$. We write $\tau : \mathcal{G} \Rightarrow \mathcal{S}_{2\varepsilon}(\mathcal{G})$ for the analogous natural transformation for $\mathcal{G}$.

Definition 5.1 (Interleaving distance between Categorical Reeb spaces)**.** *An ε-interleaving between functors $\mathcal{F}, \mathcal{G} : \mathbf{Open}(\mathbb{R}^d) \to \mathbf{Set}$ is a pair of natural transformations, $\varphi : \mathcal{F} \Rightarrow \mathcal{S}_\varepsilon(\mathcal{G})$ and $\psi : \mathcal{G} \Rightarrow \mathcal{S}_\varepsilon(\mathcal{F})$ such that the diagrams below commute.*

$$
\begin{tikzcd}
\mathcal{F} \arrow[r, Rightarrow, "\varphi"] \arrow[dr, Rightarrow, "\eta"'] & \mathcal{S}_\varepsilon(\mathcal{G}) \arrow[d, Rightarrow, "\mathcal{S}_\varepsilon[\psi]"] \\
& \mathcal{S}_{2\varepsilon}(\mathcal{F})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{G} \arrow[r, Rightarrow, "\psi"] \arrow[dr, Rightarrow, "\tau"'] & \mathcal{S}_\varepsilon(\mathcal{F}) \arrow[d, Rightarrow, "\mathcal{S}_\varepsilon[\varphi]"] \\
& \mathcal{S}_{2\varepsilon}(\mathcal{G})
\end{tikzcd}
$$

Given two functors $\mathcal{F}, \mathcal{G} : \mathbf{Open}(\mathbb{R}^d) \to \mathbf{Set}$, the interleaving distance is defined to be

$$d_I(\mathcal{F}, \mathcal{G}) = \inf\{\varepsilon \in \mathbb{R}_{\geq 0} \mid \mathcal{F}, \mathcal{G} \text{ are } \varepsilon\text{-interleaved}\}.$$

We define $d_I(F, G) = \infty$ if the set on the right-hand side is empty.

We have the following property of d_I.

Theorem 5.2. *The interleaving distance d_I, between two categorical representations of Reeb spaces, is an extended pseudometric on $\mathbf{Set}^{\mathbf{Open}(\mathbb{R}^d)}$.*

Proof. We will use Theorem 3.21 of [1], which states that as long as T_ε has the properties we need, that is, if maps T_ε comprise a superlinear family in the poset $\mathbf{Open}(\mathbb{R}^d)$, then the interleaving distance d_I defined using T_ε in Definition 5.1 is guaranteed to be an extended pseudometric.

Let $\mathbf{P} = (P, \leq)$ be a poset category. A translation of $\mathbf{P}$ is a functor $\Gamma : \mathbf{P} \to \mathbf{P}$ which has a natural transformation $\eta_\Gamma : \mathbf{1} \Rightarrow \Gamma$ where $\mathbf{1}$ is the identity functor [1]. Let $\mathbf{Trans_P}$ denote the category that contains the collection of translations of $\mathbf{P}$. The functor T_ε is a translation of the poset category $\mathbf{Open}(\mathbb{R}^d)$ since we can choose $\eta_I : \mathbf{1}(I) \to T_\varepsilon(I)$ to be induced by the inclusion $I \subseteq I^\varepsilon$.

A superlinear family of translations is a function $\Omega : [0, \infty) \to \mathbf{Trans_P}$ defined by $\Omega(\varepsilon) = \Gamma_\varepsilon$ such that $\Gamma_{\alpha+\beta} \leq \Gamma_\alpha \Gamma_\beta$ for all $\alpha, \beta \geq 0$. The family of translations T_ε satisfies this requirement since $T_{\alpha+\beta}(I) = I^{\alpha+\beta}$ and $T_\alpha T_\beta(I) = \left(I^\beta\right)^\alpha = I^{\alpha+\beta}$. So since T_ε is a superlinear family of translations in $\mathbf{Open}(\mathbb{R}^d)$, the interleaving distance is an extended pseudometric. $\square$

Special case for Reeb graphs. When $d = 1$ we have much more control of the situation. In particular, [6] gives us that the category of Reeb graphs, defined to be finite graphs with real valued functions that are strictly monotone on the edges, is equivalent to a well-behaved subcategory of $\mathbf{Set}^{\mathbf{Open}(\mathbb{R})}$. Theorem 5.3 (as a direct consequence of Corollary 4.9 in [6]) says that the above defined interleaving distance d_I is an extended metric, not just a pseudometric, when restricted to these objects.

Theorem 5.3 ([6])**.** *When $d = 1$, $d_I(\mathcal{C}(\mathbb{X}, f), \mathcal{C}(\mathbb{Y}, g))$ is an extended metric on the categorical Reeb spaces.*

Theorem 5.3 means that for $d = 1$, if $d_I(\mathcal{C}(\mathbb{X}, f), \mathcal{C}(\mathbb{Y}, g)) = 0$ (that is, when the categorical mapper converges to the categorical Reeb graph), then $\mathcal{C}(\mathbb{X}, f)$ and $\mathcal{C}(\mathbb{Y}, g)$ are isomorphic as functors. This implies that, in the special case when $d = 1$, the mapper converges to the Reeb space not only categorically but also geometrically. This is discussed in Section 8.

While recent work is beginning to elucidate the case where $d > 1$, the technical finesse needed to make a similar statement to Theorem 5.3 is beyond the scope of this paper. Thus, we will stick to statements about the categorical representations for Reeb spaces when $d > 1$, and make concrete geometric statements when they are available for $d = 1$ (see Section 8).

6 Categorical Representation of Mapper and its Construction

The beauty of working with category theory is that we can store a categorical representation of the mapper as sets over the nerve of a cover, rather than working directly with its complicated topological definition (given in Section 2). Given a choice of finite open cover for image$(f) \subseteq \mathbb{R}^d$, $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$, let $K = \mathrm{Nrv}(\mathcal{U})$. In order to ensure that K faithfully represents the underlying structure, we will assume that $\mathcal{U}$ is a *good* cover. This ensures that the nerve lemma applies, that is, K has the homotopy type of image$(f) \subseteq \mathbb{R}^d$ (see, e.g., Corollary 4G.3 [11] or Theorem 15.21 [13]).

For simplicity of notation, we denote $\mathcal{U}_\sigma = \bigcap_{\alpha \in \sigma} U_\alpha$ to be the open set in $\mathbb{R}^d$ associated to the simplex $\sigma \in K$. One important property of this construction is that for $\sigma \leq \tau$ in K, the associated inclusion of spaces is reversed: $\mathcal{U}_\sigma \supseteq \mathcal{U}_\tau$. So, if we wish to represent the connected components for a particular $\mathcal{U}_\sigma$ for $\sigma \in K$, we can still consider $\pi_0 f^{-1}(\mathcal{U}_\sigma)$, however, the face relation $\sigma \leq \tau$ induces a "backwards" mapping $\pi_0 f^{-1}(\mathcal{U}_\tau) \to \pi_0 f^{-1}(\mathcal{U}_\sigma)$. We keep track of this switch using the opposite category. Recall $\mathbf{Cell}(K)$ is a category with simplices of K as objects and a unique arrow $\sigma \to \tau$ given by the face relation $\sigma \leq \tau$. Then the opposite category, $\mathbf{Cell}(K)^{\mathrm{op}}$, has the simplices of K as objects and a unique arrow $\tau \to \sigma$ given by the face relation $\sigma \leq \tau$.

Thus, given an object $(\mathbb{X}, f)$ in $\mathbb{R}^d$-$\mathbf{Top}$, we have a functor $\mathcal{C}_K^f : \mathbf{Cell}(K)^{\mathrm{op}} \to \mathbf{Set}$ that maps every σ to $\mathcal{C}_K^f(\sigma) := \pi_0 f^{-1}(\mathcal{U}_\sigma)$. We are required to use the opposite cell category so that $\mathcal{C}_K^f$ maps the morphism $\sigma \leq \tau$ (equivalently notated $\tau \to \sigma$ in the opposite category) to the set map $\pi_0 f^{-1}(\mathcal{U}_\tau) \to \pi_0 f^{-1}(\mathcal{U}_\sigma)$ induced by the inclusion $\mathcal{U}_\tau \subseteq \mathcal{U}_\sigma$ as discussed above. This functor is used to represent the categorical mapper of $(\mathbb{X}, f)$ for the cover $\mathcal{U}$.

Note that the functor $\mathcal{C}_K^f$ is an object of the category of functors $\mathbf{Set}^{\mathbf{Cell}(K)^{\mathrm{op}}}$. The process of building the mapper is thus represented itself by the functor $\mathcal{C}_K : \mathbb{R}^d$-$\mathbf{Top} \to \mathbf{Set}^{\mathbf{Cell}(K)^{\mathrm{op}}}$, which is defined as follows. For the objects, $\mathcal{C}_K$ maps an $\mathbb{R}^d$-space $(\mathbb{X}, f)$ in $\mathbb{R}^d$-$\mathbf{Top}$ to the functor $\mathcal{C}_K(\mathbb{X}, f) := \mathcal{C}_K^f$ as given above. For the morphisms, it sends a function preserving map $\nu : (\mathbb{X}, f) \to (\mathbb{Y}, g)$ to a natural transformation (which is an arrow in $\mathbf{Set}^{\mathbf{Cell}(K)^{\mathrm{op}}}$), $\mathcal{C}_K[\nu] : \mathcal{C}_K^f \to \mathcal{C}_K^g$.

$$
\begin{tikzcd}
\mathcal{C}_K^f(\tau) = \pi_0 f^{-1}(\mathcal{U}_\tau) \arrow[r, "\mathcal{C}_K[\nu]_\tau"] \arrow[d, "\mathcal{C}_K^f[\sigma \leq \tau]"'] & \mathcal{C}_K^g(\tau) = \pi_0 g^{-1}(\mathcal{U}_\tau) \arrow[d, "\mathcal{C}_K^g[\sigma \leq \tau]"] \\
\mathcal{C}_K^f(\sigma) = \pi_0 f^{-1}(\mathcal{U}_\sigma) \arrow[r, "\mathcal{C}_K[\nu]_\sigma"] & \mathcal{C}_K^g(\sigma) = \pi_0 g^{-1}(\mathcal{U}_\sigma)
\end{tikzcd}
$$

Figure 6: Commutative diagram for $\mathcal{C}_k[v]$ being a natural transformation.

Checking that $\mathcal{C}_K[\nu]$ is indeed a natural transformation amounts to observing that the diagram in Figure 6 commutes by functoriality of π_0. Recall $\mathcal{C}_K[\nu]_\sigma$ is defined by setting $\mathcal{C}_K[\nu]_\sigma : \pi_0 f^{-1}(\mathcal{U}_\sigma) \to \pi_0 g^{-1}(\mathcal{U}_\sigma)$ to be the map induced by the restriction $\nu : f^{-1}(\mathcal{U}_\sigma) \to g^{-1}(\mathcal{U}_\sigma)$, which is well-defined since ν is function preserving.

7 Convergence between Mapper and Reeb Space

In order to compare the discrete mapper with the continuous Reeb space, we must move them both into the same category. At the moment, for data given as $(\mathbb{X}, f)$ in $\mathbb{R}^d$-**Top**, we have the categorical Reeb space representation $\mathcal{C}(\mathbb{X}, f)$ in $\mathbf{Set}^{\mathbf{Open}(\mathbb{R}^d)}$, and the categorical mapper representation $\mathcal{C}_K(\mathbb{X}, f)$ in $\mathbf{Set}^{\mathbf{Cell}(K)^{\mathrm{op}}}$. Thus we must first define the functor $\mathcal{P}_K$ in order to push the mapper representation into the $\mathbf{Set}^{\mathbf{Open}(\mathbb{R}^d)}$ category, then prove the convergence result there using the interleaving distance from Section 5.

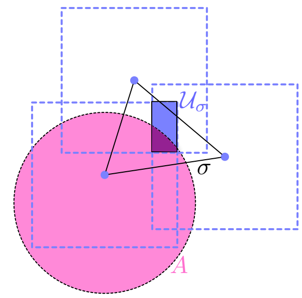

Figure 7: The simplex (triangle) σ is in K_A since $\mathcal{U}_\sigma \cap A \neq \emptyset$. The set $A \subseteq \mathbb{R}^2$ is the pink circle, $\mathcal{U}_\sigma$ is the solid blue rectangle, and the blue squares with dotted boundaries represent the cover elements in $\mathcal{U}$ associated with σ.

Given a simplicial complex K which is the nerve of the cover $\mathcal{U}$, we define K_A for a open set $A \subseteq \mathbb{R}^d$ to be the collection of simplices in K such that the associated intersection $\mathcal{U}_\sigma$ intersects A, $K_A = \{\sigma \in K \mid \mathcal{U}_\sigma \cap A \neq \emptyset\}$ (see Figure 7 for an example when $d = 2$). Now we can construct the functor $\mathcal{P}_K : \mathbf{Set}^{\mathbf{Cell}(K)^{\mathrm{op}}} \to \mathbf{Set}^{\mathbf{Open}(\mathbb{R}^d)}$ as follows. Given a functor $F : \mathbf{Cell}(K)^{\mathrm{op}} \to \mathbf{Set}$, $\mathcal{P}_K$ sends it to a functor $\mathcal{P}_K(F) : \mathbf{Open}(\mathbb{R}^d) \to \mathbf{Set}$ by defining

$$\mathcal{P}_K(F)(I) = \operatorname*{colim}_{\sigma \in K_I} F(\sigma)$$

for every I in $\mathbf{Open}(\mathbb{R}^d)$. Here, the colimit construction can be thought of as a set representing the connected components over the collection of open sets $\mathcal{U}_\sigma$ for the simplices $\sigma \in K_I$, or equivalently, over the union $\bigcup_{\sigma \in K_I} \mathcal{U}_\sigma$. The morphisms in the two functor categories $\mathbf{Set}^{\mathbf{Cell}(K)^{\mathrm{op}}}$ and $\mathbf{Set}^{\mathbf{Open}(\mathbb{R}^d)}$ are natural transformations; $\mathcal{P}_K$ sends arrows to arrows in a well-defined way via the colimit as discussed at the end of Section 3, since if $I \subseteq J$, then $K_I \subseteq K_J$. Additionally, we must check that $\mathcal{P}_K$ sends a natural transformation $\eta : F \Rightarrow G$ to a natural transformation $\mathcal{P}_K(F) \to \mathcal{P}_K(G)$; we omit this bookkeeping here. Since the mapper depends on the choice of a cover, it makes sense that the cover, in particular, its resolution, will be a key factor in understanding the convergence. With all of this machinery, we have our main result, Theorem 4.1.

Theorem 4.1 implies that if we have a sequence of covers $\mathcal{U}_i$ such that $\mathrm{res}(\mathcal{U}_i) \to 0$, then the categorical representations of the mapper converge to the Reeb space in the interleaving distance. Its proof relies on a main technical result, Lemma 7.1 below, which relates the functor $\mathcal{P}_K\mathcal{C}_K(\mathbb{X}, f)$ to one which avoids the combinatorial structure of K as much as possible and instead works with inverse images of subsets of $\mathbb{R}^d$.

Lemma 7.1. *Let $\mathcal{F} : \mathbf{Open}(\mathbb{R}^d) \to \mathbf{Set}$ be a functor which maps an open set I, to a set $\pi_0 f^{-1}(\bigcup_{\sigma \in K_I} \mathcal{U}_\sigma)$ with morphisms induced by π_0 on the inclusions. Then, the functor $\mathcal{P}_K\mathcal{C}_K(\mathbb{X}, f)$ is equivalent to $\mathcal{F}$.*

Proof. The functor $\mathcal{C}_K(\mathbb{X}, f) = \mathcal{C}_K^f : \mathbf{Cell}(K)^{\mathrm{op}} \to \mathbf{Set}$ is given by sending a cell σ to $\pi_0 f^{-1}(\mathcal{U}_\sigma)$, and its composition with $\mathcal{P}_K$ is given by $\mathcal{P}_K\mathcal{C}_K(\mathbb{X}, f) = \mathcal{P}_K(\mathcal{C}_K^f) : \mathbf{Open}(\mathbb{R}^d) \to \mathbf{Set}$ defined by $\mathcal{P}_K(\mathcal{C}_K^f)(I) = \mathrm{colim}_{\sigma \in K_I} \mathcal{C}_K^f(\sigma)$. To establish natural equivalence of functors, we will construct a natural transformation $\psi : \mathcal{F} \Rightarrow \mathcal{P}_K\mathcal{C}_K(\mathbb{X}, f)$ which is an isomorphism for each ψ_I.

As a roadmap, we can refer to the following diagram:

$$
\begin{tikzcd}
\mathcal{C}_K^f(\sigma) = \pi_0 f^{-1}(\mathcal{U}_\sigma) \arrow[dd, "\mathcal{C}_K^f{[\tau \leq \sigma]}"'] \arrow[dr, "\varphi_\sigma"'] \arrow[drr, bend left, "\eta_\sigma"] & & \\
& \mathcal{F}(I) = \pi_0 f^{-1}(\bigcup_{\sigma \in K_I} \mathcal{U}_\sigma) \arrow[r, dashed, "\psi_I"] & \mathcal{P}_K \mathcal{C}_K^f(I) = \mathrm{colim}_{\sigma \in K_I} \mathcal{C}_K^f(\sigma) \\
\mathcal{C}_K^f(\tau) = \pi_0 f^{-1}(\mathcal{U}_\tau) \arrow[ur, "\varphi_\tau"] \arrow[urr, bend right, "\eta_\sigma"'] & &
\end{tikzcd}
$$

By definition of $\mathcal{F}$, $\mathcal{F}(I) = \pi_0 f^{-1}(\bigcup_{\sigma \in K_I} \mathcal{U}_\sigma)$ so there are obvious maps induced by inclusions $\varphi_\sigma : \pi_0 f^{-1}(\mathcal{U}_\sigma) \to \mathcal{F}(I)$ which all commute; this gives us a cone $(\mathcal{F}(I), \varphi_\sigma)$ for the diagram $\{\mathcal{C}_K^f(\sigma)\}_{\sigma \in K_I}$. The colimit of this same diagram is a cocone denoted by $(\mathcal{P}_K \mathcal{C}_K^f(I), \eta_\sigma)$. We will construct a map $\psi_I : \pi_0 f^{-1}(\bigcup_{\sigma \in K_I} \mathcal{U}_\sigma) \to \mathrm{colim}_{\sigma \in K_I} \mathcal{C}_K^f(\sigma)$ such that the colimit cocone factors through the cocone $(\mathcal{F}(I), \varphi_\sigma)$ using ψ_I; that is, $\psi_I \circ \varphi_\sigma = \eta_\sigma$ for all $\sigma \in K_I$. The universality of the colimit then implies that ψ_I is an isomorphism.

To construct ψ_I, consider any u in $\pi_0 f^{-1}(\bigcup_{\sigma \in K_I} \mathcal{U}_\sigma)$. This set element represents a connected component in $f^{-1}(\bigcup_{\sigma \in K_I} \mathcal{U}_\sigma)$, and thus there is at least one σ with an element $u_\sigma \in \mathcal{C}_K^f(\sigma)$ such that $\varphi_\sigma(u_\sigma) = u$. Now we define $\psi_I(u) = \eta_\sigma(u_\sigma)$. Ensuring that ψ_I above is well defined corresponds to ensuring that different elements mapping to u from possibly different cells τ must map to the same element of the colimit, but this is an immediate consequence of the colimit properties.

$$
\begin{tikzcd}
\pi_0 f^{-1}(\bigcup_{\sigma \in K_I} \mathcal{U}_\sigma) \arrow[r, "\psi_I"] \arrow[d] & \mathcal{P}_K \mathcal{C}_K^f(I) \arrow[d] \\
\pi_0 f^{-1}(\bigcup_{\sigma \in K_J} \mathcal{U}_\sigma) \arrow[r, "\psi_J"] & \mathcal{P}_K \mathcal{C}_K^f(J)
\end{tikzcd}
$$

Figure 8: The diagram showing that $\varphi = \{\varphi_I\}$ defines a natural transformation.

Finally, we prove that the collection $\{\psi_I\}$ defines a natural transformation. Since if $I \subseteq J$, then $K_I \subseteq K_J$. Then an exercise in colimit properties ensures that the diagram in Figure 8 commutes, where the arrow on the left is the map induced by inclusions, and the map on the right is induced by the colimit definition. $\square$

Proof of Theorem 4.1. Let $\varepsilon = \mathrm{res}(\mathcal{U})$. Combined with Lemma 7.1, we will construct an ε-interleaving, $\varphi : \mathcal{F} \Rightarrow \mathcal{C}(\mathbb{X}, f) \circ T_\varepsilon$ and $\psi : \mathcal{C}(\mathbb{X}, f) \Rightarrow \mathcal{F} \circ T_\varepsilon$. The constructed interleaving proves the desired inequality following Definition 5.1.

First, we prove the following statement: if $\mathcal{U}_\sigma \cap I \neq \emptyset$, then $\mathcal{U}_\sigma \subset I^\varepsilon$. Indeed, for any $x \in \mathcal{U}_\sigma$, if $x \in I$ then $x \in I^\varepsilon$. If $x \notin I$, then because there exists a $y \in \mathcal{U}_\sigma \cap I$, such that $\|x - y\| \leq \mathrm{diam}(\mathcal{U}_\sigma) \leq \mathrm{res}(\mathcal{U}) = \varepsilon$, so $x \in I^\varepsilon$. This statement implies that we have the inclusion $\bigcup_{\sigma \in K_I} \mathcal{U}_\sigma \hookrightarrow I^\varepsilon$.

We also have inclusions $I \hookrightarrow \bigcup_{\sigma \in K_I} \mathcal{U}_\sigma \hookrightarrow \bigcup_{\sigma \in K_{I^\varepsilon}} \mathcal{U}_\sigma$, since any point $x \in I$ is contained in some $\mathcal{U}_\alpha$, for some vertex $\alpha \in K_I \subseteq K_{I^\varepsilon}$. We define $\varphi_I : \pi_0 f^{-1}\left(\bigcup_{\sigma \in K_I} \mathcal{U}_\sigma\right) \to \pi_0 f^{-1}(I^\varepsilon)$ and $\psi_I : \pi_0 f^{-1}(I) \to \pi_0 f^{-1}\left(\bigcup_{\sigma \in K_{I^\varepsilon}} \mathcal{U}_\sigma\right)$ to be the maps induced by the inclusions. Then applying the

$$
\begin{tikzcd}
\mathcal{F}(I) \arrow[r, "\varphi_I"] \arrow[d, "\mathcal{F}{[I \subseteq J]}"] & \mathcal{C}(I^\varepsilon) \arrow[d, "\mathcal{C}{[I^\varepsilon \subseteq J^\varepsilon]}"] \\
\mathcal{F}(J) \arrow[r, "\varphi_J"] & \mathcal{C}(J^\varepsilon)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}(I) \arrow[r, "\psi_I"] \arrow[d, "\mathcal{C}{[I \subseteq J]}"] & \mathcal{F}(I^\varepsilon) \arrow[d, "\mathcal{F}{[I^\varepsilon \subseteq J^\varepsilon]}"] \\
\mathcal{C}(J) \arrow[r, "\psi_J"] & \mathcal{F}(J^\varepsilon)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{F}(I) \arrow[r, "\varphi_I"] \arrow[dr, "\mathcal{F}{[I \subseteq I^{2\varepsilon}]}"'] & \mathcal{C}(I^\varepsilon) \arrow[d, "\psi_{I^\varepsilon}"] \\
& \mathcal{F}(I^{2\varepsilon})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}(I) \arrow[r, "\psi_I"] \arrow[dr, "\mathcal{C}{[I \subseteq I^{2\varepsilon}]}"'] & \mathcal{F}(I^\varepsilon) \arrow[d, "\varphi_{I^\varepsilon}"] \\
& \mathcal{C}(I^{2\varepsilon})
\end{tikzcd}
$$

Figure 9: Communicative diagrams showing φ and ψ being natural transformations (top) and forming an ε interleaving.

functor $\pi_0 f^{-1}$ to a diagram of these inclusions gives us the commutative diagrams in Figure 9 top. Hence $\varphi = \{\varphi_I\}$ and $\psi = \{\psi_I\}$ are natural transformations as desired.

Similarly, the inclusions $\bigcup_{\sigma \in K_I} \mathcal{U}_\sigma \subseteq I^\varepsilon \subseteq \bigcup_{\sigma \in K_{I^{2\varepsilon}}} \mathcal{U}_\sigma$ and $I \subseteq \bigcup_{\sigma \in K_{I^\varepsilon}} \mathcal{U}_\sigma \subseteq I^{2\varepsilon}$ imply that the diagrams in Figure 9 bottom also commute, hence φ and ψ are an ε-interleaving. □

8 Geometric Representations

We now leverage the results of [6] to make geometric statements connecting the mapper and the Reeb space for $d = 1$. The main idea is to define a mapping that recovers the geometric representation of the mapper from its categorical representation, and to establish convergence between the mapper and the Reeb graph geometrically. Such a mapping relies on well behaved data, made precise by the notion of constructibility.

$$\begin{tikzcd}
\mathbb{R}\text{-}\mathbf{Top}^{\mathrm{c}} \arrow[r, "\mathcal{R}"] \arrow[d, "\mathcal{C}_K"] \arrow[dr, "\mathcal{C}"] & \mathbf{Reeb} \arrow[d, shift right, "\mathcal{C}"'] \\
\mathbf{Set}^{\mathbf{Cell}(K)^{\mathrm{op}}} \arrow[r, "\mathcal{P}_K"] & \mathbf{Csh}^{c} \arrow[u, shift right, "\mathcal{D}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\text{Well behaved data} \arrow[r] \arrow[d] \arrow[dr] & \text{Geometric Reeb graph} \arrow[d, shift right] \\
\text{Categorical mapper} \arrow[r] & \text{Categorical Reeb graph} \arrow[u, shift right]
\end{tikzcd}$$

Figure 10: The diagram for connecting geometric representations of the Reeb graph and the mapper.

Review of prior results. We will follow the notations of [6] which occasionally can be technical. The categories and functors we will discuss can be summed up in the roadmap of Figure 10. Notice its lower left triangle resembles that of Figure 4 with further restrictions. Recall the notation from Section 4; when $d = 1$, the category $\mathbb{R}$-**Top** is exactly the category $\mathbb{R}^d$-**Top**: an object of $\mathbb{R}$-**Top** is an $\mathbb{R}$-space (a pair of a topological space $\mathbb{X}$ and a continuous map $f : \mathbb{X} \to \mathbb{R}$), and an arrow in $\mathbb{R}$-**Top** is a function-preserving map. An $\mathbb{R}$-space $(\mathbb{X}, f)$ is *constructible* if it is given by the following data[5]:

- A finite set of critical values $S \subset \mathbb{R}$, $S = \{a_1 < \cdots < a_n\}$.
- A locally path connected space $\mathbb{V}_i$ for each $i = 1, \cdots, n$.
- A locally path connected space $\mathbb{E}_i$ for each $i = 1, \cdots, n-1$.
- A pair of attaching maps $\ell_i : \mathbb{E}_i \to \mathbb{V}_i$ and $r_i : \mathbb{E}_i \to \mathbb{V}_{i+1}$ for $i = 1, \cdots, n-1$.
- $\mathbb{X}$ is isomorphic to the disjoint union of $\mathbb{V}_i \times \{a_i\}$ and $\mathbb{E}_i \times [a_i, a_{i+1}]$ by making the identifications $(x, a_i) \sim (\ell_i(x), a_i)$ and $(x, a_{i+1}) \sim (\ell_{i+1}(x), a_{i+1})$ for all i and all $x \in \mathbb{E}_i$.

Since the geometric Reeb graph of a general $\mathbb{R}$-space may be badly behaved, we restrict to special classes of spaces [6], that is, we focus on well behaved subcategories. In particular, we define the full subcategory $\mathbb{R}$-**Top**$^{\mathrm{c}}$ of $\mathbb{R}$-**Top** where the objects are constructible $\mathbb{R}$-spaces. This collection includes, e.g., PL functions on triangulations of manifolds and Morse functions. Then we define the full subcategory **Reeb** of $\mathbb{R}$-**Top**$^{\mathrm{c}}$ where each $\mathbb{V}_i$ and $\mathbb{E}_i$ is a finite, discrete set. The **Reeb** category is exactly the category of Reeb graphs, viewed as a pair of a graph with a real valued function which is monotone on edges, with arrows given by function preserving maps. Subsequently, the construction of a (geometric) Reeb graph from well behaved data (a constructible $\mathbb{R}$-space) is captured by the functor $\mathcal{R} : \mathbb{R}\text{-}\mathbf{Top}^{\mathrm{c}} \to \mathbf{Reeb}$.

We can similarly restrict our objects of interest in $\mathbf{Set}^{\mathbf{Open}(\mathbb{R})}$ to be well behaved. A *cosheaf* is a functor $F : \mathbf{Open}(\mathbb{R}) \to \mathbf{Set}$ such that for any open cover $\mathcal{U}$ of a set U, the unique map $\mathrm{colim}_{U_\alpha \in \mathcal{U}} F(U_\alpha) \to F(U)$ is an isomorphism. We further restrict the cosheaves to constructible

[5] See Section 2.2 and Figure 5 of [6] for illustrations and technical details, the definition of a constructible $\mathbb{R}$-space is given here for completeness.

cosheaves; a cosheaf is *constructible* if there is a finite set $S \subset \mathbb{R}$ such that if $A, B \in \mathbf{Open}(\mathbb{R})$ with $A \subseteq B$ and $S \cap A = S \cap B$, then $F(A) \to F(B)$ is an isomorphism. In addition, we require that if $A \cap S = \emptyset$ then $F(A) = \emptyset$. The category of constructible cosheaves with natural transformations is denoted $\mathbf{Csh}^c$.

The work of [6] gives the equivalence of categories $\mathbf{Reeb} \equiv \mathbf{Csh}^c$. In Figure 10, when $d = 1$, the functor $\mathcal{C} : \mathbb{R}^d\text{-}\mathbf{Top} \to \mathbf{Set}^{\mathbf{Open}(\mathbb{R}^d)}$ (given in Figure 4) restricts to a functor $\mathcal{C} : \mathbb{R}\text{-}\mathbf{Top}^{\mathrm{c}} \to \mathbf{Csh}^c$. Its further restriction $\mathcal{C} : \mathbf{Reeb} \to \mathbf{Csh}^c$ is exactly the functor used in [6] to give the equivalence of categories. In addition, $\mathcal{C}$ has an "inverse" functor $\mathcal{D} : \mathbf{Csh}^c \to \mathbf{Reeb}$ which can turn a constructible cosheaf back into a geometric object through the display locale construction [25]. This construction also satisfies the equality $\mathcal{R} = \mathcal{D}\mathcal{C}$ due to the commutativity of the upper right triangle in Figure 10 (as proved in Section 3.5 of [6]). Therefore constructing the (geometric) Reeb graph from well behaved data is the same as creating its categorical representation, and then turning it back into a geometric object.

Our result. The above result implies that because we can turn *any* constructible cosheaf back into a geometric Reeb graph, we can now turn the mapper, defined previously as a categorical object, back into a geometric object. In this spirit, let $\mathcal{M}_K(\mathbb{X}, f) := \mathcal{D}\mathcal{P}_K\mathcal{C}_K(\mathbb{X}, f)$ be the geometric representation of the mapper object, referred to as the *geometric mapper* (following the rectangular diagram in Figure 10), and let $\mathcal{R}(\mathbb{X}, f)$ be the geometric Reeb graph. Then, the equivalence of categories gives us the following immediate corollary to Theorem 4.1.

Corollary 8.1. *Given a constructible* $\mathbb{R}$*-space* $(\mathbb{X}, f)$ *with* $f : \mathbb{X} \to \mathbb{R}$*, let* $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ *be a good cover of* $f(\mathbb{X}) \subseteq \mathbb{R}$*, and let* K *be the nerve of the cover. Then*

$$d_I(\mathcal{R}(\mathbb{X}, f), \mathcal{M}_K(\mathbb{X}, f)) \leq \mathrm{res}(\mathcal{U}).$$

In particular, because the interleaving distance is an extended metric when $d = 1$, this implies that a sequence of mappers for more refined covers $\mathcal{U}$ converges to the Reeb graph geometrically. Recent work has also investigated this convergence problem using the bottleneck distance for the extended persistence diagrams [4]; instead, we use the interleaving distance.

Algorithm for geometric mapper. Constructing the geometric representation of 1-dimensional mapper from its categorical representation follows a simple algorithm (as illustrated in Figure 5). For the purpose of exposition, we assume that the mapper is constructed with a connected, minimal cover; that is, a cover with no subcover. We further assume that the open sets (intervals) in $\mathcal{U} = \{U_i = (a_i, b_i)\}_{i=1}^n$ can be ordered and satisfy $a_1 < a_2 < b_1 < a_3 < b_2 < \cdots < a_{n-1} < b_{n-1} < b_n$. For ease of notation, we assume there are extra intervals $U_0 = (a_0, b_0)$ with $a_0 < a_1 < b_0 < b_1$ and $U_{n+1} = (a_{n+1}, b_{n+1})$ with $b_{n-1} < a_{n+1} < b_n < b_{n+1}$ and such that $f^{-1}(U_0) = f^{-1}(U_{n+1}) = \emptyset$. Let $M := M(\mathcal{U}, f)$ be the mapper with the added property that for any cover element U_i, we store the vertices corresponding to connected components of $f^{-1}(U_i)$ in the set $F(i)$. Furthermore, let $M[i]$ be the subgraph of M induced by the collection of vertices $F(i)$, and let $M[i, i+1]$ be the subgraph of M induced by the vertices $F(i) \cup F(i+1)$.

Note that for any small enough interval $I \subset (a_{i+1}, b_i)$, the colimit construction for I gives exactly the connected components over the union $U_i \cup U_{i+1}$, which is equivalently represented by the connected components of $M[i, i+1]$. For any small enough interval $I \subset (b_{i-1}, a_{i+1})$, the colimit construction for I gives the connected components over U_i, and thus is represented by the connected components of $M[i]$, which are just the vertices.

Thus, the geometric mapper, $\mathcal{M}_K(\mathbb{X}, f) = (\mathbb{X}', f')$, a graph $\mathbb{X}'$ equipped with a function f', can be constructed based on a combinatorial structure described below. For each interval $[b_{i-1}, a_{i+1}]$, add an edge uv with two new pink vertices for each vertex in $M[i]$ (see Figure 5 Algorithm). Set $f'(u) = b_{i-1}$ and set $f'(v) = a_{i+1}$. For each interval $[a_{i+1}, b_i]$, add an edge wx with two new yellow vertices for each connected component in $M[i, i+1]$. Set $f'(w) = a_{i+1}$ and $f'(x) = b_i$. Now, we have a combinatorial structure which consists of a collection of disjoint edges spread across each of the intervals defined by the cover, and each edge has a top vertex and a bottom vertex given by the function values. A pink and a yellow vertex are called equivalent if the vertex sets corresponding to them in $M[i]$ and $M[i, i+1]$ respectively have a nontrivial intersection. The graph $\mathbb{X}'$ resulting from identifying (i.e. gluing) equivalent vertices with the same function value of f' is the geometric mapper. Such an algorithm relies on subroutines of union-find, therefore it inherits the complexity of union-find that varies depending on naive or advanced implementations.

9 Discussion

In this paper, we provided formal proofs that the categorical representation of mapper converges to that of the Reeb space in terms of their interleaving distances (Theorem 4.1). In particular, we showed that their interleaving distance is bounded by the resolution of the cover used to construct the mapper. In addition, we gave the first definition of an interleaving distance on Reeb spaces (Definition 5.1 and Theorem 5.2). When $d = 1$, we turned these category theoretic results into concrete geometric ones. In particular, we showed that 1-dimensional mapper converges to the Reeb graph as spaces, not only in the interleaving distance (Corollary 8.1). We also provided an algorithm for constructing the newly defined, geometric mapper from its categorical representation.

The authors of [4] asked whether it is possible to describe the mapper as a particular constructible cosheaf. We addressed this question for $d = 1$ in Section 8 with our geometric results: we described the mapper as a constructible cosheaf when it is passed to the continuous version. We suspect that our geometric results hold in the case $d > 1$. That is, with the proper notion of constructibility for $\mathbb{R}^d$-spaces and cosheaves, we will have both an equivalence of categories, and a proof that the interleaving distance is an extended metric, not just a pseudometric; and therefore the mapper converges to the Reeb space on the space level. Additionally, the algorithm strategy for building the associated geometric mapper may be generalized by considering k-dimensional cover elements and their intersections. Our results are first steps towards providing a theoretical justification for the use of discrete objects (mapper and JCN) as approximations to the Reeb space with guarantees. Some future directions include creating categorical interpretation of multiscale mapper [7] and studying distance metrics between Jacobi sets in the categorical setting.

References

[1] Peter Bubenik, Vin de Silva, and Jonathan Scott. Metrics for generalized persistence modules. *Foundations of Computational Mathematics*, 15(6):1501–1531, 2015.

[2] Hamish Carr and David Duke. Joint contour nets. *IEEE Transactions on Visualization and Computer Graphics*, 20(8):1100–1113, 2014.

[3] Hamish Carr, Jack Snoeyink, and Michiel van de Panne. Flexible isosurfaces: Simplifying and displaying scalar topology using the contour tree. *Computational Geometry*, 43:42–58, 2010.

[4] Mathieu Carriére and Steve Oudot. Structure and stability of the 1-dimensional mapper. arxiv 1511.05823, 2015.

[5] Frédéric Chazal and Jian Sun. Gromov-Hausdorff approximation of filament structure using Reeb-type graph. *Proceedings 13th Annual Symposium on Computational Geometry*, pages 491–500, 2014.

[6] Vin de Silva, Elizabeth Munch, and Amit Patel. Categorification of Reeb graphs. arXiv:1501.04147, 2015.

[7] Tamal K. Dey, Facundo Mémoli, and Yusu Wang. Mutiscale mapper: A framework for topological summarization of data and maps. arXiv:1504.03763, 2015.

[8] Herbert Edelsbrunner and John Harer. Jacobi sets of multiple Morse functions. In F. Cucker, R. DeVore, P. Olver, and E. Süli, editors, *Foundations of Computational Mathematics, Minneapolis 2002*, pages 37–57. Cambridge University Press, 2002.

[9] Herbert Edelsbrunner, John Harer, and Amit K. Patel. Reeb spaces of piecewise linear mappings. *Proceedings 24th Annual Symposium on Computational Geometry*, pages 242–250, 2008.

[10] William Harvey, Yusu Wang, and Rephael Wenger. A randomized $O(m \log m)$ algorithm for computing Reeb graphs of arbitrary simplicial complexes. *ACM Symposium on Computational Geometry*, pages 267–276, 2010.

[11] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[12] Masaki Hilaga, Yoshihisa Shinagawa, Yoshihisa Shinagawa, and Tosiyasu L. Kunii. Topology matching for fully automatic similarity estimation of 3D shapes. *Proceedings 28th annual conference on Computer graphics and interactive techniques*, pages 203–212, 2001.

[13] Dimitry Kozlov. *Combinatorial Algebraic Topology*. Springer, 2008.

[14] P. Y. Lum, G. Singh, A. Lehman, T. Ishkanov, M. Vejdemo-Johansson, M. Alagappan, J. Carlsson, and G. Carlsson. Extracting insights from the shape of complex data using topology. *Scientific Reports*, 3, 2013.

[15] Saunders Mac Lane. *Categories for the Working Mathematician*. Springer-Verlag, New York, NY, 2 edition, 1978.

[16] Monica Nicolaua, Arnold J. Levineb, and Gunnar Carlsson. Topology based data analysis identifies a subgroup of breast cancers with a unique mutational profile and excellent survival. *Proceedings National Academy of Sciences of the United States of America*, 108(17):7265–7270, 2011.

[17] Salman Parsa. A deterministic $O(m \log m)$ time algorithm for the Reeb graph. *Proceedings 29th Annual Symposium on Computational Geometry*, pages 269–276, 2012.

[18] Amit Patel. *Reeb Spaces and the Robustness of Preimages*. PhD thesis, Duke University, 2010.

[19] Michael A. Penna. On the geometry of combinatorial manifolds. *Pacific Journal of Mathematics*, 77(2):499–522, 1978.

[20] G. Reeb. Sur les points singuliers d'une forme de pfaff completement intergrable ou d'une fonction numerique [on the singular points of a complete integral pfaff form or of a numerical function]. *Comptes Rendus de l'Académie des Sciences*, 222:847–849, 1946.

[21] Gurjeet Singh, Facundo Mémoli, and Gunnar Carlsson. Topological methods for the analysis of high dimensional data sets and 3D object recognition. In *Eurographics Symposium on Point-Based Graphics*, 2007.

[22] Roar Bakken Stovner. On the mapper algorithm: A study of a new topological method for data analysis. Master's thesis, Norwegian University of Science and Technology, 2012.

[23] Marc van Kreveld, René van Oostrum, Chandrajit Bajaj, Valerio Pascucci, and Dan Schikore. Contour trees and small seed sets for isosurface traversal. *Proceedings 13th Annual Symposium on Computational Geometry*, pages 212–220, 1997.

[24] Gunther H. Weber, Scott E. Dillard, Hamish Carr, Valerio Pascucci, and Bernd Hamann. Topology-controlled volume rendering. *IEEE Transactions on Visualization and Computer Graphics*, 13(2):330–341, 2007.

[25] Jon Woolf. The fundamental category of a stratified space. arXiv:0811.2580, 2013.