Discrete Stratified Morse Theory: A User's Guide

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Abstract

Inspired by the works of Forman on discrete Morse theory, which is a combinatorial adaptation to cell complexes of classical Morse theory on manifolds, we introduce a discrete analogue of the stratified Morse theory of Goresky and MacPherson [17]. We describe the basics of this theory and prove fundamental theorems relating the topology of a general simplicial complex with the critical simplices of a discrete stratified Morse function on the complex. We also provide an algorithm that constructs a discrete stratified Morse function out of an arbitrary function defined on a finite simplicial complex; this is different from simply constructing a discrete Morse function on such a complex. We borrow Forman's idea of a "user's guide," where we give simple examples to convey the utility of our theory.

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1 Introduction

It is difficult to overstate the utility of classical Morse theory in the study of manifolds. A Morse function $f : \mathbb{M} \to \mathbb{R}$ determines an enormous amount of information about the manifold $\mathbb{M}$: a handlebody decomposition, a realization of $\mathbb{M}$ as a CW-complex whose cells are determined by the critical points of f, a chain complex for computing the integral homology of $\mathbb{M}$, and much more.

With this as motivation, Forman developed discrete Morse theory on general cell complexes [13]. This is a combinatorial theory in which function values are assigned not to points in a space but rather to entire cells. Such functions are not arbitrary; the defining conditions require that function values generically increase with the dimensions of the cells in the complex. Given a cell complex with set of cells K, a discrete Morse function $f : K \to \mathbb{R}$ yields information about the cell complex similar to what happens in the smooth case.

While the category of manifolds is rather expansive, it is not sufficient to describe all situations of interest. Sometimes one is forced to deal with singularities, most notably in the study of algebraic varieties. One approach to this is to expand the class of functions one allows, and this led to the development of stratified Morse theory by Goresky and MacPherson [17]. The main objects of study in this theory are *Whitney stratified spaces*, which decompose into pieces that are smooth manifolds. Such spaces are triangulable.

The goal of this paper is to generalize stratified Morse theory to finite simplicial complexes, much as Forman did in the classical smooth case. Given that stratified spaces admit simplicial structures, and any simplicial complex admits interesting discrete Morse functions, this could be the end of the story. However, we present examples in this paper illustrating that the class of discrete stratified Morse functions defined here is much larger than that of discrete Morse functions. Moreover, there exist discrete stratified Morse functions that are nontrivial and interesting from a data analysis point of view. Our motivations are three-fold.

1. **Generating discrete stratified Morse functions from point cloud data.** Consider the following scenario. Suppose K is a simplicial complex and that f is a function defined on the 0-skeleton of K. Such functions arise naturally in data analysis where one has a sample of function values on a space. Algorithms exist to build discrete Morse functions on K extending f (see, for example, [20]). Unfortunately, these are often of potentially high computational complexity and might not behave as well as we would like. In our framework, we may take this input and generate a discrete stratified Morse function which will not be a global discrete Morse function in general, but which will allow us to obtain interesting information about the underlying complex.
2. **Filtration-preserving reductions of complexes in persistent homology and parallel computation.** As discrete Morse theory is useful for providing a filtration-preserving reduction of complexes in the computation of both persistent homology [7, 24, 28] and multi-parameter persistent homology [1], we believe that discrete stratified Morse theory could help to push the computational boundary even further. First, given any real-valued function $f : K \to \mathbb{R}$, defined on a simplicial complex, our algorithm generates a stratification of K such that the restriction of f to each stratum is a discrete Morse function. Applying *Morse pairing* to each stratum reduces K to a smaller complex of the same homotopy type. Second, if such a reduction can be performed in a filtration-preserving way with respect to each stratum, it would lead to a faster computation of persistent homology in the setting where the function is not required to be Morse. Finally, since discrete Morse theory can be applied independently to each stratum of K, we can design a parallel algorithm that computes persistent homology pairings by strata and uses the stratification (i.e. relations among strata) to combine the results.

3. **Applications in imaging and visualization.** Discrete Morse theory can be used to construct discrete Morse complexes in imaging (e.g. [6, 28]), as well as Morse-Smale complexes [10, 9] in visualization (e.g. [19, 18]). In addition, it plays an essential role in the visualization of scalar fields and vector fields (e.g. [27, 26]). Since discrete stratified Morse theory leads naturally to stratification-induced domain partitioning where discrete Morse theory becomes applicable, we envision our theory to have wide applicability for the analysis and visualization of large complex data.

Contributions. Throughout the paper, we hope to convey via simple examples the usability of our theory. It is important to note that our discrete stratified Morse theory is *not* a simple reinterpretation of discrete Morse theory; it considers a larger class of functions defined on any finite simplicial complex and has potentially many implications for data analysis. Our contributions are:

1. We describe the basics of a discrete stratified Morse theory and prove fundamental theorems that relate the topology of a finite simplicial complex with the critical simplices of a discrete stratified Morse function defined on the complex.
2. We provide an algorithm that constructs a discrete stratified Morse function on any finite simplicial complex equipped with a real-valued function.

A Simple Example. We begin with an example from [15], where we demonstrate how a discrete stratified Morse function can be constructed from a function that is not a discrete Morse function. As illustrated in Figure 1, the function on the left is a discrete Morse function where the green arrows can be viewed as its discrete gradient vector field; function f in the middle is not a discrete Morse function, as the vertex $f^{-1}(5)$ and the edge $f^{-1}(0)$ both violate the defining conditions a discrete Morse function. However, we can equip f with a stratification s by treating such violators as their own independent strata, therefore converting it into a discrete stratified Morse function.

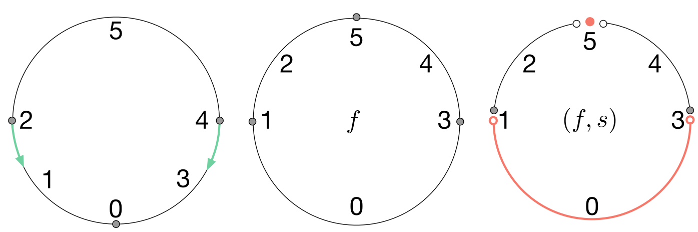

Figure 1: The function on the left is a discrete Morse function. The function f in the middle is not a discrete Morse function; however, it can be converted into a discrete stratified Morse function when it is equipped with an appropriate stratification s.

2 Preliminaries on Discrete Morse Theory

We review the most relevant definitions and results on discrete Morse theory and refer the reader to Appendix A for a review of classical Morse theory. Discrete Morse theory is a combinatorial version of Morse theory [13, 15]. It can be defined for any CW complex but in this paper we will restrict our attention to simplicial complexes.

Discrete Morse functions. Let K be any finite simplicial complex, where K need not be a triangulated manifold nor have any other special property [14]. When we write K we mean the set of simplices of K; by $|K|$ we mean the underlying topological space. Let $\alpha^{(p)} \in K$ denote a simplex of dimension p. Let $\alpha < \beta$ denote that simplex α is a face of simplex β. If $f : K \to \mathbb{R}$ is a function define $U(\alpha) = \{\beta^{(p+1)} > \alpha \mid f(\beta) \leq f(\alpha)\}$ and $L(\alpha) = \{\gamma^{(p-1)} < \alpha \mid f(\gamma) \geq f(\alpha)\}$. In other words, $U(\alpha)$ contains the higher-dimensional cofaces of α with lower (or equal) function values, while $L(\alpha)$ contains the lower-dimensional faces of α with higher (or equal) function values. Let $|U(\alpha)|$ and $|L(\alpha)|$ be their sizes.

Definition 2.1. *A function $f : K \to \mathbb{R}$ is a* discrete Morse function *if for every $\alpha^{(p)} \in K$, (i) $|U(\alpha)| \leq 1$ and (ii) $|L(\alpha)| \leq 1$.*

Forman showed that conditions (i) and (ii) are exclusive – if one of the sets $U(\alpha)$ or $L(\alpha)$ is nonempty then the other one must be empty ([13], Lemma 2.5). Therefore each simplex $\alpha \in K$ can be paired with at most one exception simplex: either a face γ with larger function value, or a coface β with smaller function value. Formally, this means that if K is a simplicial complex with a discrete Morse function f, then for any simplex α, either (i) $U(\alpha) = 0$ or (ii) $L(\alpha) = 0$ ([15], Lemma 2.4).

Definition 2.2. *A simplex $\alpha^{(p)}$ is* critical *if (i) $|U(\alpha)| = 0$ and (ii) $|L(\alpha)| = 0$. A* critical value *of f is its value at a critical simplex.*

Definition 2.3. *A simplex $\alpha^{(p)}$ is* noncritical *if either of the following conditions holds: (i) $|U(\alpha)| = 1$; (ii) $|L(\alpha)| = 1$; as noted above these conditions can not both be true ([13], Lemma 2.5).*

Given $c \in \mathbb{R}$, we have the *level subcomplex* $K_c = \cup_{f(\alpha) \leq c} \cup_{\beta \leq \alpha} \beta$. That is, K_c contains all simplicies α of K such that $f(\alpha) \leq c$ along with all of their faces.

Results. We have the following two combinatorial versions of the main results of classical Morse theory (see Appendix A).

Theorem 2.1 (DMT Part A, [14])**.** *Suppose the interval $(a, b]$ contains no critical value of f. Then K_b is homotopy equivalent to K_a. In fact, K_b simplicially collapses onto K_a.*

A key component in the proof of Theorem 2.1 is the following fact [13]: for a simplicial complex equipped with an arbitrary discrete Morse function, when passing from one level subcomplex to the next, the noncritical simplices are added in pairs, each of which consists of a simplex and its free face.

Theorem 2.2 (DMT Part B, [14])**.** *Suppose $\sigma^{(p)}$ is a critical simplex with $f(\sigma) \in (a, b]$, and there are no other critical simplices with values in $(a, b]$. Then $K(b)$ is homotopy equivalent to attaching a p-cell $e^{(p)}$ along its entire boundary; that is, $K_b = K_a \cup_{\dot{e}^{(p)}} e^{(p)}$.*

The associated gradient vector field. Given a discrete Morse function $f : K \to \mathbb{R}$ we may associate a discrete gradient vector field as follows. Since any noncritical simplex $\alpha^{(p)}$ has at most one of the sets $U(\alpha)$ and $L(\alpha)$ nonempty, there is a unique face $\nu^{(p-1)} < \alpha$ with $f(\nu) \geq f(\alpha)$ or a unique coface $\beta^{(p+1)} > \alpha$ with $f(\beta) \leq f(\alpha)$. Denote by V the collection of all such pairs $\{\sigma < \tau\}$. Then every simplex in K is in at most one pair in V and the simplices not in any pair are precisely the critical cells of the function f. We call V the *gradient vector field associated to f*. We visualize V by drawing an arrow from α to β for every pair $\{\alpha < \beta\} \in V$. Theorems 2.1 and 2.2 may then be visualized in terms of V by collapsing the pairs in V using the arrows. Thus a discrete gradient (or equivalently a discrete Morse function) provides a collapsing order for the complex K, simplifying it to a complex L with potentially fewer cells but having the same homotopy type.

The collection V has the following property. By a *V-path*, we mean a sequence

$$\alpha_0^{(p)} < \beta_0^{(p+1)} > \alpha_1^{(p)} < \beta_1^{(p+1)} > \cdots < \beta_r^{(p+1)} > \alpha_{r+1}^{(p)}$$

where each $\{\alpha_i < \beta_i\}$ is a pair in V. Such a path is *nontrivial* if $r > 0$ and *closed* if $\alpha_{r+1} = \alpha_0$. Forman proved the following result.

Theorem 2.3 ([13]). *If V is a gradient vector field associated to a discrete Morse function f on K, then V has no nontrivial closed V-paths.*

In fact, if one defines a discrete vector field W to be a collection of pairs of simplices of K such that each simplex is in at most one pair in W, then one can show that if W has no nontrivial closed W-paths there is a discrete Morse function f on K whose associated gradient is precisely W.

3 A Discrete Stratified Morse Theory

Our goal is to describe a combinatorial version of stratified Morse theory. To do so, we need to: (a) define a discrete stratified Morse function; and (b) prove the combinatorial versions of the relevant fundamental results. Our results are very general as they apply to any finite simplicial complex K equipped with a real-valued function $f : K \to \mathbb{R}$. However, our work is motivated by relevant concepts from (classical) stratified Morse theory [17], whose details are found in Appendix B.

3.1 Background

Open simplices. To state our main results, we need to consider open simplices (as opposed to the closed simplices of Section 2). Let $\{a_0, a_1, \cdots, a_k\}$ be a geometrically independent set in $\mathbb{R}^N$, a *closed k-simplex* $[\sigma]$ is the set of all points x of $\mathbb{R}^N$ such that $x = \sum_{i=0}^{k} t_i a_i$, where $\sum_{i=0}^{k} t_i = 1$ and $t_i \geq 0$ for all i [25]. An *open simplex* (σ) is the interior of the closed simplex $[\sigma]$.

A *simplicial complex* K is a finite set of open simplices such that: (a) If $(\sigma) \in K$ then all open faces of $[\sigma]$ are in K; (b) If $(\sigma_1), (\sigma_2) \in K$ and $(\sigma_1) \cap (\sigma_2) \neq \emptyset$, then $(\sigma_1) = (\sigma_2)$. For the remainder of this paper, we always work with a finite open simplicial complex K.

Unless otherwise specified, we work with open simplices σ and define its boundary $\dot{\sigma}$ to be the boundary of its closure. We will often need to talk about a "half-open" or "half-closed" simplex, consisting of the open simplex σ along with some of the open faces in its boundary $\dot{\sigma}$. We denote such objects ambiguously as $[\sigma)$ or $(\sigma]$, specifying particular pieces of the boundary as necessary.
Remarks. We include a few facts concerning open simplices [29]:

- A vertex is a 0-dimensional closed face; it is also an open face.
- An open simplex (σ) is an open set in the closed simplex $[\sigma]$; its closure is $[\sigma]$.
- The closed simplex $[\sigma]$ is the union of its open faces.
- Distinct open faces of a simplex are disjoint.
- The open simplex (σ) is the interior of the closed simplex $[\sigma]$; that is, it is the close simplex minus its proper open faces.
- If $[\sigma]$ is a closed simplex, the collection of its open faces is a simplicial complex.

Stratified simplicial complexes. A simplicial complex K equipped with a stratification is referred to as a *stratified simplicial complex.* [1] A *stratification* of a simplicial complex K is a finite filtration

$$\emptyset = K^0 \subset K^1 \subset \cdots \subset K^m = K,$$

[1] Our notion of a stratified simplicial complex can be considered as a relaxed version of the notion in [3].

such that for each i, $K^i - K^{i-1}$ is a locally closed subset of K. [2] We say a subset $L \subset K$ is locally closed if it is the intersection of an open and a closed set in K. We will refer to a connected component of the space $K^i - K^{i-1}$ as a *stratum*; and the collection of all strata is denoted by $\mathcal{S} = \{S_j\}$. We may consider a stratification as an assignment from K to the set $\mathcal{S}$, denoted $s : K \to \mathcal{S}$.

In our setting, each S_j is the union of finitely many open simplices (that may not form a subcomplex of K); and each open simplex σ in K is assigned to a particular stratum $s(\sigma)$ via the mapping s.

Adjunction spaces. Let X and Y be topological spaces with $A \subseteq X$. Let $f : A \to Y$ be a continuous map called the *attaching* map. The *adjunction space* $X \cup_f Y$ is obtained by taking the disjoint union of X and Y by identifying x with $f(x)$ for all x in A. That is, Y is *glued* onto X via a quotient map, $X \cup_f Y = (X \amalg Y)/\{f(A) \sim A\}$.

Gluing theorem for homotopy equivalences. In homotopy theory, a continuous mapping $i : A \to X$ is a *cofibration* if it satisfies the homotopy extension property with respect to all spaces Y. In other words, i is a cofibration if and only if there is a retraction from $X \times I$ to $(A \times I) \cup (X \times \{0\})$. In particular, this holds if X is a cell complex and A is a subcomplex of X; it follows that the inclusion $i : A \to X$ a closed cofibration.

Theorem 3.1 (Gluing theorem for adjunction spaces ([5], Theorem 7.5.7))**.** *Suppose we have the following commutative diagram of topological spaces and continuous maps:*

$$\begin{tikzcd}
Y \arrow[d, "\varphi_Y"] & A \arrow[l, "f"'] \arrow[r, hook, "i"] \arrow[d, "\varphi_A"] & X \arrow[d, "\varphi_X"] \\
Y' & A' \arrow[l, "f'"'] \arrow[r, hook, "i'"] & X'
\end{tikzcd}$$

where φ_A, φ_X and φ_Y are homotopy equivalences and inclusions i and i' are closed cofibrations, then the map $\phi : X \cup_f Y \to X' \cup_f Y'$ induced by ϕ_A, ϕ_X ad ϕ_Y is a homotopy equivalence.

In our setting, since we are not in general dealing with closed subcomplexes of simplicial complexes, this theorem does not apply directly. However, the condition that the maps i, i' be closed cofibrations is not necessary (see [30], 5.3.2, 5.3.3), and in our setting it will be the case that our various pairs (X, A) will satisfy the property that $X \times \{0\} \cup A \times [0, 1]$ is a retract of $X \times [0, 1]$.

Stratum-preserving homotopies. If X and Y are two filtered spaces, we call a map $f : X \to Y$ *stratum-preserving* if the image of each component of a stratum of X lies in a stratum of Y [16]. A map $f : X \to Y$ is a *stratum-preserving homotopy equivalence* if there exists a stratum-preserving map $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the identity [16].

3.2 A Primer

Discrete stratified Morse function. Let K be a simplicial complex equipped with a stratification s and a discrete stratified Morse function $f : K \to \mathbb{R}$. We define

$$U_s(\alpha) = \{\beta^{(p+1)} > \alpha \mid s(\beta) = s(\alpha) \text{ and } f(\beta) \leq f(\alpha)\},$$
$$L_s(\alpha) = \{\gamma^{(p-1)} < \alpha \mid s(\gamma) = s(\alpha) \text{ and } f(\gamma) \geq f(\alpha)\}.$$

[2]Technically we should speak of the geometric realization $|K^i - K^{i-1}|$ being a locally closed subspace of $|K|$; we often confuse these notations as it should be clear from context.

Definition 3.1. *Given a simplicial complex K equipped with a stratification $s : K \to \mathcal{S}$, a function $f : K \to \mathbb{R}$ (equipped with s) is a* discrete stratified Morse function *if for every $\alpha^{(p)} \in K$, (i) $|U_s(\alpha)| \leq 1$ and (ii) $|L_s(\alpha)| \leq 1$.*

In other words, a discrete stratified Morse function is a pair (f, s) where $f : K \to \mathbb{R}$ is a discrete Morse function when restricted to each stratum $S_j \in \mathcal{S}$. We omit the symbol s whenever it is clear from the context.

Definition 3.2. *A simplex $\alpha^{(p)}$ is* critical *if (i) $|U_s(\alpha)| = 0$ and (ii) $|L_s(\alpha)| = 0$. A* critical value *of f is its value at a critical simplex.*

Definition 3.3. *A simplex $\alpha^{(p)}$ is* noncritical *if exactly one of the following two conditions holds: (i) $|U_s(\alpha)| = 1$ and $|L_s(\alpha)| = 0$; or (ii) $|L_s(\alpha)| = 1$ and $|U_s(\alpha)| = 0$.*

The two conditions in Definition 3.3 mean that, within the same strata as $s(\alpha)$: (i) $\exists \beta^{(p+1)} > \alpha$ with $f(\beta) \leq f(\alpha)$ or (ii) $\exists \gamma^{(p-1)} < \alpha$ with $f(\gamma) \geq f(\alpha)$; conditions (i) and (ii) cannot both be true.

Note that a classical discrete Morse function $f : K \to \mathbb{R}$ is a discrete stratified Morse function. We will present several examples in Section 4 illustrating that the class of discrete stratified Morse functions is much larger.

Violators. The following definition is central to our algorithm in constructing a discrete stratified Morse function from any real-valued function defined on a simplicial complex.

Definition 3.4. *Given a simplicial complex K equipped with a real-valued function, $f : K \to \mathbb{R}$. A simplex $\alpha^{(p)}$ is a* violator *of the conditions associated with a discrete Morse function if one of these conditions hold: (i) $|U(\alpha)| \geq 2$; (ii) $|L(\alpha)| \geq 2$; (iii) $|U(\alpha)| = 1$ and $|L(\alpha)| = 1$. These are referred to as type I, II and III violators; the sets containing such violators are not necessarily mutually exclusive.*

3.3 Main Results

To describe our main results, we work with the *sublevel set* of an open simplicial complex K, where $K_c = \cup_{f(\alpha) \leq c} \alpha$, for any $c \in \mathbb{R}$. That is, K_c contains all open simplices α of K such that $f(\alpha) \leq c$. Note that K_c is not necessarily a subcomplex of K. Suppose that K is a simplicial complex equipped with a stratification s and a discrete stratified Morse function $f : K \to \mathbb{R}$. We now state our two main results which will be proved in Section 5.

Theorem 3.2 (DSMT Part A)**.** *Suppose the interval $(a, b]$ contains no critical value of f. Then K_b is strata-preserving homotopy equivalent to K_a.*

Theorem 3.3 (DSMT Part B)**.** *Suppose $\sigma^{(p)}$ is a critical simplex with $f(\sigma) \in (a, b]$, and there are no other critical simplices with values in $(a, b]$. Then K_b is homotopy equivalent to attaching a p-cell $e^{(p)}$ along its boundary in K_a; that is, $K_b = K_a \cup_{\dot{e}^{(p)}|_{K_a}} e^{(p)}$.*

Remarks. K_c as defined above falls under a nonclassical notion of a "simplicial complex" as defined in [21]: K is a "simplicial complex" if it is the union of finitely many open simplices σ_1, σ_2, ...σ_t in some $\mathbb{R}^N$ such that the intersection of the closure of any two simplices σ_i and σ_j is either a common face of them or empty. Thus the closure $[K] = \{[\sigma_i]\}_{i=1}^t$ of K is a classical finite simplicial complex; and K is obtained from $[K]$ by omitting some open faces.

3.4 Algorithm

We give an algorithm to construct a discrete stratified Morse function from any real-valued function on a simplicial complex.

Given a simplicial complex K equipped with a real-valued function, $f : K \to \mathbb{R}$, define a collection of strata $\mathcal{S}$ as follows. Each violator $\sigma^{(p)}$ is an element of the collection $\mathcal{S}$. Let $\mathcal{V}$ denote the set of violators and denote by S_j the connected components of $K \setminus \mathcal{V}$. Then we set $\mathcal{S} = \mathcal{V} \cup \{S_j\}$. Denote by $s : K \to \mathcal{S}$ the assignment of the simplices of K to their corresponding strata.

We realize this as a stratification of K by taking $K^1 = \bigcup_j S_j$ and then adjoining the elements of $\mathcal{V}$ one simplex at a time by increasing function values (we may assume that f is injective). This filtration is unimportant for our purposes; rather, we shall focus on the strata themselves. We have the following theorem whose proof is delayed to Section 5.

Theorem 3.4. *The function f equipped with the stratification s produced by the algorithm above is a discrete stratified Morse function.*

4 Discrete Stratified Morse Theory by Example

We apply the algorithm described in 3.4 to a collection of examples to demonstrate the utility of our theory. For each example, given an $f : K \to \mathbb{R}$ that is not necessarily a discrete Morse function, we equip f with a particular stratification s, thereby converting it to a discrete stratified Morse function (f, s). These examples help to illustrate that the class of discrete stratified Morse functions is much larger than that of discrete Morse functions.

Example 1: upside-down pentagon. As illustrated in Figure 2, $f : K \to \mathbb{R}$ defined on the boundary of an upside-down pentagon is not a discrete Morse function, as it contains a set of violators: $\mathcal{V} = \{f^{-1}(10), f^{-1}(1), f^{-1}(2)\}$, since $|U(f^{-1}(10))| = 2$ and $|L(f^{-1}(1))| = |L(f^{-1}(2))| = 2$, respectively.

We construct a stratification s by considering elements in $\mathcal{V}$ and connected components in $K \setminus \mathcal{V}$ as their own strata. It is then easy to verify that all simplicies in s now satisfy the defining condition of a discrete stratified Morse function. The resulting discrete stratified Morse function (f, s) is a discrete Morse function when restricted to each stratum.

Recall that a simplex is critical for (f, s) if it is neither the source nor the target of a discrete gradient vector. The critical values of (f, s) are therefore $1, 2, 3, 4, 9$ and 10. The vertex $f^{-1}(3)$ is noncritical for f since $|U(f^{-1}(3))| = 1$ and $|L(f^{-1}(3))| = 0$; however it is critical for (f, s) since $|U_s(f^{-1}(3))| = L_s(f^{-1}(3))| = 0$.

One of the primary uses of classical discrete Morse theory is *simplification*. In this example, we can collapse a portion of each stratum following the discrete gradient field (illustrated by green arrows, see Section 2). Removing the Morse pairs $(f^{-1}(7), f^{-1}(5))$ and $(f^{-1}(8), f^{-1}(6))$ simplifies the original complex as much as possible without changing its homotopy type, see Figure 2 (right).

Example 2: pentagon. For our second pentagon example, f can be made into a discrete stratified Morse function (f, s) by making $f^{-1}(0)$ (a type II violator) and $f^{-1}(9)$ (a type I violator) their own strata (Figure 3). The critical values of (f, s) are $0, 1, 3, 7, 8$ and 9. The simplicial complex can be reduced to one with fewer cells by canceling the Morse pairs, as shown in Figure 3 (right).

Example 3: split octagon. The split octagon example (Figure 4) begins with a function f defined on a triangulation of a stratified space that consists of two 0-dimensional and three 1-dimensional strata. The violators are $f^{-1}(0)$, $f^{-1}(10)$, $f^{-1}(24)$, $f^{-1}(30)$ and $f^{-1}(31)$. The result of canceling Morse pairs yields the simpler complex shown on the right.

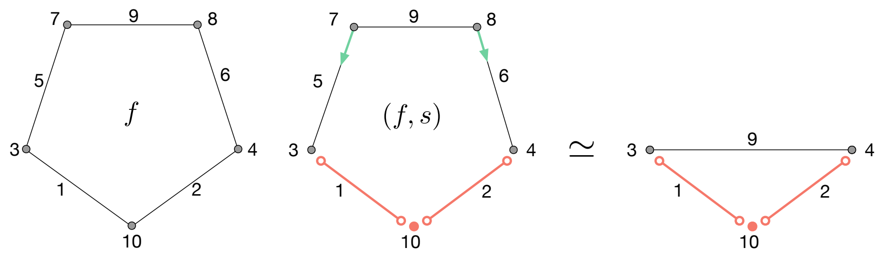

Figure 2: Example 1: upside-down pentagon. Left: f is not a discrete Morse function. Middle: (f, s) is a discrete stratified Morse function where violators are in red. Right: the simplified simplicial complex following the discrete gradient vector field (green arrows).

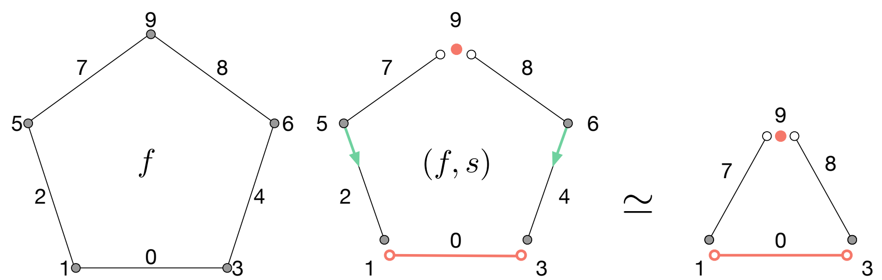

Figure 3: Example 2: pentagon. Middle: there are four strata pieces associated with the discrete stratified Morse function (f, s).

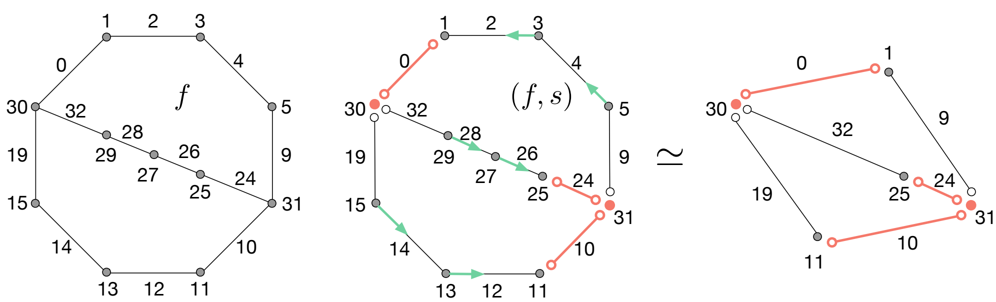

Figure 4: Example 3: split octagon. f is defined on the triangulation of a stratified space.

Example 4: tetrahedron. In Figure 5, the values of the function f defined on the simplices of a tetrahedron are specified for each dimension; that is, we have vertices $f^{-1}(1)$, $f^{-1}(3)$, $f^{-1}(10)$ and $f^{-1}(14)$; edges $f^{-1}(2)$, $f^{-1}(4)$, $f^{-1}(7)$, $f^{-1}(8)$, $f^{-1}(11)$ and $f^{-1}(12)$; triangles $f^{-1}(6)$, $f^{-1}(9)$, $f^{-1}(5)$ and $f^{-1}(13)$. For each simplex $\alpha \in K$, we list the elements of its corresponding $U(\alpha)$ and $L(\alpha)$ in Table 1. We also classify each simplex in terms of its criticality in the setting of classical discrete Morse theory. According to Table 1, violators with function values of $10, 14$ (type I), 6 (type II), $7, 8, 11, 12$ (type III) form their individual strata in (f, s). Given such a stratification s, every simplex is critical except for $f^{-1}(2)$ and $f^{-1}(3)$. Observing that the space is homeomorphic to S^2 and collapsing the single Morse pair $(f^{-1}(2), f^{-1}(3))$ yields a space of the same homotopy type (in fact still homeomorphic to S^2, but not a triangulation of S^2).

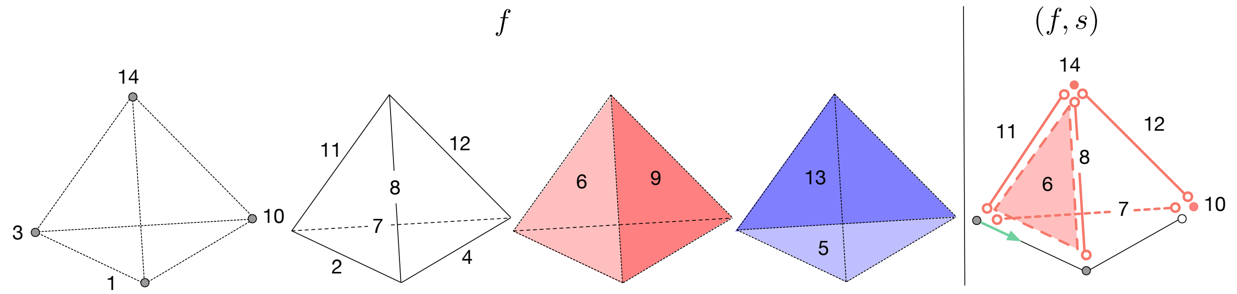

Figure 5: Example 4: tetrahedron. Left: f is defined on the simplices of increasing dimensions. Right: violators are highlighted in red; not all simplicies are shown for (f, s).

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$U(\alpha)$	$\emptyset$	$\emptyset$	$\{2\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{15\}$	$\{6\}$	$\emptyset$	$\{4,7\}$	$\{6\}$	$\{9\}$	$\emptyset$	$\{8,11,12\}$
$L(\alpha)$	$\emptyset$	$\{3\}$	$\emptyset$	$\{10\}$	$\{7\}$	$\{8,11\}$	$\{10\}$	$\{14\}$	$\{12\}$	$\emptyset$	$\{14\}$	$\{14\}$	$\emptyset$	$\emptyset$
Type	C	R	R	R	R	II	III	III	R	I	III	III	C	I

Table 1: Example 4: tetrahedron. For simplicity, a simplex α is represented by its function value $f(\alpha)$ (as f is 1-to-1). In terms of criticality for each simplex: C means critical; R means regular; I, II and III correspond to type I, II and III violators.

Example 5: split solid square. As illustrated in Figure 6, the function f defined on a split solid square is not a discrete Morse function; there are three type I violators $f^{-1}(9)$, $f^{-1}(10)$, and $f^{-1}(11)$. Making these violators their own strata helps to convert f into a discrete stratified Morse function (f, s). In this example, all simplices are considered critical for (f, s). For instance, consider the open 2-simplex $f^{-1}(4)$, we have $L(f^{-1}(4)) = \{f^{-1}(11)\}$ and $U(f^{-1}(4)) = \emptyset$; with the stratification s in Figure 6 (right), $L_s(f^{-1}(4)) = \emptyset$ and so 4 is not a critical value for f but it is a critical value for (f, s). Since every simplex is critical for (f, s), there is no simplification to be done.

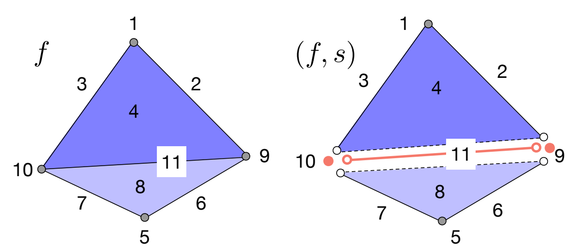

Figure 6: Example 5: split solid square. Every simplex is critical for (f, s).

5 Proofs of Main Results

We now provide the proofs of our main results, Theorem 3.2, Theorem 3.3, and Theorem 3.4. To better illustrate our ideas, we construct "filtrations" by sublevel sets based upon the upside-down pentagon example (Figure 7) and the split solid square example (Figure 8).

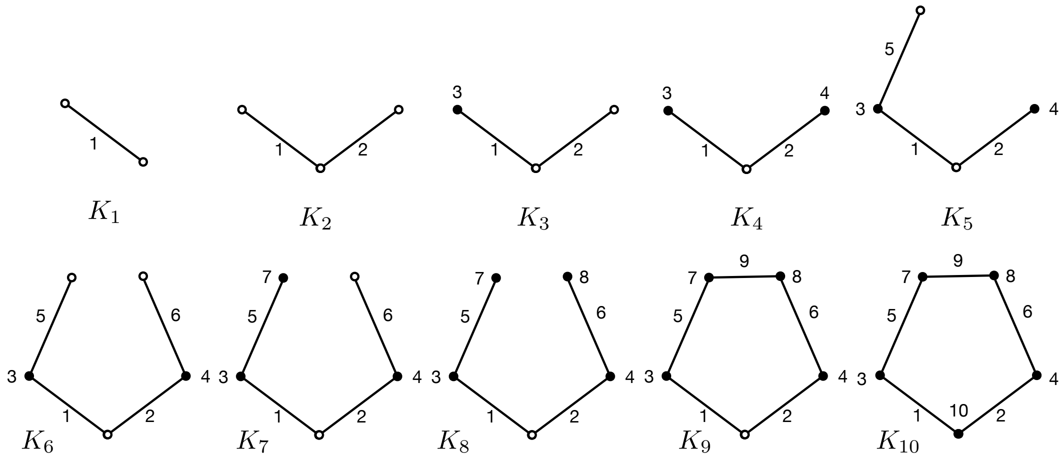

Figure 7: Example 1: upside-down pentagon. We show K_c as c increases from 1 to 10.

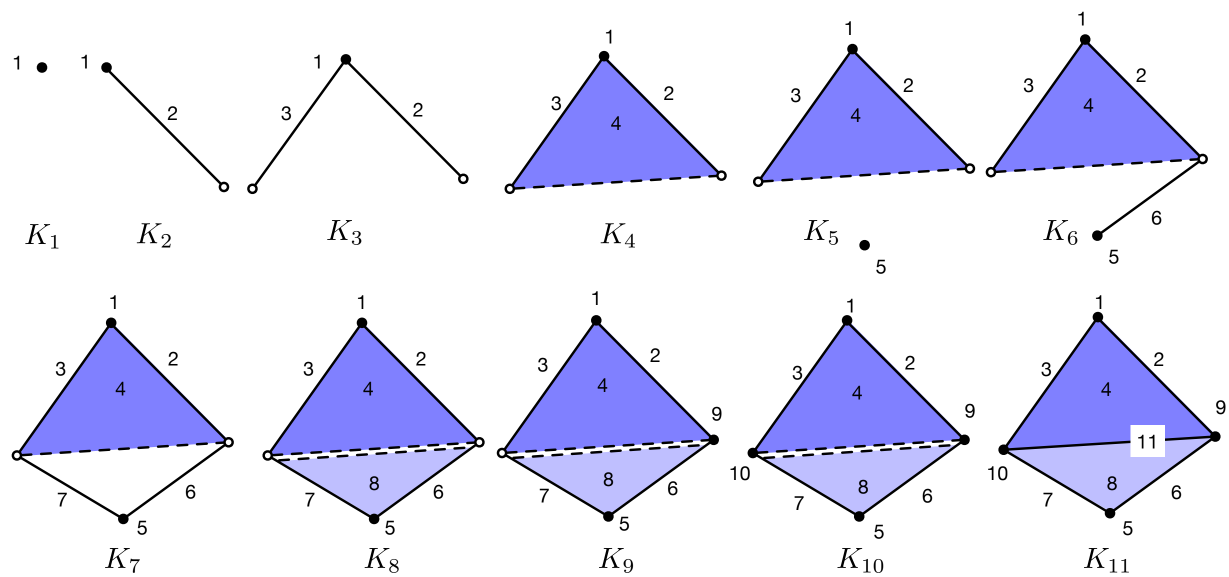

Figure 8: Example 4: split solid square. We show K_c as c increases from 1 to 11.

5.1 Proof of Theorem 3.2

Proof. Based on the principle of simulation of simplicity [11], we may perturb f slightly without changing which cells are critical in K_a or K_b so that $f : K \to \mathbb{R}$ is 1-to-1. By partitioning $(a, b]$ into smaller intervals if necessary, we may assume there is a single noncritical cell σ with $f(\sigma) \in (a, b]$. Since σ is noncritical, either (a) $|L_s(\sigma)| = 1$ and $|U_s(\sigma)| = 0$ or (b) $|U_s(\sigma)| = 1$ and $|L_s(\sigma)| = 0$.

Since case (a) requires that $p \geq 1$, we assume for now that is the case. There exists a single $\nu^{(p-1)} < \sigma$ with $f(\nu) > f(\sigma)$; such a $\nu \notin K_b$. Meanwhile, any other $(p-1)$-face $\tilde{\nu}^{(p-1)} < \sigma$ satisfies $f(\tilde{\nu}) < f(\sigma)$, implying $\tilde{\nu} \in K_a$. The set $\{\tilde{\nu}\}$ of such $\tilde{\nu}$ corresponds to the portion of the boundary of σ that lies in K_a, that is, $K_b = K_a \cup_{\{\tilde{\nu}\}} [\sigma)$, where $\tilde{\nu}$ are open faces of σ. Note that we use the half-closed simplex $[\sigma)$ to emphasize its boundary $\tilde{\nu}$ in K_a. We now apply Theorem 3.1 by setting $A = A' = Y = \{\tilde{\nu}\}$, $X = X' = K_a$, $Y' = [\sigma)$; i, i', φ_Y and f' the corresponding inclusions, and all other maps the identity. Since the diagram commutes and the pairs $(K_a, \{\tilde{\nu}\})$ and $(\sigma, \{\tilde{\nu}\})$ both satisfy the homotopy extension property, the maps $i = i'$ and φ_Y are cofibrations. It follows that $K_a = K_a \cup_{\{\tilde{\nu}\}} \{\tilde{\nu}\}$ and $K_b = K_a \cup_{\{\tilde{\nu}\}} [\sigma)$ are homotopy equivalent.

For case (b), σ has a single coface $\tau^{(p+1)} > \sigma$ with $f(\tau) < f(\sigma)$. Thus $\tau \in K_a$ and any other coface $\tilde{\tau} > \sigma$ must have a larger function value; that is, $\tilde{\tau} \notin K_b$. Denote by K'_a the set $K_a \setminus \tau$. Let $\{\omega\}$ denote the boundary of τ in K'_a. Then $K_a = K'_a \cup_{\{\omega\}} [\tau)$, and $K_b = K'_a \cup_{\{\omega\}} ([\tau) \cup \sigma)$ and σ is a free face of τ. We apply Theorem 3.1 by setting $A = A' = \{\omega\}$, $X = X' = K'_a$, $Y = [\tau)$, $Y' = [\tau) \cup \sigma$. The maps i, i', φ_Y and f' are inclusions and also cofibrations, while all other maps are the identity. Attaching σ to τ is clearly a homotopy equivalence and so we see that K_a and K_b are homotopy equivalent in this case as well.

Finally, it is clear that the above homotopy equivalence leaves the strata alone; in particular, the retract associated with the inclusion/cofibration $\varphi_Y : \{\tilde{\nu}\} \to [\sigma)$ for case (a), and $\varphi_Y : [\tau) \to [\tau) \cup \sigma$ for case (b) are both completely contained within their own strata. Therefore K_a and K_b are strata-preserving homotopy equivalent. $\square$

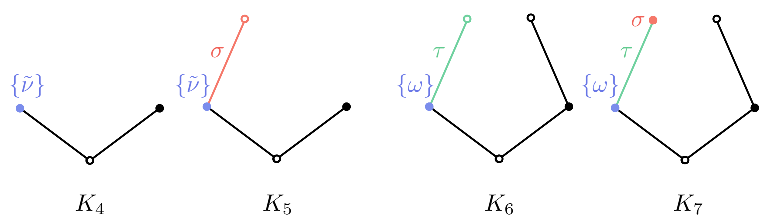

Figure 9: Using upside-down pentagon to illustrate the proof of Theorem 3.2.

Examples of attaching regular simplices. Let's examine how this works in our upside-down pentagon example (Figure 7). Applying Theorem 3.2 going from K_4 to K_5, we attach the open simplex $f^{-1}(5)$ to its boundary in K_4, which consists of the single vertex $f^{-1}(4)$. The simplex $f^{-1}(5)$ is a regular simplex and so $K_4 \simeq K_5$. This is precisely case (a) in the proof of Theorem 3.2, see also Figure 9 (left). Similarly, $K_6 \simeq K_7$, as $f^{-1}(6)$ is a regular simplex in its stratum, and this corresponds to case (b) in the proof of Theorem 3.2, see Figure 9 (right).

5.2 Proof of Theorem 3.3

Proof. Again, we may assume that f is 1-to-1. We may further assume that σ is the only simplex with a value between $(a, b]$ and prove that K_b is homotopy equivalent to $K_a \cup_{\dot{\sigma}|_{K_a}} \sigma$. Based on the definition of K_c, since $f(\sigma) > a$, we know that $\sigma \cap K_a = \emptyset$. We now consider several cases. Let σ and (σ) denote open simplices and $[\sigma]$ denote the closure.

Case (a), suppose σ is not on the boundary of a stratum. Since σ is critical in its own stratum $s(\sigma)$, then for every $\nu^{(p-1)} < \sigma$ in the same stratum as σ (i.e. $s(\nu) = s(\sigma)$), we have $f(\nu) < f(\sigma)$, so that $f(\nu) < a$, which implies $\nu \in K_a$. In addition any such ν is not on the boundary of a stratum (otherwise σ would be part of the boundary). This means that all $(p-1)$-dimensional open faces of σ lying in $s(\sigma)$ are in K_a; this is precisely the boundary of σ in K_a, denoted $\dot{\sigma}|_{K_a}$. Therefore $K_b = K_a \cup_{\dot{\sigma}|_{K_a}} \sigma$.

Case (b), suppose σ is on the boundary of a stratum. There are two subcases: (i) σ is a violator in the sense of Definition 3.4 and therefore forms its own stratum; or (ii) σ is not a violator.

Case (b)(i), suppose σ is a type I violator; that is, globally $|U(\sigma)| \geq 2$. Then for any $\tau^{(p+1)} > \sigma$ in $U(\sigma)$ we have $f(\tau) \leq f(\sigma)$. If follows that $f(\tau) < a$, implying $\tau \in K_a$. Denote the set of such τ as $\{\tau\}$. Meanwhile, if $|L(\sigma)| = 0$, then for all $\nu^{(p-1)} < \sigma$ we have $f(\nu) < f(\sigma)$; that is, all the $(p-1)$-dimensional faces of σ are in K_a. Denote the set of such ν as $\{\nu\}$. The set $\{\nu\}$ is precisely

$\dot{\sigma}|_{K_a}$. Therefore, $K_b = K_a \cup_{\dot{\sigma}|_{K_a}} \sigma$, where we are attaching σ along its whole boundary (which lies in K_a) and realizing it as a portion of $\dot{\tau}$ for each $\tau \in \{\tau\}$. On the other hand, if $|L(\sigma)| \neq 0$, let $\mu^{(p-1)} < \sigma$ denote any face of σ *not* in $L(\sigma)$. Again denote the set of such μ as $\{\mu\}$. The remaining $(p-1)$ faces $\nu < \sigma$ all lie in K_a; denote these by $\{\nu\}$. Note that $\{\nu\} = \dot{\sigma}|_{K_a}$. Then $\dot{\sigma} = \{\nu\} \cup \{\mu\}$ and $K_b = K_a \cup_{\dot{\sigma}|_{K_a}} \sigma$.

Now suppose σ is a type II violator, thus globally $|L(\sigma)| \geq 2$. The simplices $\nu^{(p-1)} < \sigma$ not in $L(\sigma)$ satisfy $f(\nu) < f(\sigma)$, thus such $\nu \in K_a$ form the (possibly empty) set $\{\nu\}$. The simplices $\tau^{(p+1)} > \sigma$ in $U(\sigma)$ satisfy $f(\tau) < f(\sigma)$ thus such $\tau \in K_a$ form the (possibly empty) set $\{\tau\}$. The set $\{\nu\}$ is precisely $\dot{\sigma}|_{K_a}$ and we again have $K_b = K_a \cup_{\dot{\sigma}|_{K_a}} \sigma$. Finally, suppose σ is a type III violator, the proof in this case is similar is therefore omitted.

Case (b)(ii): σ is not a violator. Since σ is critical for a discrete stratified Morse function, it is either critical globally (i.e. $|U(\sigma)| = |L(\sigma)| = 0$) or locally (i.e. $|U_s(\sigma)| = |L_s(\sigma)| = 0$). Suppose σ is critical locally but not globally, meaning that either $|U(\sigma)| = 1, |L(\sigma)| = 0$, or $|U(\sigma)| = 0, |L(\sigma)| = 1$. If $|U(\sigma)| = 1$ and $|L(\sigma)| = 0$ globally, then $|U_s(\sigma)|$ becomes 0. If $\tau^{(p+1)} > \sigma$ is the unique element in $U(\sigma)$, then $f(\tau) < f(\sigma)$ and τ is in K_a. All cells $\nu^{(p-1)} < \sigma$ satisfy $f(\nu) < f(\sigma)$ and therefore are in K_a. The set $\{\nu\}$ again is precisely $\dot{\sigma}|_{K_a}$ and we have $K_b = K_a \cup_{\dot{\sigma}|_{K_a}} \sigma$, where we are attaching σ as a free face of τ. The cases when $|U(\sigma)| = 0, |L(\sigma)| = 1$, or $|U(\sigma)| = 0, |L(\sigma)| = 0$ are proved similarily.

In summary, when passing through a single, unique critical cell $\sigma^{(p)}$ with a function value in $(a, b]$, $K_b = K_a \cup_{\dot{\sigma}|_{K_a}} \sigma$. Since σ is homeomorphic to $e^{(p)}$, $K_b = K_a \cup_{\dot{e}^{(p)}|_{K_a}} e^{(p)}$. □

Examples of attaching critical simplices. Returning to the upside-down pentagon, we have a few critical cells, namely those with critical values $1, 2, 3, 4, 9$, and 10. Attaching $f^{-1}(2)$ to K_1, for example, changes the homotopy type, yielding a space with two connected components. Note that the boundary of this cell, restricted to K_1 is empty. When we attach $f^{-1}(9)$, we do so along its entire boundary (which lies in K_8), joining the two components together. Finally, attaching the vertex $f^{-1}(10)$ to K_9 changes the homotopy type yet again, yielding a circle.

In Example 4 (Figure 8, the split solid square), all the cells are critical in their strata. Attaching some of them does not change the homotopy type of the sublevel sets (e.g., $f^{-1}(3)$, $f^{-1}(8)$) while the addition of others can change the homotopy type (e.g. $f^{-1}(1)$, $f^{-1}(5)$, $f^{-1}(9)$, $f^{-1}(10)$, and $f^{-1}(11)$). Observe that the difference between these two types is that the latter consists of violators and global critical simplices, while the former consists of non-violators.

5.3 Proof of Theorem 3.4

Proof. We assume K is connected. If f itself is a discrete Morse function, then there are no violators in K. The algorithm produces the trivial stratification $\mathcal{S} = \{K\}$ and since f is a discrete Morse function on the entire complex, the pair (f, s) trivially satisfies Definition 3.1.

If f is not a discrete Morse function, let $\mathcal{S} = \mathcal{V} \cup \{S_j\}$ denote the stratification produced by the algorithm. Since each violator α forms its own stratum $s(\alpha)$, the restriction of f to $s(\alpha)$ is trivially a discrete Morse function in which α is a critical simplex. It remains to show that the restriction of f to each S_j is a discrete Morse function.

If σ is a simplex in S_j, that is, $s(\sigma) = S_j$, consider the sets $U_s(\sigma)$ and $L_s(\sigma)$. Since σ is not a violator, the global sets $U(\sigma)$ and $L(\sigma)$ already satisfy the conditions required of an ordinary discrete Morse function. Restricting attention to the stratum $s(\sigma)$ can only reduce their size; that is, $|U_s(\sigma)| \leq |U(\sigma)|$ and $|L_s(\sigma)| \leq |L(\sigma)|$. It follows that the restriction of f to S_j is a discrete Morse function. □

Remark. When we restrict the function $f : K \to \mathbb{R}$ to one of the strata S_j, a non-violator σ that is regular globally (that is, σ forms a gradient pair with a unique simplex τ) may become a critical simplex for the restriction of f to S_j, e.g. $f^{-1}(3)$ in Example 1. The edge $f^{-1}(3)$ in Example 4 also has such a property.

6 Discussion

In this paper we have identified a reasonable definition of a discrete stratified Morse function and demonstrated some of its fundamental properties. Many questions remain to be answered; we plan to address these in future work.

Relation to classical stratified Morse theory. An obvious question to ask is how our theory relates to the smooth case. Suppose X is a Whitney stratified space and $F : X \to \mathbb{R}$ is a stratified Morse function. One might ask the following: is there a triangulation K of X and a discrete stratified Morse function (f, s) on K that mirrors the behavior of F? That is, can we define a discrete stratified Morse function so that its critical simplices contain the critical points of the function F? This question has a positive answer in the setting of discrete Morse theory [4], so we expect the same to be true here as well.

Morse inequalities. Forman proved the discrete version of the Morse inequalities in [13]. Does our theory produce similar inequalities?

Discrete dynamics. Forman developed a more general theory of discrete vector fields [12] in which closed V-paths are allowed (analogous to recurrent dynamics). This yields a decomposition of a cell complex into pieces and an associated Lyaponov function (constant on the recurrent sets). This is not the same as a stratification, but it would be interesting to uncover any connections between our theory and this general theory. In particular, one might ask if there is some way to glue together the discrete Morse functions on each piece of a stratification into a global discrete vector field.

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A Classical Morse Theory

Let X be a compact, smooth d-manifold and f a real valued function on X, $f : X \to \mathbb{R}$. For a given value $a \in \mathbb{R}$, let $X_a = f^{-1}(-\infty, a] = \{x \in X \mid f(x) \leq a\}$ be the *sublevel set.* The classical Morse theory studies the topological change of X_a as a varies.

Morse function. Let f be a smooth function on X, $f : X \to \mathbb{R}$. A point $x \in X$ is *critical* if the derivative at x equals zero. The value of f at a critical point is a *critical value.* All other points are *regular points* and all other values are *regular values* of f. A critical point x is *non-degenerate* if the Hessian, that is, the matrix of second partial derivatives at the point, is invertible. The *Morse index* of the non-degenerate critical point x is the number of negative eigenvalues in the Hessian matrix, denoted as $\lambda(x)$. f is a *Morse function* if all critical points are non-degenerate and its values at the critical points are distinct.

Results. We now review two fundamental results of classical Morse theory (CMT).

Theorem A.1 (CMT Part A)**.** *([17], page 4; [8], page 129) Let $f : X \to \mathbb{R}$ be a differentiable function on a compact smooth manifold X. Let $a < b$ be real values such that $f^{-1}[a, b]$ is compact and contains no critical points of f. Then X_a is diffeomorphic to X_b.*

On the other hand, let f be a Morse function on X. We consider two regular values $a < b$ such that $f^{-1}[a, b]$ is compact but contains one critical point u of f, with index λ. Then X_b has the homotopy type of X_a with a λ-cell (or λ-handle, the smooth analogue of a λ-cell) attached along its boundary ([17], page 5; [8], page 129). We define *Morse data* for f at a critical point u in X to be a pair of topological spaces (A, B) where $B \subset A$ with the property that as a real value c increases from a to b (by crossing the critical value $f(u)$), the change in X_c can be described by gluing in A along B [17] (page 4). Morse data measures the topological change in X_c as c crosses critical value $f(u)$. We have the second fundamental result of classical Morse theory,

Theorem A.2 (CMT Part B)**.** *([17], page 5; [22], page 77) Let f be a Morse function on X. Consider two regular values $a < b$ such that $f^{-1}[a, b]$ is compact but contains one critical point u of f, with index λ. Then X_b is homotopy equivalent (diffeomorphic) to the space $X_a \cup_B A$, that is, by attaching A along B, where the Morse data $(A, B) = (D^\lambda \times D^{d-\lambda}, (\partial D^\lambda) \times D^{d-\lambda})$, where d is the dimension of X and λ is the Morse index of u, D^k denotes the closed k-dimensional disk and ∂D^k is its boundary.*

B Stratified Morse Theory

Morse theory can be generalized to certain singular spaces, in particular to Whitney stratified spaces [17, 23].

Stratified Morse function. Let X be a compact d-dimensional Whitney stratified space embedded in some smooth manifold $\mathbb{M}$. A function on X is *smooth* if it is the restriction to X of a smooth function on $\mathbb{M}$. Let $\bar{f} : \mathbb{M} \to \mathbb{R}$ be a smooth function. The restriction f of $\bar{f}$ to X is *critical* at a point $x \in X$ iff it is critical when restricted to the particular manifold piece which contains x [2]. A *critical value* of f is its value at a critical point. f is a *stratified Morse function* iff ([2], [17] page 13):

1. All critical values of f are distinct.
2. At each critical point u of f, the restriction of f to the stratum S containing u is non-degenerate.

3. The differential of f at a critical point $u \in S$ does not annihilate (destroy) any limit of tangent spaces to any stratum S' other than the stratum S containing u.

Condition 1 and 2 imply that f is a Morse function when restricted to each stratum in the classical sense. Condition 1 is a non-degeneracy requirement in the tangential directions to S. Condition 2 is a non-degeneracy requirement in the directions normal to S [17] (page 13).

Results. Now we state the two fundamental results of stratified Morse theory.

Theorem B.1 (SMT Part A)**.** *([17], page 6) Let X be a Whitney stratified space and $f : X \to \mathbb{R}$ a stratified Morse function. Suppose the interval $[a, b]$ contains no critical values of f. Then X_a is diffeomorphic to X_b.*

Theorem B.2 (SMT Part B)**.** *([17], page 8 and page 64) Let f be a stratified Morse function on a compact Whitney stratified space X. Consider two regular values $a < b$ such that $f^{-1}[a, b]$ is compact but contains one critical point u of f. Then X_b is diffeomorphic to the space $X_a \cup_B A$, that is, by attaching A along B, where the Morse data (A, B) is the product of the normal Morse data at u and the tangential Morse data at u.*

To define tangential and normal Morse data, we have the following set-up. Let X be a Whitney stratified subset of some smooth manifold $\mathbb{M}$. Let $f : X \to \mathbb{R}$ be a stratified Morse function with a critical point u. Let S denote the stratum of X which contains the critical point u. Let N be a *normal slice* at u, that is, $N = X \cap N' \cap B_\delta^{\mathbb{M}}(u)$, where N' is a sub-manifold of $\mathbb{M}$ which is traverse to each stratum of X, intersects the stratum S in a single point u, and satisfies $\dim S + \dim N' = \dim \mathbb{M}$. $B_\delta^{\mathbb{M}}(u)$ is a closed ball of radius δ in $\mathbb{M}$ based on a Riemannian metric on $\mathbb{M}$. By Whitney's condition, if δ is sufficiently small then $\partial B_\delta^{\mathbb{M}}(u)$ will be transverse to each stratum of X, and to each stratum in $X \cap N'$, fix such a $\delta > 0$ [17] (page 40).

The *tangential Morse data* for f at u is homotopy equivalent to the pair

$$(P, Q) = (D^\lambda \times D^{s-\lambda}, (\partial D^\lambda) \times D^{s-\lambda}),$$

where λ is the (classical) Morse index of f restricted to S, $f|S$, at u, and s is the dimensional of stratum S [17] (page 65).

The *normal Morse data* is the pair

$$(J, K) = (N \cap f^{-1}[v - \varepsilon, v + \varepsilon], N \cap f^{-1}(v - \varepsilon)),$$

where $f(u) = v$ and $\varepsilon > 0$ is chosen such that $f|N$ has no critical values other than v in the interval $[v - \varepsilon, v + \varepsilon]$ [17] (page 65).

The *Morse data* is homotopy equivalent to the topological product of tangential and normal Morse Data, where the notion of product of pairs is defined as $(A, B) = (P, Q) \times (J, K) = (P \times J, P \times K \cup Q \times J)$.