Math 6630: Numerical Solutions of Partial Differential Equations
Interpolation with Fourier Series

See Hesthaven, S. Gottlieb, and D. Gottlieb 2007, Chapters 2-3,
Canuto et al. 2011, Chapter 2.1,

Shen, Tang, and Wang 2011, Chapter 2

Akil Narayan[1]

[1] Department of Mathematics, and Scientific Computing and Imaging (SCI) Institute
University of Utah

March 15, 2023

THE UNIVERSITY OF UTAH

SCI www.sci.utah.edu

Fourier Series approximations

We have established that Fourier series approximations u_N,

$$u_N(x) = \sum_{|k| \leqslant N} \hat{u}_k \phi_k(x), \qquad \phi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}, \qquad \hat{u}_k = \langle u, \phi_k \rangle,$$

have orders of convergence that depend on the smoothness of u:

$$u \in H_p^s \quad \Longrightarrow \quad \|u - u_N\|_{L^2} \leqslant N^{-s} \|u\|_{H^s}.$$

I.e.,

$$\text{Smoothness} \Longrightarrow \text{Compressibility}$$

Fourier Series approximations

We have established that Fourier series approximations u_N,

$$u_N(x) = \sum_{|k| \leqslant N} \widehat{u}_k \phi_k(x), \qquad \phi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}, \qquad \widehat{u}_k = \langle u, \phi_k \rangle,$$

have orders of convergence that depend on the smoothness of u:

$$u \in H_p^s \quad \Longrightarrow \quad \|u - u_N\|_{L^2} \leqslant N^{-s} \|u\|_{H^s}.$$

I.e.,

$$\text{Smoothness} \Longrightarrow \text{Compressibility}$$

One major outstanding question is *how* we actually compute $\widehat{u}_k$ in practice.

Quadrature

The expansion coefficients require computing an integral,

$$\widehat{u}_k = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u(x) e^{-ikx} \mathrm{d}x,$$

which can be intractable, even if an explicit formula for u is available.

Quadrature

The expansion coefficients require computing an integral,

$$\widehat{u}_k = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u(x) e^{-ikx} \mathrm{d}x,$$

which can be intractable, even if an explicit formula for u is available.

A standard recourse is to approximate the integral with quadrature:

$$\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u(x) e^{-ikx} \mathrm{d}x \approx \sum_{j=1}^{M} w_{k,j} u(x_j), \qquad w_{k,j} = \frac{\sqrt{2\pi}}{M} e^{-ikx_j}, \qquad x_j = \frac{2\pi(j-1)}{M},$$

where we have made particular *choices*:

- x_j are equispaced on $[0, 2\pi]$ for $j \in [M]$
- $w_{k,j}$ correspond to a uniform quadrature rule

Quadrature

The expansion coefficients require computing an integral,

$$\widehat{u}_k = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u(x) e^{-ikx} \mathrm{d}x,$$

which can be intractable, even if an explicit formula for u is available.

A standard recourse is to approximate the integral with quadrature:

$$\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u(x) e^{-ikx} \mathrm{d}x \approx \sum_{j=1}^{M} w_{k,j} u(x_j), \qquad w_{k,j} = \frac{\sqrt{2\pi}}{M} e^{-ikx_j}, \qquad x_j = \frac{2\pi(j-1)}{M},$$

where we have made particular *choices*:

- x_j are equispaced on $[0, 2\pi]$ for $j \in [M]$
- $w_{k,j}$ correspond to a uniform quadrature rule
- We'll also assume that $M = 2N + 1$. (Quadrature nodes = expansion coefficients)

Note that this is just the trapezoid rule on $[0, 2\pi]$ with periodic boundary conditions.

One can make other choices, but these choices are most convenient for discussing the major concepts surrounding theory and computation.

Quadrature as linear algebra

$$\widehat{u}_k \approx \widetilde{u}_k := \sum_{j=1}^{M} w_{k,j} u(x_j), \qquad w_{k,j} = \frac{\sqrt{2\pi}}{M} e^{-ikx_j}, \qquad x_j = \frac{2\pi(j-1)}{M}.$$

We compute these coefficients for all $|k| \leqslant N$, with $M = 2N + 1$.

Quadrature as linear algebra

$$\widehat{u}_k \approx \widetilde{u}_k := \sum_{j=1}^{M} w_{k,j} u(x_j), \qquad w_{k,j} = \frac{\sqrt{2\pi}}{M} e^{-ikx_j}, \qquad x_j = \frac{2\pi(j-1)}{M}.$$

We compute these coefficients for all $|k| \leqslant N$, with $M = 2N+1$.

A simple implementation quadrature of amounts to matrix-vector algebra:

$$\boldsymbol{u} := \begin{pmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_M) \end{pmatrix}, \qquad \widetilde{\boldsymbol{u}} := \begin{pmatrix} \widetilde{u}_{-N} \\ \widetilde{u}_{-N+1} \\ \vdots \\ \widetilde{u}_N \end{pmatrix} \implies \widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{V}}^* \boldsymbol{u},$$

where $\widetilde{\boldsymbol{V}}^*$ is the conjugate transpose of $\widetilde{\boldsymbol{V}}$, which in turn is given by,

$$\widetilde{\boldsymbol{V}} = \sqrt{\frac{2\pi}{M}} \boldsymbol{V}, \qquad \boldsymbol{V} = \begin{pmatrix} | & | & & | \\ \boldsymbol{v}_{-N} & \boldsymbol{v}_{-N+1} & \cdots & \boldsymbol{v}_N \\ | & | & & | \end{pmatrix}, \qquad \boldsymbol{v}_k = \sqrt{\frac{2\pi}{M}} \phi_k(\boldsymbol{x}),$$

and $\boldsymbol{x} = (x_1, x_2, \ldots, x_M)^T$.

Duality of Fourier Series with quadrature

$$\widetilde{\boldsymbol{V}} = \sqrt{\frac{2\pi}{M}} \boldsymbol{V}, \qquad \boldsymbol{V} = \begin{pmatrix} | & | & & | \\ \boldsymbol{v}_{-N} & \boldsymbol{v}_{-N+1} & \cdots & \boldsymbol{v}_{N} \\ | & | & & | \end{pmatrix}, \qquad \boldsymbol{v}_k = \sqrt{\frac{2\pi}{M}} \phi_k(\boldsymbol{x}),$$

A somewhat straightforward computation shows:

$$\langle \boldsymbol{v}_\ell, \boldsymbol{v}_k \rangle = \frac{1}{M} \sum_{j=1}^{M} e^{i(\ell-k)2\pi(j-1)/M} = \frac{1}{M} \sum_{j=0}^{M-1} \left(e^{i(\ell-k)2\pi/M} \right)^j,$$

Duality of Fourier Series with quadrature

$$\widetilde{\boldsymbol{V}} = \sqrt{\frac{2\pi}{M}}\boldsymbol{V}, \qquad \boldsymbol{V} = \begin{pmatrix} | & | & & | \\ \boldsymbol{v}_{-N} & \boldsymbol{v}_{-N+1} & \cdots & \boldsymbol{v}_N \\ | & | & & | \end{pmatrix}, \qquad \boldsymbol{v}_k = \sqrt{\frac{2\pi}{M}}\phi_k(\boldsymbol{x}),$$

A somewhat straightforward computation shows:

$$\langle \boldsymbol{v}_\ell, \boldsymbol{v}_k \rangle = \frac{1}{M}\sum_{j=1}^{M} e^{i(\ell-k)2\pi(j-1)/M} = \frac{1}{M}\sum_{j=0}^{M-1}\left(e^{i(\ell-k)2\pi/M}\right)^j,$$

Thus, in particular if $\ell = k$ then $\langle \boldsymbol{v}_\ell, \boldsymbol{v}_k \rangle = 1$, and for $\ell \neq k$ and $|\ell - k| \leqslant M - 1$:

$$\langle \boldsymbol{v}_\ell, \boldsymbol{v}_k \rangle = \frac{1}{M}\frac{1 - \left(e^{i(\ell-k)2\pi/M}\right)^M}{1 - e^{i(\ell-k)2\pi/M}} = 0$$

I.e., $\{\boldsymbol{v}_k\}_{|k|\leqslant N}$ are *orthonormal vectors*.

Duality of Fourier Series with quadrature

$$\widetilde{\boldsymbol{V}} = \sqrt{\frac{2\pi}{M}} \boldsymbol{V}, \qquad \boldsymbol{V} = \begin{pmatrix} | & | & & | \\ \boldsymbol{v}_{-N} & \boldsymbol{v}_{-N+1} & \cdots & \boldsymbol{v}_N \\ | & | & & | \end{pmatrix}, \qquad \boldsymbol{v}_k = \sqrt{\frac{2\pi}{M}} \phi_k(\boldsymbol{x}),$$

A somewhat straightforward computation shows:

$$\langle \boldsymbol{v}_\ell, \boldsymbol{v}_k \rangle = \frac{1}{M} \sum_{j=1}^{M} e^{i(\ell-k)2\pi(j-1)/M} = \frac{1}{M} \sum_{j=0}^{M-1} \left(e^{i(\ell-k)2\pi/M} \right)^j,$$

Thus, in particular if $\ell = k$ then $\langle \boldsymbol{v}_\ell, \boldsymbol{v}_k \rangle = 1$, and for $\ell \neq k$ and $|\ell - k| \leqslant M - 1$:

$$\langle \boldsymbol{v}_\ell, \boldsymbol{v}_k \rangle = \frac{1}{M} \frac{1 - \left(e^{i(\ell-k)2\pi/M}\right)^M}{1 - e^{i(\ell-k)2\pi/M}} = 0$$

I.e., $\{\boldsymbol{v}\}_{|k| \leqslant N}$ are *orthonormal vectors*.

This shows the important property that $\boldsymbol{V}$ is a unitary matrix:

$$\boldsymbol{V}^* \boldsymbol{V} = \boldsymbol{I} \quad \Longrightarrow \quad \boldsymbol{V}^{-1} = \boldsymbol{V}^*.$$

Inverting the Fourier Series

Putting everything together:

$$\widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{V}}^* \boldsymbol{u}, \qquad \widetilde{\boldsymbol{V}} = \sqrt{\frac{2\pi}{M}} \boldsymbol{V}, \qquad \boldsymbol{V}^{-1} = \boldsymbol{V}^*.$$

This implies that:

$$\boldsymbol{u} = \left(\widetilde{\boldsymbol{V}}^*\right)^{-1} \widetilde{\boldsymbol{u}} = \sqrt{\frac{M}{2\pi}} \left(\boldsymbol{V}^*\right)^{-1} \widetilde{\boldsymbol{u}} = \sqrt{\frac{M}{2\pi}} \boldsymbol{V} \widetilde{\boldsymbol{u}} = \frac{M}{2\pi} \widetilde{\boldsymbol{V}} \widetilde{\boldsymbol{u}}.$$

Inverting the Fourier Series

Putting everything together:

$$\widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{V}}^* \boldsymbol{u}, \qquad \widetilde{\boldsymbol{V}} = \sqrt{\frac{2\pi}{M}} \boldsymbol{V}, \qquad \boldsymbol{V}^{-1} = \boldsymbol{V}^*.$$

This implies that:

$$\boldsymbol{u} = \left(\widetilde{\boldsymbol{V}}^*\right)^{-1} \widetilde{\boldsymbol{u}} = \sqrt{\frac{M}{2\pi}} \left(\boldsymbol{V}^*\right)^{-1} \widetilde{\boldsymbol{u}} = \sqrt{\frac{M}{2\pi}} \boldsymbol{V} \widetilde{\boldsymbol{u}} = \frac{M}{2\pi} \widetilde{\boldsymbol{V}} \widetilde{\boldsymbol{u}}.$$

I.e., the map between $\boldsymbol{u}$ and $\widetilde{\boldsymbol{u}}$ is invertible and quite explicit:

$$\widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{V}}^* \boldsymbol{u}, \qquad \boldsymbol{u} = \frac{M}{2\pi} \widetilde{\boldsymbol{V}} \widetilde{\boldsymbol{u}}.$$

This invertible map is called the Discrete Fourier Transform (DFT). As a consequence of $\boldsymbol{V}$ being unitary, we have also shown that the DFT is a (scaled) isometry,

$$\int_0^{2\pi} |u(x)|^2 \mathrm{d}x \approx \frac{2\pi}{M} \|\boldsymbol{u}\|_2^2 = \|\widetilde{\boldsymbol{u}}\|_2^2,$$

which is the discrete analogue of Parseval's identity.

The Fast Fourier Transform, I

The inverse/DFT is relatively expensive:

$$\boldsymbol{u} \xrightarrow{\mathcal{O}(M^2)} \widetilde{\boldsymbol{V}}^* \boldsymbol{u}, \qquad\qquad \widetilde{\boldsymbol{u}} \xrightarrow{\mathcal{O}(M^2)} \frac{M}{2\pi} \widetilde{\boldsymbol{V}}.$$

One of the most well-known algorithms is the *fast Fourier transform*, which is a fast algorithm for accomplishing the particular matrix-vector multiplication $\widetilde{\boldsymbol{V}}^* \boldsymbol{u}$.

It is simpler to explain the basic idea if M is even, in which case we have:

$$\begin{aligned}
\frac{M}{\sqrt{2\pi}} \widetilde{u}_k &= \sum_{j=1}^{M} u(x_j) e^{-ikx_j} = \sum_{j=1}^{M} u(x_j) e^{-ik2\pi(j-1)/M} \\
&= \sum_{j=1}^{M/2} u(x_{2j}) e^{-ik2\pi 2(j-1)/M} + \sum_{j=1}^{M/2} u(x_{2j-1}) e^{-ik2\pi(2j-1)/M}
\end{aligned}$$

The Fast Fourier Transform, I

The inverse/DFT is relatively expensive:

$$\boldsymbol{u} \xrightarrow{\mathcal{O}(M^2)} \widetilde{\boldsymbol{V}}^* \boldsymbol{u}, \qquad\qquad \widetilde{\boldsymbol{u}} \xrightarrow{\mathcal{O}(M^2)} \frac{M}{2\pi} \widetilde{\boldsymbol{V}}.$$

One of the most well-known algorithms is the *fast Fourier transform*, which is a fast algorithm for accomplishing the particular matrix-vector multiplication $\widetilde{\boldsymbol{V}}^* \boldsymbol{u}$.

It is simpler to explain the basic idea if M is even, in which case we have:

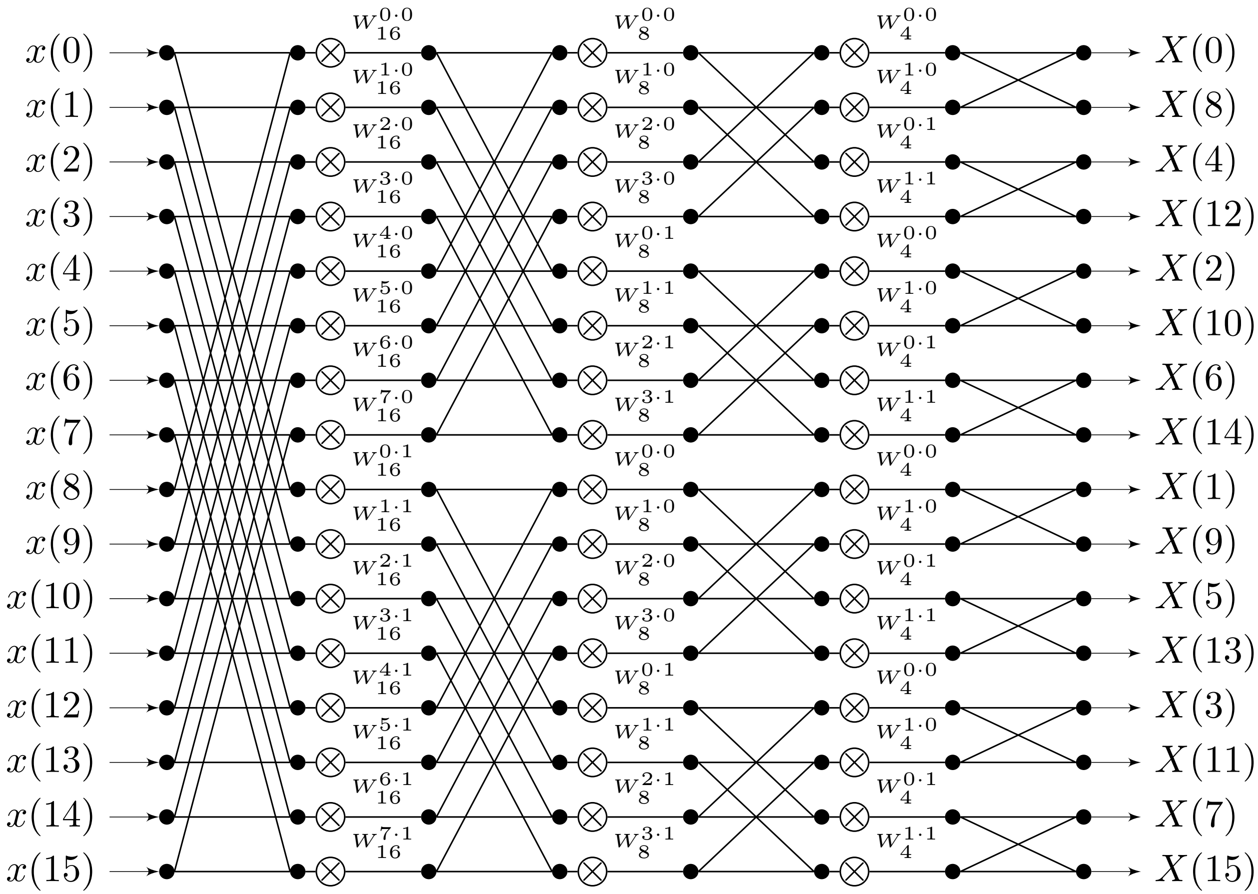
$$\begin{aligned}
\frac{M}{\sqrt{2\pi}} \widetilde{u}_k &= \sum_{j=1}^{M} u(x_j) e^{-ikx_j} = \sum_{j=1}^{M} u(x_j) e^{-ik2\pi(j-1)/M} \\
&= \sum_{j=1}^{M/2} u(x_{2j}) e^{-ik2\pi 2(j-1)/M} + \sum_{j=1}^{M/2} u(x_{2j-1}) e^{-ik2\pi(2j-1)/M} \\
&= \sum_{j=1}^{M/2} u(x_{2j}) e^{-ik2\pi 2(j-1)/M} + e^{ik2\pi/M} \sum_{j=1}^{M/2} u(x_{2j-1}) e^{-ik2\pi(2j-2)/M}.
\end{aligned}$$

Note that the last two sums are $M/2$-point DFT coefficients associated with half the data (either at x_{2j} or at x_{2j-1}).

I.e., with some book-keeping, we can compute the M-point DFT using 2 $M/2$-point DFT's.

The Fast Fourier Transform, II

This logic can be repeated, showing that actually we can compute the M-point DFT using J (M/J)-point DFT's, where J is a power of two. This yields the simplest, *radix 2 fast Fourier transform* (FFT) algorithm.



Through this divide-and-conquer strategy, an M-point DFT that naively requires $\mathcal{O}(M^2)$ complexity can be accomplished in $\mathcal{O}(M \log M)$ time.

Interpolation

$$\widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{V}}^* \boldsymbol{u}, \qquad\qquad \boldsymbol{u} = \frac{M}{2\pi} \widetilde{\boldsymbol{V}} \widetilde{\boldsymbol{u}}.$$

We have introduced the DFT via quadrature, but an alternative and illustrative viewpoint is *interpolation*.

Note that the coefficients $\widetilde{\boldsymbol{u}}$ are determined by the conditions,

$$\frac{M}{2\pi} \widetilde{\boldsymbol{V}} \widetilde{\boldsymbol{u}} = \boldsymbol{u} \quad \Longrightarrow \quad \begin{pmatrix} | & | & & | \\ \boldsymbol{\phi}_{-N} & \boldsymbol{\phi}_{-N+1} & \cdots & \boldsymbol{\phi}_N \\ | & | & & | \end{pmatrix} \widetilde{\boldsymbol{u}} = \boldsymbol{u}.$$

Note that these are "just" interpolation conditions for the $\widetilde{\boldsymbol{u}}$ at the data points x_j, $j \in [M]$.

Interpolation

$$\widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{V}}^* \boldsymbol{u}, \qquad\qquad \boldsymbol{u} = \frac{M}{2\pi} \widetilde{\boldsymbol{V}} \widetilde{\boldsymbol{u}}.$$

We have introduced the DFT via quadrature, but an alternative and illustrative viewpoint is *interpolation*.

Note that the coefficients $\widetilde{\boldsymbol{u}}$ are determined by the conditions,

$$\frac{M}{2\pi} \widetilde{\boldsymbol{V}} \widetilde{\boldsymbol{u}} = \boldsymbol{u} \quad \Longrightarrow \quad \begin{pmatrix} | & | & & | \\ \boldsymbol{\phi}_{-N} & \boldsymbol{\phi}_{-N+1} & \cdots & \boldsymbol{\phi}_N \\ | & | & & | \end{pmatrix} \widetilde{\boldsymbol{u}} = \boldsymbol{u}.$$

Note that these are "just" interpolation conditions for the $\widetilde{\boldsymbol{u}}$ at the data points x_j, $j \in [M]$.

Hence, $u_N(x) = \sum_{|k| \leqslant N} \widetilde{u}_k \phi_k(x)$ interpolates the data $\boldsymbol{u}$.

We have already concluded that this interpolation problem is *unisolvent*. Hence, there are *cardinal basis functions* $\ell_j(x)$, $j \in [M]$ such that,

$$u_N(x) = \sum_{j \in [M]} u(x_j) \ell_j(x), \qquad\qquad \ell_j(x_r) = \delta_{j,r}.$$

I.e., u_N has a Lagrange form.

Cardinal Lagrange basis

The cardinal Lagrange functions yield insight into the interpolation process.

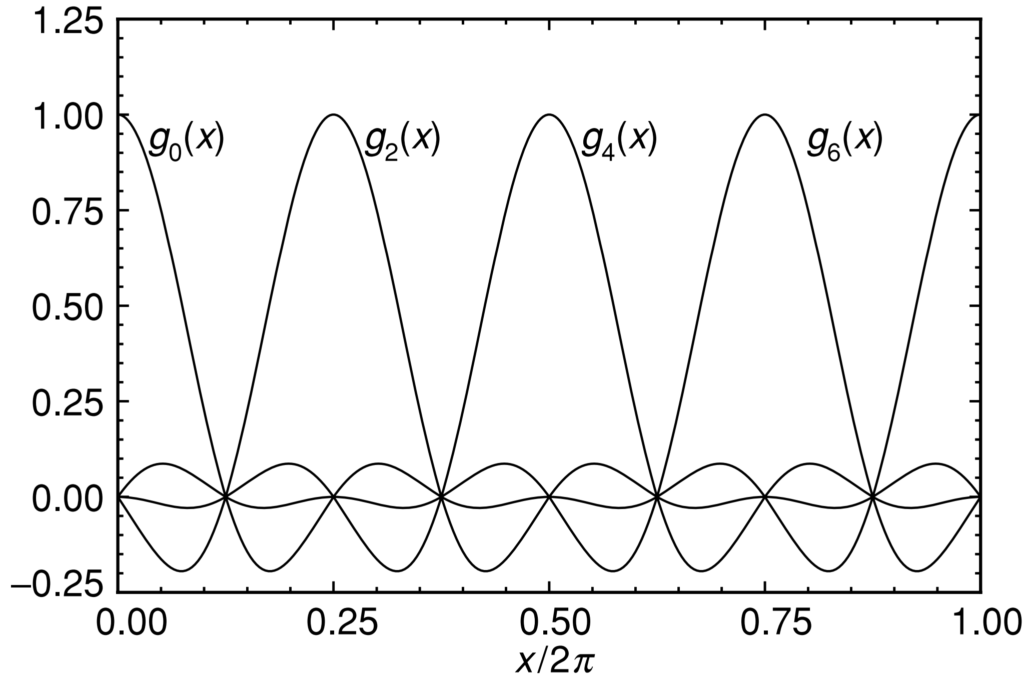


Figure 2.3 of Hesthaven, S. Gottlieb, and D. Gottlieb 2007

Note that interpolation implies

$$u(x) \in V_N := \operatorname{span}\left\{e^{ikx}\right\}_{|k| \leqslant N} \quad \Longrightarrow \quad I_N u := \sum_{|k| \leqslant N} \widetilde{u}_k \phi_k(x) = u(x).$$

Aliasing

The fact that our DFT is an interpolation process reveals a significant issue that we must be cognizant of: aliasing error.

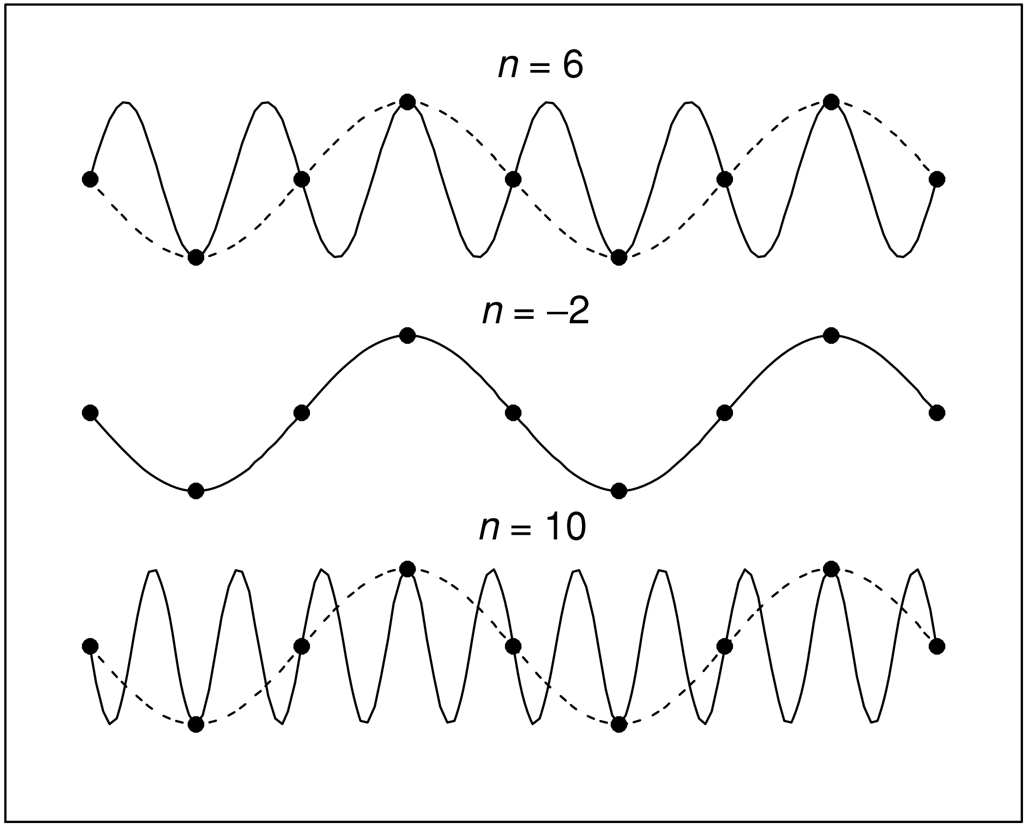


Figure 2.7 of Hesthaven, S. Gottlieb, and D. Gottlieb 2007

Aliasing

The fact that our DFT is an interpolation process reveals a significant issue that we must be cognizant of: aliasing error.

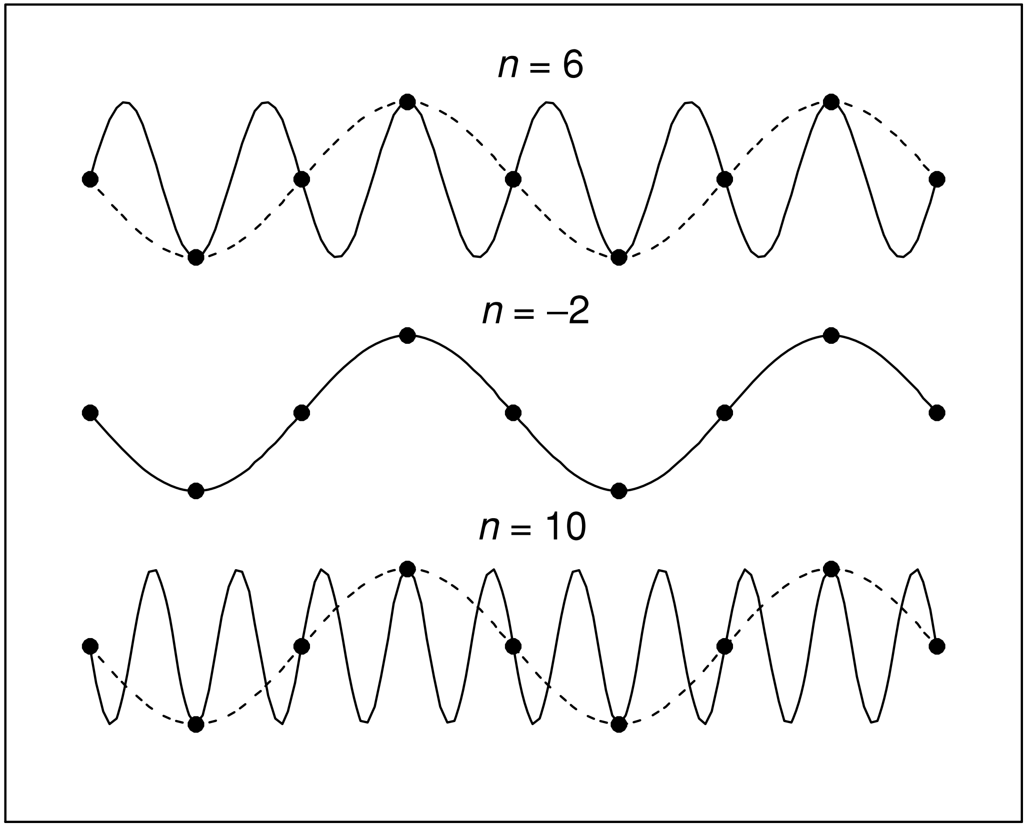


Figure 2.7 of Hesthaven, S. Gottlieb, and D. Gottlieb 2007

So, for example, even if $\left\langle e^{i\ell x}, \phi_k(x)\right\rangle = 0$ for $\ell > N$, it's possible that $I_N u \neq 0$.

I.e., the interpolation/DFT procedure *is* a projection operator, it's just an oblique one.

Aliasing error

Aliasing is not just an academic curiosity: with P_N the L^2-orthogonal projection operator onto

$$V_N = \operatorname{span}\left\{e^{ikx}\right\}_{|k| \leqslant N},$$

recall that $u \in H_p^s$ implies that $\|u - P_N u\|_{L^2} \lesssim N^{-s}$.

Ok, but what about $I_N u$?

Aliasing error

Aliasing is not just an academic curiosity: with P_N the L^2-orthogonal projection operator onto

$$V_N = \operatorname{span}\left\{e^{ikx}\right\}_{|k| \leqslant N},$$

recall that $u \in H_p^s$ implies that $\|u - P_N u\|_{L^2} \lesssim N^{-s}$.

Ok, but what about $I_N u$?

The main strategy to understanding this is to estimate the aliasing error. Note that for the L^2 norm,

$$\begin{aligned}\|u - I_N u\| = \|\,(u - P_N u) + (P_N u - I_N u)\,\| &\leqslant \|u - P_N u\| + \|P_N u - I_N u\| \\ &=: \|u - P_N u\| + \|A_N u\|,\end{aligned}$$

where we have defined the aliasing error $A_N u$.

Bounding the aliasing error

$$A_N u = P_N u - I_N u$$

The following observations are crucial:

– If $u \in V_N$, then $I_N u = P_N u = u$, so $A_N u = 0$. Therefore, $A_N u = A_N(I - P_N)u$.
  The aliasing error is only affected by truncation error.

Bounding the aliasing error

$\|u - P_N u\| \sim N^{-s}$

$$A_N u = P_N u - I_N u$$

The following observations are crucial:

- If $u \in V_N$, then $I_N u = P_N u = u$, so $A_N u = 0$. Therefore, $A_N u = A_N \underline{(I - P_N) u}$.
  The aliasing error is only affected by truncation error.
- We know that $(I - P_N)u$ is small. Therefore, if A_N is "behaves well", then $A_N u$ will be small.
  The truncation error is small, but does A_N amplify small inputs?

Bounding the aliasing error

$$A_N u = P_N u - I_N u$$

The following observations are crucial:

- If $u \in V_N$, then $I_N u = P_N u = u$, so $A_N u = 0$. Therefore, $A_N u = A_N (I - P_N) u$.
  The aliasing error is only affected by truncation error.
- We know that $(I - P_N)u$ is small. Therefore, if A_N is "behaves well", then $A_N u$ will be small.
  The truncation error is small, but does A_N amplify small inputs?
- A_N is well-behaved: for $|k| \leqslant N$,

  $$I_N \cancel{A_N} e^{i(k+(2N+1))x} = e^{ikx}, \qquad P_N e^{i(k+(2N+1))x} = 0$$

  and thus in particular,

  $$u = \sum_{\cancel{|k| \leqslant N} \; k \in \mathbb{Z}} \widehat{u}_k \phi_k(x) \qquad \Longrightarrow \qquad \widetilde{u}_k = \sum_{\ell \in \mathbb{Z}} \widehat{u}_{k+\ell(2N+1)}.$$

  A_N does not amplify small inputs.

Bounding the aliasing error

$$A_N u = P_N u - I_N u$$

The following observations are crucial:

- If $u \in V_N$, then $I_N u = P_N u = u$, so $A_N u = 0$. Therefore, $A_N u = A_N (I - P_N) u$.
  The aliasing error is only affected by truncation error.
- We know that $(I - P_N) u$ is small. Therefore, if A_N is "behaves well", then $A_N u$ will be small.
  The truncation error is small, but does A_N amplify small inputs?
- A_N is well-behaved: for $|k| \leqslant N$,

$I_N \hat{u}_k \varphi_k \sim \hat{u}_k \varphi_\ell$

$$\underset{I_N}{\cancel{A_N}} e^{i(k+(2N+1))x} = e^{ikx}, \qquad I_N \varphi_k \sim e^{i\ell x}$$

$|\ell| \leq N$

  and thus in particular,

$$u = \sum_{\substack{\cancel{|k| \leqslant N} \\ k \in \mathbb{Z}}} \hat{u}_k \phi_k(x) \qquad \Longrightarrow \qquad \underset{||}{\tilde{u}_k} = \sum_{\ell \in \mathbb{Z}} \hat{u}_{k+\ell(2N+1)}.$$

  A_N does not amplify small inputs.

$|\ell| \leq N$: $\langle I_N u, \varphi_\ell \rangle = \sum_{k \in \mathbb{Z}} \hat{u}_k \langle I_N \varphi_k, \varphi_\ell \rangle$

Therefore, if $\hat{u}_{k+\ell(2N+1)}$ decays quickly for large $|\ell|$, then we can expect the aliased coefficients $\tilde{u}_k$ to be "close" to $\hat{u}_k$.

$= \sum_{k \in \mathbb{Z}} \hat{u}_k \, \delta_{k, \ell + (2N+1) q}$

Interpolation estimates

While we have only discussed the high-level ideas, going through the details produces the following estimate:

Theorem

Assume $u \in H_p^s$ *with* $s > 1/2$. *Then*

$$\|u - I_N u\|_{L^2} \lesssim N^{-s} \|u\|_{H^s} .$$

$$\|u - I_N u\|_{H^r} \lesssim N^{-(s-r)} \|u\|_{H^s}, \qquad r < s.$$

Note that this is exactly the asymptotic behavior for the exact orthogonal projector P_N. Thus, one can expect the DFT to produce good results.

The DFT in practice

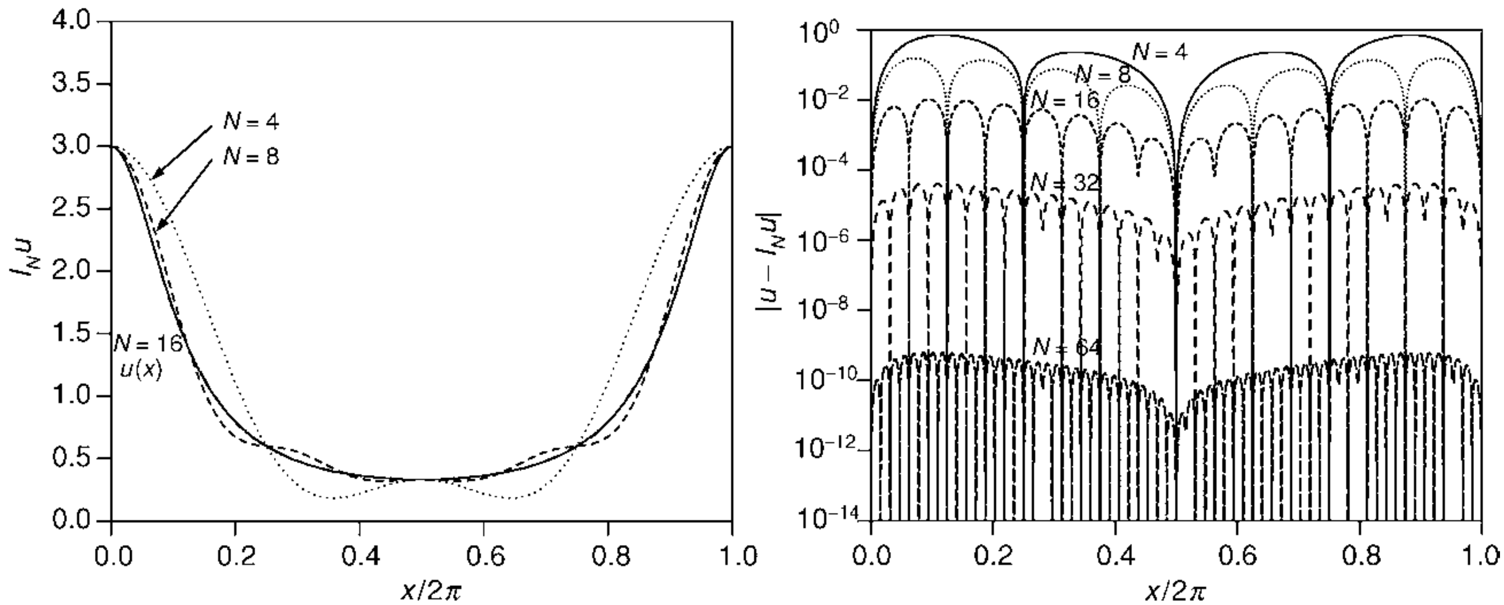


Figure 2.4 of Hesthaven, S. Gottlieb, and D. Gottlieb 2007

$$u(x) = \frac{3}{5 - 4\cos x}$$

The DFT in practice

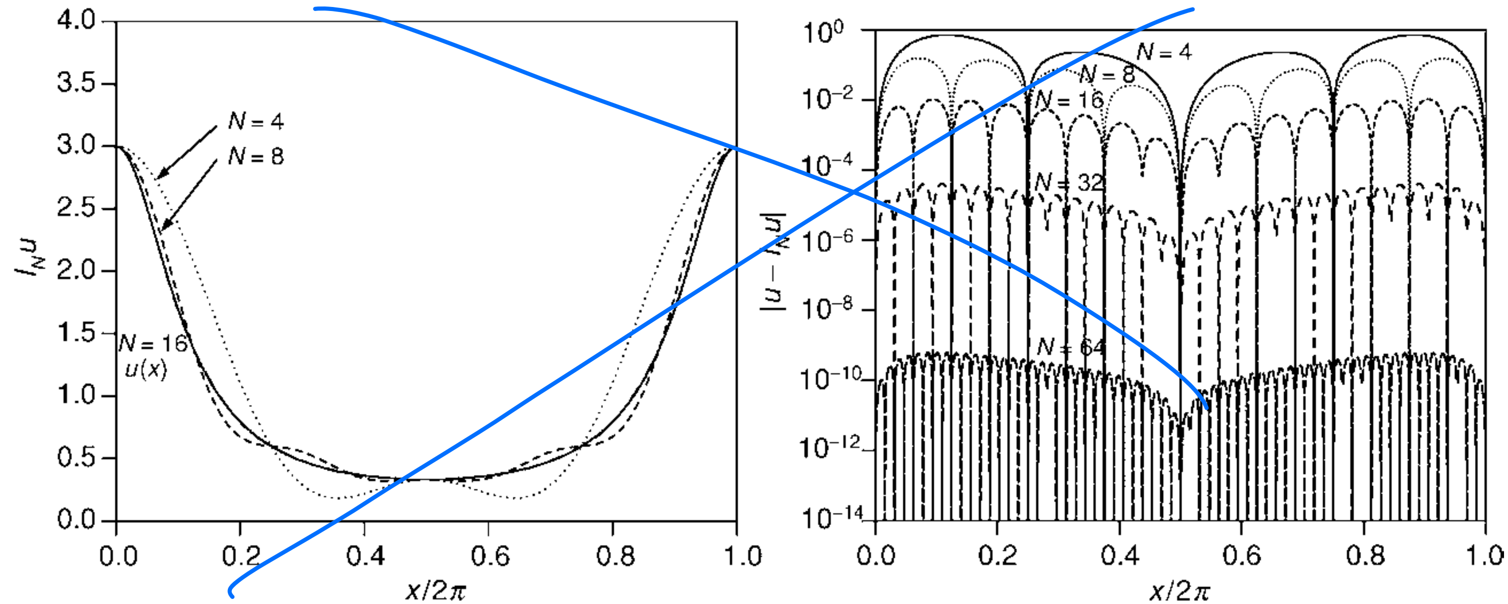
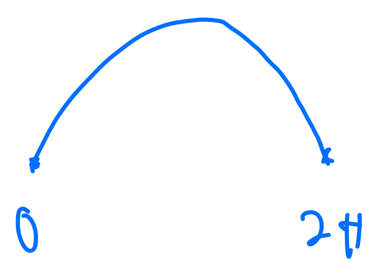


Figure 2.4 of Hesthaven, S. Gottlieb, and D. Gottlieb 2007

$$u(x) = \sin(x/2)$$



References I

Canuto, Claudio et al. (2011). *Spectral Methods: Fundamentals in Single Domains*. 1st ed. 2006. Corr. 4th printing 2010 edition. Berlin ; New York: Springer. ISBN: 978-3-540-30725-9.

Hesthaven, Jan S., Sigal Gottlieb, and David Gottlieb (2007). *Spectral Methods for Time-Dependent Problems*. Cambridge University Press. ISBN: 0-521-79211-8.

Shen, Jie, Tao Tang, and Li-Lian Wang (2011). *Spectral Methods: Algorithms, Analysis and Applications*. Springer Science & Business Media. ISBN: 978-3-540-71041-7.