Math 6630: Numerical Solutions of Partial Differential Equations
Approximation with Fourier Series

See Hesthaven, P. S. Gottlieb, and D. Gottlieb 2007, Chapters 1-2,
Canuto et al. 2011, Chapters 2.1, 5.1,

Shen, Tang, and Wang 2011, Chapter 2

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High order approximations

We are now very familiar with our rather standard approximation to u_{xx} on an equidistant grid:

$$D_0 u_j^n \approx u_{xx} + \mathcal{O}(h^2)$$

Note that the h^2 truncation error is a direct result of our choice of 3-point stencil.

Using more points in the stencil allows us to attain higher order truncation errors.

$$\frac{1}{12h^2}\left[-u_{j-2}^n + 16u_{j-1}^n - 30u_j^n + 16u_{j+1}^n - u_{j+2}^n\right] \approx u_{xx} + \cancel{\mathcal{O}h^2}. \quad \mathcal{O}(h^4)$$

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In general, using $2k+1$ points allows us to achieve $\mathcal{O}(h^{2k})$ LTE.

Why stop here? Why not take k as large as possible?

This requires a stencil spreading over the whole domain, globally coupling all degrees of freedom.

Is it worth it?

Fourier Series, I

Before solving differential equations, let's answer some basic approximation theory questions first.

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The simplest example of an approximation scheme that globally couples all degrees of freedom is a Fourier Series.

Consider a given $u : [0, 2\pi] \to \mathbb{C}$, which we represent as a sum of complex exponentials,

$$u(x) \approx \sum_{k \in \mathbb{Z}} \widehat{u}_k \phi_k(x), \qquad \phi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}.$$

The most straightforward strategy to identify $\widehat{u}_k$ is to choose them to minimize a loss,

$$\widehat{u}_k = \operatorname*{arg\,min}_{\widehat{u}_k, k \in \mathbb{Z}} \left\| u(x) - \sum_{k \in \mathbb{Z}} \widehat{u}_k \phi_k(x) \right\|_2^2,$$

where we have introduced the norm and a corresponding inner product,

$$\langle f, g \rangle := \int_0^{2\pi} f(x) \overline{g}(x) \mathrm{d}x, \qquad \|f\|_2^2 := \langle f, f \rangle,$$

where $\overline{z}$ is the complex conjugate of z. [1]

[1] We are mostly interested in real-valued functions, so the introduction of complex arithmetic is somewhat artificial here. We could write the basis as real-valued $\sin kx$ and $\cos kx$ functions with real coefficients. This achieves the same results but uses somewhat more technical formulas.

Fourier Series, II

We have conveniently chosen the basis ϕ_k so that,

$$\langle \phi_k, \phi_\ell \rangle = \begin{cases} 1, & k = \ell \\ 0, & k \neq \ell \end{cases}$$

Such basis functions are orthonormal.

There is a unique solution for the $\widehat{u}_k$ that minimizes the loss, and using basis orthonormality the solution has a fairly simple expression,

$$\widehat{u}_k = \langle u, \phi_k \rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u(x) e^{-ikx} \mathrm{d}x.$$

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This gives us a first taste of some functional analysis: Define,

$$L^2 = L^2\left([0, 2\pi]; \mathbb{C}\right) = \left\{ f : [0, 2\pi] \to \mathbb{C} \;\middle|\; \|f\|_2^2 < \infty \right\}.$$

Then Fourier Series representations are complete in L^2:

$$u \in L^2 \quad \Longrightarrow \quad \lim_{N \to \infty} \left\| u(x) - \sum_{k=-N}^{N} \widehat{u}_k \phi_k(x) \right\|_2 = 0,$$

and orthonormality of the basis results in Parseval's identity,

$$u \in L^2 \quad \Longrightarrow \quad \|u\|_2^2 = \sum_{k \in \mathbb{Z}} \left| \widehat{u}_k \right|^2.$$

Fourier approximation

$$u(x) \stackrel{L^2}{=} \sum_{k \in \mathbb{Z}} \widehat{u}_k \phi_k(x), \qquad \widehat{u}_k = \langle u, \phi_k \rangle$$

This is all well and good, but how does this serve us *computationally*?

With finite storage, we have to truncate the infinite series,

$$u(x) \approx u_N(x) := \sum_{|k| \leqslant N} \widehat{u}_k \phi_k(x)$$

How well does u_N approximate u?

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But let's focus on one sin at a time....

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So our question regards how *compressible* the infinite series is with respect to the truncation N:

$$\|u - u_N\|_2^2 \overset{?}{\lesssim} h(N),$$

for some function $h(N)$.

- h decays quickly with $N \rightarrow u$ is very compressible
- h decays slowly with $N \rightarrow u$ is not very compressible

Projections

Before investigating Fourier approximation results, it's worthwhile to introduce additional concepts: Projections.

Given an operator $P : L^2 \to V$, where $V \subset L^2$ is some subspace of L^2, then P is a projection operator if

$$P^2 = P.$$

The action $u \mapsto Pu$ projects u onto V.
The action $u \mapsto (I - P)u$ projects u onto some subspace W such that $V \bigoplus W = L^2$.

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A projection operator P is orthogonal if $W \perp V$, equivalently if for every $u, v \in L^2$:

$$P = P^*, \qquad \langle P^* u, v\rangle := \langle u, Pv\rangle.$$

$u = Pu + (I-P)u$

$P = P^* \Rightarrow \|u\|^2 = \|Pu\|^2 + \|(I-P)u\|^2$

$\forall\ K>1,\ \exists\ P\ (P^2 = P)$ s.t. $\|Pu\| > K\|u\|$

Truncation and projection

We are considering the truncation,

$$\sum_{k \in \mathbb{Z}} \widehat{u}_k \phi_k(x) \overset{L^2}{=} u \approx u_N = \sum_{|k| \leqslant N} \widehat{u}_k \phi_k(x).$$

This truncation is an orthogonal projector.

Theorem

Define P_N as the operator, , $N \in \mathbb{N}_0$

$$P_N u = u_N = \sum_{|k| \leqslant \cancel{\mathbb{Z}}\, N} \widehat{u}_k \phi_k(x), \qquad\qquad u \overset{L^2}{=} \sum_{k \in \mathbb{Z}} \widehat{u}_k \phi_k.$$

Then P_N is an orthogonal projection operator.

$\|u - P_N u\| \leq ?$

A basic approximation estimate, I

Can we bound $\|u - P_N\|_2$? First note that,

$$\|u - P_N u\|_2^2 = \sum_{|k|>N} |\widehat{u}_k|^2 .$$

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Integration by parts is our friend, and note that,

$$\begin{aligned} \langle u, \phi_k \rangle = \widehat{u}_k &= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u(x) e^{-ikx} \mathrm{d}x \\ &= \frac{i}{k\sqrt{2\pi}} u(x) e^{-ikx} \Big|_0^{2\pi} - \frac{i}{k\sqrt{2\pi}} \int_0^{2\pi} u'(x) e^{-ikx} \mathrm{d}x. \end{aligned}$$

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\end{aligned}$$

Note that, conveniently, the first term vanishes if $u(0) = u(2\pi)$.
This is, of course, quite reasonable since we are approximating with periodic functions.

Note also that the remaining integral is the Fourier series coefficient for the derivative, $u'(x)$:

$$u'(x) = \sum_{|k| \in \mathbb{Z}} \widehat{u'}_k \phi_k(x), \qquad \widehat{u'}_k = \left\langle u', \phi_k \right\rangle.$$

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Thus, if u is periodic and $u' \in L^2$ (so that $\widehat{u}_k$ is well-defined), then

$$\widehat{u}_k = -\frac{i}{k} \widehat{u'}_k.$$

A basic approximation estimate, II

$$\left\| u - P_N u \right\|_2^2 = \sum_{|k|>N} \left| \widehat{u}_k \right|^2,$$

$$\widehat{u}_k = -\frac{i}{k} \widehat{u'}_k.$$

This very basic estimate for Fourier series coefficients implies:

$$\begin{aligned}\left\| u - P_N u \right\|_2^2 = \sum_{|k|>N} \frac{1}{|k|^2} \left| \widehat{u'}_k \right|^2 \leqslant \frac{1}{N^2} \sum_{|k|>N} \left| \widehat{u'}_k \right|^2 \leqslant \frac{1}{N^2} \sum_{k\in\mathbb{Z}} \left| \widehat{u'}_k \right|^2 \\ = \frac{1}{N^2} \left\| u' \right\|_2^2,\end{aligned}$$

where the last relation is Parseval's identity.

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where the last relation is Parseval's identity.
We have just proven the following:

Theorem

Suppose $u, u' \in L^2$*, and that* $u(0) = u(2\pi)$*. Then,*

$$\|u - P_N u\|_2 \leqslant \frac{1}{N}\|u'\|_{L^2}$$

Sobolev spaces

To generalize this result, some additional notation will be helpful.

Definition (Sobolev spaces)

Given $s \in \mathbb{N}_0 = \{0, 1, \ldots, \}$, the ($L^2$ periodic) Sobolev space of functions is given by,

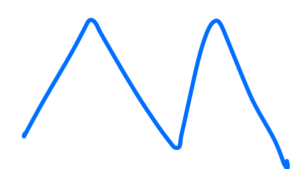
$$H_p^s([0, 2\pi]; \mathbb{C}) := \big\{ f : [0, 2\pi] \to \mathbb{C} \;\big|\; f^{(k)} \in L^2([0, 2\pi]; \mathbb{C}) \text{ for all } 0 \leqslant k \leqslant s, \\ f^{(k)}(0) = f^{(k)}(2\pi) \text{ for all } 0 \leqslant k \leqslant s - 1 \big\}$$

The *norm* on H^s is defined as,

$$\|u\|_{H^s}^2 := \sum_{k=0}^{s} \left\| u^{(k)} \right\|_2^2 .$$

Some specializations of interest:

- $s = 0 \implies H^0 = L^2$
- $s > 0 \implies$ continuous functions $\cancel{\subset}$ $\supset$ H^s

$f(x) =$ 

$f \in H^1$

$f \notin H^2$

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$\ast$ $H^s([0,2\pi];\mathbb{C})$

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- $s = 0 \implies H^0 = L^2$
- $s > 0 \implies$ continuous functions $\subset H^s$

The parameter s encodes the "amount" of smoothness that functions have, and the following inclusions hold:

$$H_p^r \subset H_p^s, \qquad r > s \geqslant 0.$$

General approximation results

The language of Sobolev spaces is the standard language in which to technically describe convergence rates of Fourier Series approximations.

Theorem

If $u \in H_p^s$, *then*

$$\|u - P_N u\|_{L^2} \leqslant N^{-s} \|u\|_{H^s}$$

Note that $s = 1$ is our previous result.

In terms of degrees of freedom, M, $\|u - P_N u\|_{L^2} \lesssim (M/2)^{-s} \|u\|_{H^s}$, which is *fantastic* for large s.

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If $u \in H^s_p$, *then*

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Actually, something even stronger is true about Fourier approximation:

Theorem

If $u \in H^s_p$, *then for every* $0 \leqslant r < s$,

$$\|u - P_N u\|_{H^r} \leqslant N^{-(s-r)} \|u\|_{H^s}.$$

This result demonstrates tradeoff between smoothness of the function versus the strength of the norm under which convergence is sought.

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